BIFURCATIONS, DYNKIN DIAGRAMS, AND MODALITY OF ISOLATED SINGULARITIES

A. M. Gabriélov

Introduction

Let F(x) be an analytic function in $\mathbf{C}^n$ with an isolated singularity of multiplicity μ at the origin,* $F(0) = 0$. The modality of the singularity of F(x) (compare [2]) is the maximum dimension of the set of orbits of the group G of germs of biholomorphic transformations $(\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ in a neighborhood of the orbit of the function F(x). (We have in mind the orbits of the natural action of the group G on the space of germs of analytic functions which reduce to zero in the first derivatives at the origin.)

In the present paper, we prove that the modality of the singularity of F(x) coincides with its proper modality, i.e., with the dimension of the set of those values of the parameter λ of a versal deformation of this singularity for which the corresponding functions $F_\lambda(x)$ have a singular point of the same multiplicity as F(x) and are equal to zero at the origin.

In § 1, we establish the connection between the bifurcation diagram Σ of a versal deformation of the singularity of F(x) and the transversal T to the orbit of the group G which passes through F(x). Namely, the natural mapping $\tau: T \to \Sigma$ proves to be proper and bimeromorphic. From this follow, in particular, the irreducibility of the bifurcation diagram and the indecomposability of the covering $\Sigma \to \mathbf{C}^{\mu-1}$. In § 2, we deduce from this the connectedness of the Dynkin diagram of the singularity of F(x). As a corollary of the connectedness of the Dynkin diagram, we prove that an isolated singular point cannot be distributed under a deformation over several singular points all of the critical values at which coincide.†

In § 3, we deduce from the results of the preceding paragraphs the semicontinuity of the proper modality and, finally, the coincidence of modality and proper modality.

The author is deeply grateful to V. I. Arnol'd for numerous helpful discussions. Theorems 5 and 6 of the present paper were originally formulated by Arnol'd as hypotheses. The author is also grateful to A. G. Kushnirenko and V. P. Palamodov for helpful discussions.

§ 1. The Bifurcation Diagram‡

Let F(x) be an analytic function in $\mathbf{C}^n$ with an isolated singular point x^0 of multiplicity μ, $F(x^0) = u^0$. Let r^0 be an admissible radius for the singularity $(F(x), x^0)$; i.e., let for all r, $0 < r \leq r^0$, the set $F(x) = u^0$ transversally intersect the sphere $S_r = \{|x - x^0| = r\}$. In the sequel, in all statements containing x, it is assumed that $|x - x^0| \leq r^0$, and all deformations are assumed to be so small that the condition of transversality of the intersection with the sphere S_{r^0} is not violated.

Let $F_\lambda(x)$ $(\lambda = (\lambda_0, \ldots, \lambda_{\mu-1}),\ F_0(x) = F(x))$ be a minimal versal deformation of the singularity $(F(x), x^0)$, $\Sigma \subset \mathbf{C}^\mu_\lambda$ be the bifurcation diagram of this deformation, i.e., the set of those values λ for which $F_\lambda(x)$ has critical value equal to u^0. We may assume that $F_\lambda(x) = F_{(0,\lambda')}(x) - \lambda_0$, where $\lambda' = (\lambda_1, \ldots, \lambda_{\mu-1})$,

*Here and below, multiplicity means the Milnor number of the singularity, i.e., the multiplicity of the gradient mapping.
†Another proof of this assertion, based on a theorem of A'Campo [5], was obtained independently by Lê Dũng Tráng [4].
‡The notation introduced at the beginning of the paragraph is used in the sequel without reference.

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 8, No. 2, pp. 7-12, April-June, 1974. Original article submitted October 29, 1973.

and $F_{(0,\lambda')}(x^0) = u^0$. Below, in place of $F_{(0,\lambda')}$, we will write simply $F_{\lambda'}$. Let $\rho: C^{\mu}_{\lambda} \to C^{\mu-1}_{\lambda'}$ be a projection. Then $\rho_\Sigma: \Sigma \to C^{\mu-1}_{\lambda'}$ is a μ-sheeted ramified covering. Let $\Delta \subset C^{\mu-1}_{\lambda'}$ be the discriminant set of this covering. The set Δ corresponds to the non-Morse functions in the deformation $F_{\lambda'}$ and has two components, one of which (the "caustic") corresponds to functions having a degenerate singular point and the other, to functions with two nondegenerate singular points with the same critical values.

Let Z be the space of germs at x^0 of analytic functions which have at this point a singularity with critical value u^0. The group G of germs of biholomorphic transformations $(C^n, x^0) \to (C^n, x^0)$ acts on Z in a natural way. Let $\check{F}_\alpha(x)$ $(\alpha = (\alpha_1, \ldots, \alpha_{\mu-1}),\ \check{F}_0(x) = F(x))$ be the transversal in Z to that orbit of the group G which passes through F(x). We consider $\check{F}_\alpha(x)$ as a deformation of the function F(x), and let $\tau: C^{\mu-1}_\alpha \to C^{\mu}_\lambda$ be the mapping under which the deformation $\check{F}_\alpha$ is induced from the versal deformation F_λ. Clearly, $\tau(C^{\mu-1}_\alpha) \subseteq \Sigma$.

THEOREM 1. The mapping $\tau: C^{\mu-1}_\alpha \to \Sigma$ is proper and bimeromorphic.

Proof. We may assume that $x^0 = u^0 = 0$ and $\partial^2 F(0) = 0$. Then as a minimal versal deformation of the singularity of F(x), we can take

$$F_\lambda(x) = F(x) - \lambda_0 - \sum_{i=1}^{n} \lambda_i x_i + \sum_{j=n+1}^{\mu-1} \lambda_j \varphi_j(x),$$

where $(\varphi_j(x))$ is a basis of the space $\mathfrak{m}^2/(\partial F/\partial x_i)$. As the transversal to the orbit of the group G we can take

$$\check{F}_\alpha(x) = F(x+a) - \sum_{j=n+1}^{\mu-1} \alpha_j \varphi_j(x+a) - \Big(F(a) + \sum_{j=n+1}^{\mu-1} \alpha_j \varphi_j(a)\Big) - \sum_{i=1}^{n} x_i \Big(\frac{\partial F}{\partial x_i}(a) + \sum_{j=n+1}^{\mu-1} \alpha_j \frac{\partial \varphi_j}{\partial x_i}(a)\Big),$$

where $a = (\alpha_1, \ldots, \alpha_n)$. [We recall that the family F_λ is a versal deformation if its tangent vectors form a basis of the space $C\{x\}/(\partial F/\partial x_i)$, and the family $\check{F}_\alpha$ of functions having a singularity at the origin with critical value zero is transversal to the orbit of G if its tangent vectors form a basis of the space $\mathfrak{m}^2/\mathfrak{m}(\partial F/\partial x_i)$.]

For such a choice of the families F_λ and $\check{F}_\alpha$, the mapping τ can be defined as follows:

$$\lambda_0 = F(\alpha_1, \ldots, \alpha_n) + \sum_{j=n+1}^{\mu-1} \alpha_j \varphi_j(\alpha_1, \ldots, \alpha_n) - \sum_{i=1}^{n} \alpha_i \Big(\frac{\partial F}{\partial x_i}(\alpha_1, \ldots, \alpha_n) + \sum_{j=n+1}^{\mu-1} \alpha_j \frac{\partial \varphi_j}{\partial x_i}(\alpha_1, \ldots, \alpha_n)\Big);$$

$$\lambda_i = \frac{\partial F}{\partial x_i}(\alpha_1, \ldots, \alpha_n) + \sum_{j=n+1}^{\mu-1} \alpha_j \frac{\partial \varphi_j}{\partial x_i}(\alpha_1, \ldots, \alpha_n) \text{ for } i = 1, \ldots, n;$$
$$\lambda_j = \alpha_j \text{ for } j = n+1, \ldots, \mu-1.$$

Indeed, the identity $\check{F}_\alpha(x) = F_{\tau(\alpha)}(x+a)$ is obvious. From these formulas, it is seen that $\lambda = \tau(\alpha)$ if and only if $\lambda_j = \alpha_j$ for $j = n+1, \ldots, \mu-1$ and the function $F_\lambda(x)$ has a singularity at the point $(\alpha_1, \ldots, \alpha_n)$ with critical value zero.

Since for $\lambda = 0$ the function $F_0(x) = F(x)$ has an isolated singular point, it follows that the mapping τ is proper. Since for sufficiently general values λ' the function $F_{\lambda'}$ has no more than one singular point with a given critical value, the transformation τ is bimeromorphic.

The theorem is proved.

COROLLARY 1. The set Σ is irreducible, and its normalization is nonsingular.

COROLLARY 2. The monodromy group of the covering ρ_Σ is the symmetric group $S(\mu)$.

Proof. Let $L \subset C^{\mu-1}_{\lambda'}$ be a complex line in general position. By a theorem of Zariskii (see [3]), the mapping $\pi_1((C^{\mu-1} \setminus \Delta) \cap L) \to \pi_1(C^{\mu-1} \setminus \Delta)$ is an epimorphism, and the group $\pi_1(C^{\mu-1} \setminus \Delta)$ is generated by the circuits in L around the points of the set $\Delta \cap L$. From considerations of codimension, the points $\lambda' \in \Delta \cap L$ correspond to functions $F_{\lambda'}$ which have, besides Morse singularities, at most one singular point of type A_2. The circuits around these points induce, in the monodromy group of the covering ρ_Σ, either the identity transformation or an elementary permutation (two points are transposed while the rest remain fixed). By Corollary 1, the monodromy group of ρ_Σ is transitive. The assertion now follows from the fact that a transitive subgroup of $S(\mu)$ which is generated by elementary permutations coincides with $S(\mu)$.

COROLLARY 3. The covering ρ_Σ is indecomposable (i.e., has no nontrivial factor-coverings).

For the proof, see [1], § 2, Lemma 2.

§ 2. Connectedness of the Dynkin Diagram

Let $\lambda' \notin \Delta$ and let u_m ($m = 1, \ldots, \mu$) be the critical values of the function $F_{\lambda'}$. We choose in the plane $\mathbf{C}_u$ a point u^* which is not a critical value of $F_{\lambda'}$ and construct a system of paths $\pi_m(s)$ [$s \in [0, 1]$, $\pi_m(0) = u^*$, $\pi_m(1) = u_m$] which intersect the point u^* only for $s = 0$ and do not pass through critical values of $F_{\lambda'}$ for $s < 1$. As is well known, one can construct from such a system of paths a basis of the vanishing cycles in $\tilde{H}_{n-1}(V, Z)$, where $V = \{x: |x| < r^0, F_{\lambda'}(x) = u^*\}$. [Here, $\tilde{H}_*(V, Z)$ is the reduced compact homology of the set V, i.e., the kernel of the mapping $H_*(V, Z) \to H_*(\text{point}, Z)$.] If n is odd, then from the intersection matrix of the basis obtained, one can construct a graph whose vertices are the elements of the basis, where two vertices are connected by k solid edges if the intersection index of the corresponding cycles equals $(-1)^{(n+1)/2}k$ and by k dotted edges if the intersection index is $(-1)^{(n-1)/2}k$. In the case of even n, one should carry out the same construction for the function $F(x) + z^2$. The resulting graph is called the Dynkin diagram of the singularity of F(x). We note that the Dynkin diagram is not uniquely defined: it depends on the choice of the system of paths $\pi_m(s)$ (and on the orientation of the cycles).

THEOREM 2. The Dynkin diagram of the singularity of F(x) is connected.*

Proof. We will show that the decomposition of the Dynkin diagram into connected components does not depend on the system of paths. Indeed, let $\lambda' \notin \Delta$, u_m ($m = 1, \ldots, \mu$) be the critical values of the function $F_{\lambda'}$, $\pi_m(s)$ and $\pi'_m(s)$ be two systems of paths, $\pi_m(1) = \pi'_m(1) = u_m$, (e_m) and (e'_m) be the corresponding bases of the vanishing cycles, and D and D' be the corresponding Dynkin diagrams. Since the path $\pi'_m(s)$ differs from the path $\pi_m(s)$ by the composition of simple loops corresponding to paths $\pi_j(s)$,† the cycle e'_m is obtained from the cycle e_m by means of a succession of Picard–Lefschetz transformations $T_j: e \mapsto e + (-1)^{(n+1)/2}(e, e_j)e_j$. Therefore, the cycle e'_m can differ from the cycle e_m only by a combination of cycles e_j lying in the same connected component of D as e_m. Consequently, if e_{m_1} and e_{m_2} are in different connected components of D, then e'_{m_1} and e'_{m_2} will be in different connected components of D'. Since in this argument the systems of paths $\pi_m(s)$ and $\pi'_m(s)$ can exchange places, the decompositions into connected components of the diagrams D and D' coincide, as was claimed.

Thus, the decomposition of the Dynkin diagram into connected components determines a partition, independent of the system of paths, of the set of critical values of $F_{\lambda'}(x)$, i.e., of the fibre of the covering ρ_Σ over the point λ'. Since under variation of λ' and a continuous deformation of the system of paths $\pi_m(s)$ the Dynkin diagram does not change, this partition is defined in every fibre, depends continuously on λ', and, consequently, determines a factor-covering of ρ_Σ. By Corollary 3, this factor-covering is trivial. Therefore, for the proof of the connectedness of the Dynkin diagram, it is sufficient to prove that it does not coincide with ρ_Σ, i.e., that the Dynkin diagram cannot be decomposed into connected components consisting of individual points. This, in turn, follows from the fact that every singularity of multiplicity greater than one is abutted by a singularity of type A_2 whose Dynkin diagram is connected. The theorem is proved.

Remark. Using the same arguments, one can prove the connectedness of the Dynkin diagram reduced modulo any p.

THEOREM 3. Let $F_t(x)$ ($t \in [0, t_0]$, $F_0(x) = F(x)$) be a deformation of the function F(x) for which its singular point x^0 decomposes into distinct singular points $x^1(t), \ldots, x^k(t)$ ($k \geq 2$). Then $F_t(x^i(t)) \neq F_t(x^j(t))$ for some i, j.‡

Proof. Let us assume that the assertion is false, i.e., that all critical values $F_t(x^i(t))$ of the function F_t coincide. It is easily shown that, then, the Dynkin diagram of the singularity of F(x) is not connected

---

*Remark in Proof. As has become known to the author, the proof of this theorem was also obtained by Lazzeri (see [6]).

†We recall that a simple loop corresponding to the path $\pi(s)$ ($s \in [0, 1]$) in C is a closed path which is obtained by approaching the point $\pi(1)$ from $\pi(0)$ along the path $\pi(s)$, going around $\pi(1)$, and returning to $\pi(0)$ via $\pi(s)^{-1}$.

‡Another proof of this assertion, based on a theorem of A'Campo, was obtained independently by Lê Dũng Tráng (see [5]).

(cycles corresponding to distinct singular points of F_t do not intersect and, consequently, lie in different connected components of the Dynkin diagram), which contradicts Theorem 2.

§ 3. Modality and Proper Modality

Definition 1. The modality of the singularity $(F(x), x^0)$ is the largest number m such that in any neighborhood of the point F(x) there exists in the family $F_\alpha(x)$ an m-dimensional analytic subset whose intersection with every orbit of the group G is either empty or discrete.

Definition 2. The proper modality $M(F(x), x^0)$ of the singularity $(F(x), x^0)$ is the dimension at the origin of the set of values λ' for which $F_{\lambda'}$ has a singular point of multiplicity μ.

THEOREM 4. Let X be the set of those $\lambda' \in \mathbf{C}^{\mu-1}$ over which the covering ρ_Σ has at most k sheets. If the dimension at the origin of the set X equals d, then the proper modality of the singularity $(F(x), x^0)$ is at least $d-k+1$.

Proof. It is not difficult to show that there exist analytic functions $S_j(\lambda')$ $(j = 2, \ldots, \mu)$ such that the set of those λ' over which the covering ρ_Σ has at most μ' sheets is defined by the conditions $S_\mu(\lambda') = \ldots = S_{\mu'+1}(\lambda') = 0$. If $\lambda' \in X$, then $S_\mu(\lambda') = \ldots = S_{k+1}(\lambda') = 0$. Therefore, the set Y of those $\lambda' \in X$ over which ρ_Σ has just one sheet is distinguished by the conditions $S_k(\lambda') = \ldots = S_2(\lambda') = 0$, and consequently, $\dim Y \geq d-k+1$. It follows from Theorem 3 that for $\lambda' \in Y$, the function $F_{\lambda'}(x)$ has a singular point of multiplicity μ. Consequently, the proper modality of the singularity $(F(x), x^0)$ is at least $d-k+1$, as we were required to prove.

THEOREM 5. Proper modality is upper semicontinuous.

Proof. Let us assume that the assertion of the theorem is false. Since the deformation F_λ is versal, there exist sequences $\lambda_i \to 0$ and $x_i \to x^0$ such that $M(F_{\lambda_i}(x), x_i) > M(F(x), x^0)$. We may assume that for every i the function F_{λ_i} has a singularity at the point x_i of multiplicity ν and proper modality $N > M(F, x^0)$, where ν and N do not depend on i. Let X be the set of those $\lambda' \in \mathbf{C}^{\mu-1}$ over which the covering ρ_Σ has at most $\mu-\nu+1$ sheets. We will show that $\dim X \geq N+\mu-\nu$.

We consider the deformation F_λ in a neighborhood of the point λ_i as a deformation of the singularity $(F_{\lambda_i}(x), x_i)$. It is easily shown that, for sufficiently small λ_i, this deformation is versal. Consequently, the corresponding mapping $\theta_i : \mathbf{C}^\mu_\lambda \to \mathbf{C}^\nu$, where $\mathbf{C}^\nu$ is the parameter space of a minimal versal deformation of the singularity $(F_{\lambda_i}(x), x_i)$, is a submersion. Since the proper modality of the singularity $(F_{\lambda_i}(x), x_i)$ equals N, there exists an $(N+1)$-dimensional set Y in the space $\mathbf{C}^\nu$ whose points correspond to the functions which have a singular point of multiplicity ν. The set $\theta_i^{-1}(Y) \subset \mathbf{C}^\mu_\lambda$ has dimension $N+\mu-\nu+1$, and if $\lambda \in \theta_i^{-1}(Y)$, then F_λ has a singular point of multiplicity ν, which means that the number of distinct critical values of the function F_λ is at most $\mu-\nu+1$. Consequently, $\rho(\theta_i^{-1}(Y)) \subset X$, and since λ_i can be chosen as near as desired to zero, the dimension of the set X at the origin is at least $N+\mu-\nu$, as claimed. It follows from Theorem 3 applied to X that the proper modality of the singularity $(F(x), x^0)$ is at least N, which contradicts the assumption $N > M(F(x), x^0)$. The theorem is proved.

THEOREM 6. The modality of the singularity $(F(x), x^0)$ coincides with its proper modality.

Proof. We may assume that $x^0 = 0$, $u^0 = 0$. Let X be the set of those values λ' for which $F_{\lambda'}(x)$ has singular point of multiplicity μ. We consider the set $Y = (\rho \circ \tau)^{-1}(X)$ in the space $\mathbf{C}^{\mu-1}_\alpha$ of parameters of the transversal to the orbit of the group G. It follows from Theorem 1 that $\dim Y = \dim X$.

Let $\alpha^0 \in Y$. If α^0 is sufficiently small, then the set $\check{F}_\alpha(x)$ is transversal to the orbit of the group G which passes through $\check{F}_{\alpha^0}(x)$. Since $\check{F}_{\alpha^0}(x)$ has singular point of multiplicity μ, the codimension of the orbit of this function equals $\mu-1$. Therefore, the orbit passing through $\check{F}_{\alpha^0}(x)$ intersects the set $\check{F}_\alpha(x)$, $\alpha \in Y$, discretely. Consequently, the modality of the singularity of F(x) is at least its proper modality.

We will now prove the reverse inequality. It is enough to show that there exists, as near as desired to the function F(x), a function whose proper modality is not less than the modality of F(x) and to use the semicontinuity of the proper modality.

Let $Y \subset \mathbf{C}^{\mu-1}_\alpha$ be a set whose intersection with every orbit of G is either empty or discrete, while the dimension of Y equals the modality of F(x). We may assume that all functions $\check{F}_\alpha(x)$, $\alpha \in Y$, have a singular point at the origin of multiplicity ν. Let $\alpha^0 \in Y$. It follows from the definition of Y that the mapping of Y to the transversal T to the orbit of G which passes through $\check{F}_{\alpha^0}(x)$ is proper. Therefore, the dimension of the image of Y in T equals the dimension of Y. From the theorem on the properness of the mapping τ, applied to $\check{F}_{\alpha 0}(x)$, and from the fact that all functions $\check{F}_{\alpha}(x)$, $\alpha \in Y$, have a singular point of multiplicity ν, it follows that the proper modality of the function $\check{F}_{\alpha 0}(x)$ is not less than Y, which is what was required for the proof of the theorem.

LITERATURE CITED

1. V. I. Arnol'd, "Topological invariants and algebraic functions," Funktsional. Analiz i Ego Prilozhen., 4, No. 2, 1-9 (1970).
2. V. I. Arnol'd, "Classification of unimodal critical points of functions," Funktsional. Analiz i Ego Prilozhen., 7, No. 3, 75-76 (1973).
3. Lê Dũng Tráng, "Les théorémes de Zarisky de type de Lefschetz," Centre de Math. de l'Ecole Polytechnique, Paris (1971).
4. Lê Dũng Tráng, "Une application d'un théoréme d'A'Campo a l'équisingularité," Preprint, Centre de Math. de l'École Polytechnique, Paris (1973).
5. N. A'Campo, "Le nombre de Lefschetz d'une monodromie," Indag. Math. (1973).
6. F. Lazzeri, "Some remarks on the Picard–Lefschetz monodromy," in: Quelques Jounées Singuliéres, Preprint, Centre de Math. de l'École Polytechnique, Paris (1974).