Complexity of Stratifications of Semi-Pfaffian Sets

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Abstract: An effective algorithm for a smooth (weak) stratification of a real semi-Pfaffian set is suggested, provided an oracle deciding consistency of a system of Pfaffian equations and inequalities is given. An explicit estimate of complexity of the algorithm and of the resulting stratification is given, in terms of the parameters of the Pfaffian functions defining the original semi-Pfaffian set. The algorithm is applied to sets defined by sparse polynomials and exponential polynomials.

1. Introduction.

In 1957, Whitney [19] proved that a real algebraic variety can be represented as a finite disjoint union of smooth manifolds which are semi-algebraic sets. Łojasiewicz [11, 12] extended Whitney's theorem to the class of real semi-analytic sets. His method explicitly involves the Weierstrass preparation theorem, that accounts for essential non-constructiveness of the proof and for the impossibility to restrict the class of functions defining the smooth strata. In 1993, Gabrielov [3] showed (as a part of an elementary proof of his theorem [2] on projections of semi-analytic sets) that smooth strata of a semi-analytic set X can be defined by functions belonging to the smallest extension of the family defining X which is closed under additions, multiplications and taking partial derivatives. Apart from polynomials, important classes sharing this property consist of all Pfaffian functions and of their special subclasses, such as exponential and sparse polynomials.

Pfaffian functions, introduced by Khovanskii [9, 10], define the semi-analytic (semi-Pfaffian) sets which have important global finiteness properties similar to those of semi-algebraic sets. Moreover, the characteristics that are finite (such as the number of isolated roots of a system of equations), can be effectively bounded from above, in terms of the format parameters of the defining functions.

Recently Gabrielov [4] estimated the multiplicity of intersections of Pfaffian varieties. The purpose of this paper is to show that the latter bound allows us to construct an algorithm which produces a smooth stratification for a semi-Pfaffian set and to estimate its complexity. Under stratification here we always mean a weak stratification, i.e. a subdivision into smooth non-intersecting pieces (strata) without any requirement on the boundary of a stratum to be a union of some other strata.

We always consider real semi-Pfaffian sets, although the main algorithm from the section 3 is applicable, without any change, to complex constuctable Pfaffian sets, with the inequalities "greater than" and "less than" replaced by "not equal". The estimate in [4] is valid in the complex case, too.

We are interested in the bounds on the parameters of the output of the algorithm and on its computational complexity. We shall give more precise definitions and bounds in the section 4 below. Let us mention now that the complexity turns out to be a doubly-exponential function in the number of variables n. For a fixed n, this function is singly-exponential in the maximal *order* r of Pfaffian functions involved and, for fixed n and r, polynomial in all the other parameters.

Note that the known stratification algorithms for general semi-algebraic sets ($r = 0$) have essentially the same complexity. They are based on a recursive application of a fast procedure for the quantifier elimination in the first order theory of reals (i.e., on an effective algorithmical version of the Tarski-Seidenberg principle) [8, 16]. The latter technique gives a much stronger result: a Whitney stratification of an arbitrary semi-algebraic set [14]. Let us mention also that a singly-exponential (in n) algorithm for the Whitney stratification is known for a rather broad class of real algebraic varieties [18].

On the other hand, our algorithm can handle semi-algebraic sets defined by *fewnomials*, or sparse polynomials, with the size of output estimated in terms of the number of non-zero monomials, independent of their degrees. Besides, it represents strata in a more convenient form.

The content of this paper is as follows. In section 2, the Pfaffian functions are defined (following Khovanskii [9, 10]) and certain parameters of these functions are introduced. We explain how the basic operations over the functions affect the parameters.

In section 3, an algorithm for a stratification of an elementary semi-Pfaffian set is described. Section 4 contains a complexity estimate of this algorithm.

Section 5 describes a stratification procedure for arbitrary semi-Pfaffian sets, while section 6 deals with special classes, defined by polynomials, fewnomials, exponential polynomials (dense and sparse).

2. Pfaffian functions.

Definition 1. (Sf. [9, 10, 4].) A *Pfaffian chain* of the *order* $r \geq 0$ and *degree* $\alpha \geq 1$ in an open domain $G \subset \mathbf{R}^n$ is a sequence of real analytic functions $f_1, \ldots, f_r$ in G satisfying Pfaffian equations

$$df_j(x) = \sum_{i=1}^{n} g_{ij}\big(x, f_1(x), \ldots, f_j(x)\big) dx_i, \text{ for } j = 1, \ldots, r. \tag{1}$$

Here $g_{ij}(x, y)$ are polynomials in $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_j)$ of degree not exceeding α. A function $f(x) = P(x, f_1(x), \ldots, f_r(x))$ where $P(x, y_1, \ldots, y_r)$ is a polynomial of degree not exceeding $\beta \geq 1$ is called a *Pfaffian function* of the order r and degree (α, β).

Remark 1. Note that our definition is more restrictive than the definition in [10, 4] where the Pfaffian chains were defined as sequences of nested integral manifolds of polynomial 1-forms. Our definition coincides with the definition of a special Pfaffian chain in [4]. Both definitions lead to essentially the same class of Pfaffian functions, although the orders and degrees of Pfaffian chains for the same Pfaffian function can be different according to these two definitions. We found our present definition to be more convenient to trace the behavior of parameters of Pfaffian functions under different operations. Also, it gives a better estimate for the multiplicity in [4].

Examples (a) Pfaffian functions of the order 0 and degree $(1, \beta)$ are polynomials of degree not exceeding β.

(b) The exponential function $f(x) = e^{ax}$ is a Pfaffian function of the order 1 and degree $(1, 1)$ in $\mathbf{R}$, due to the equation $df(x) = af(x)dx$.

(c) The function $f(x) = 1/x$ is a Pfaffian function of the order 1 and degree $(2, 1)$ in the domain $x \neq 0$, due to the equation $df(x) = -f^2(x)dx$.

(d) The logarithmic function $f(x) = \ln(|x|)$ is a Pfaffian function of the order 2 and degree $(2, 1)$ in the domain $x \neq 0$, due to the equations $df(x) = g(x)dx,\ dg(x) = -g^2(x)dx$, with $g(x) = 1/x$.

(e) The polynomial $f(x) = x^m$ can be considered as a Pfaffian function of the order 2 and degree $(2, 1)$ in the domain $x \neq 0$ (but not in $\mathbf{R}$), due to the equations $df(x) = mf(x)g(x)dx,\ dg(x) = -g^2(x)dx$, with $g(x) = 1/x$. The better way to deal with it, however, is to change the variable $x = \exp(u)$ reducing this case to the exponential function.

(f) The function $f(x) = \tan(x)$ is a Pfaffian function of the order 1 and degree $(2, 1)$ in the domain $x \neq \pi/2 + k\pi$, for all integer k, due to the equation $df(x) = (1 + f^2(x))dx$.

(g) The function $f(x) = \arctan(x)$ is a Pfaffian function in $\mathbf{R}$ of the order 2 and degree $(3, 1)$, due to the equations $df(x) = g(x)dx,\ dg(x) = -2xg^2(x)dx$, with $g(x) = (x^2 + 1)^{-1}$.

(h) The function $\cos(x)$ is Pfaffian of the order 2 and degree $(2, 1)$ in the domain $x \neq \pi + 2k\pi$, for all integer k, due to the equations $\cos(x) = 2f(x) - 1,\ df(x) = -f(x)g(x)dx, dg(x) = \frac{1}{2}(1 + g^2(x))dx$, with $f(x) = \cos^2\left(\frac{x}{2}\right)$ and $g(x) = \tan\left(\frac{x}{2}\right)$. Also, since $\cos(x)$ is a polynomial of degree m of $\cos\left(\frac{x}{m}\right)$, the function $\cos(x)$ is Pfaffian of the order 2 and degree $(2, m)$ in the domain $x \neq m\pi + 2km\pi$, for all integer k. The same is true, of course, for any shift of the above domain by a multiple of π. However, $\cos(x)$ is not Pfaffian in the whole real line.

The following lemmas (sf. [10]) provide additional means for construction of Pfaffian functions.

Lemma 1. *The sum (resp. product) of two Pfaffian functions, f_1 and f_2, of the orders r_1 and r_2 and degrees (α_1, β_1) and (α_2, β_2), is a Pfaffian function of the order $r_1 + r_2$ and degree $(\alpha, \max(\beta_1, \beta_2))$ (resp. $(\alpha, \beta_1 + \beta_2)$) where $\alpha = \max(\alpha_1, \alpha_2)$. If the two Pfaffian functions are defined by the same Pfaffian chain of the order r, the order of the sum and product is also r.*

Proof. We can combine the Pfaffian chains for the functions f_1 and f_2 into a Pfaffian chain for $f_1 + f_2$ and $f_1 f_2$. If a Pfaffian chain is common for the two functions, it is also a Pfaffian chain for their sum and product.

Lemma 2. *A partial derivative of a Pfaffian function of the order r and degree (α, β) is a Pfaffian function of the order r and degree $(\alpha, \alpha + \beta - 1)$.*

Proof. Let $f(x) = (f_1(x), \ldots, f_r(x))$ be a Pfaffian chain of the order r and degree α, and let $P(x, y)$ be a polynomial of degree β. The statement follows from the differentiation formula

$$\frac{\partial P(x, f(x))}{\partial x_i} = \left. \frac{\partial P(x, y)}{\partial x_i} \right|_{y=f(x)} + \sum_{j=1}^{r} \left. \frac{\partial P(x, y)}{\partial y_j} \right|_{y=f(x)} \frac{\partial f_j(x)}{\partial x_i}$$

after substitution $\partial f_j(x)/\partial x_i = g_{ij}(x, f_1(x), \ldots, f_j(x))$ from (1).

Lemma 3. *Let $z = (z_1, \ldots, z_l)$ and let $f(x, z)$ be a Pfaffian function of the order r_1 and degree (α_1, β_1) in a domain $G_1 \subset \mathbf{R}^{n+l}$. Let $h(x) = (h_1(x), \ldots, h_l(x))$ be a l-tuple of Pfaffian functions of the order r_2 and degree (α_2, β_2), with a common Pfaffian chain, defined in a domain $G_2 \subset \mathbf{R}^n$ such that $(x, h(x)) \in G_1$, for all $x \in G_2$. Then $f(x, h(x))$ is a Pfaffian function in G_2 of the order $r_1 + r_2$ and degree $(\alpha_1 \beta_2 + \alpha_2 + \beta_2 - 1, \beta_1)$.*

Proof. The Pfaffian chain for the functions h can be extended to a Pfaffian chain of the order $r_1 + r_2$ by adding the functions $f_j(x, h(x))$ where $f_j(x, y)$ constitute the Pfaffian chain for f. The statement follows from the differentiation formula

$$\frac{\partial f_j(x,h(x))}{\partial x_i} = \left.\frac{\partial f_j(x,z)}{\partial x_i}\right|_{z=h(x)} + \sum_{\nu=1}^{l} \left.\frac{\partial f_j(x,z)}{\partial z_\nu}\right|_{z=h(x)} \frac{\partial h_\nu(x)}{\partial x_i}$$

after substitution of the partial derivatives from the corresponding Pfaffian chains.

Definition 2. For a set of differentiable functions $\mathbf{h} = (h_1, \dots, h_k)$, a set of distinct indices $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \leq i_\nu \leq n$, and an index j, $1 \leq j \leq n$, different from all i_ν, we define partial differential operator

$$\partial_{\mathbf{h},\mathbf{i},j} = \det \begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} & \frac{\partial h_1}{\partial x_j} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} & \frac{\partial h_k}{\partial x_j} \\ \frac{\partial}{\partial x_{i_1}} & \cdots & \frac{\partial}{\partial x_{i_k}} & \frac{\partial}{\partial x_j} \end{pmatrix}.$$

When $k = 0$, the corresponding operator is simply $\partial_j = \partial/\partial x_j$. We define $\partial^m_{\mathbf{h},\mathbf{i},j}$ (resp. ∂^m_j) as the m-th iteration of $\partial_{\mathbf{h},\mathbf{i},j}$ (resp. ∂_j).

Lemma 4. *For a Pfaffian function f of the order r and degree (α, β), for a set $\mathbf{h} = (h_1, \dots, h_k)$ of Pfaffian functions of the order r and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_k)$ defined by the same Pfaffian chain as f, and for a set of distinct indices $\{\mathbf{i} = (i_1, \dots, i_k), j\}$, the function $\partial^m_{\mathbf{h},\mathbf{i},j} f(x)$ is a Pfaffian function of the order r and degree (α, β') where $\beta' = \beta + m[(\alpha - 1)(k + 1) + \beta_1 + \dots + \beta_k]$, defined by the same Pfaffian chain as f.*

Proof. The statement follows from the lemmas 1 and 2.

Proposition 1. *Let $\mathbf{i} = (i_1, \dots, i_k)$ be a set of distinct indices, $1 \leq i_\nu \leq n$. Let f be a Pfaffian function of the order r and degree (α, β) in an open neighborhood G of $x \in \mathbf{R}^n$, and let $\mathbf{h} = (h_1, \dots, h_k)$ be a set of Pfaffian functions of the order r and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_k)$ defined in G by the same Pfaffian chain as f, such that $h_1(x) = \dots = h_k(x) = 0$,*

$$\det \begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} \end{pmatrix} (x) \neq 0. \tag{2}$$

Let

$$M = M(k, r, \alpha, \beta, \beta_1, \ldots, \beta_k) =$$

$$= 2^{r(r-1)/2} \beta\beta_1 \cdots \beta_k \big(\min\{r,\ k+1\}\alpha - k + \beta + \beta_1 + \ldots + \beta_k + 1\big)^r. \tag{3}$$

Suppose that

$$\partial^{m_1}_{\mathbf{h},\mathbf{i},1} \cdots \partial^{m_n}_{\mathbf{h},\mathbf{i},n} f(x) = 0,\ \text{for } 0 \le m_1 + \ldots + m_n \le M,\ m_{i_1} = \ldots = m_{i_k} = 0. \tag{4}$$

Then the function f vanishes identically on $Y = \{h_1 = \ldots = h_k = 0\}$ in the neighborhood of x.

Proof. First, we want to reduce the problem to the case $n = k+1$. Suppose that $f|_Y \not\equiv 0$ in a neighborhood of x. As $f|_Y$ is analytic, the Taylor expansion of $f|_Y$ at x starts with terms of degree $\kappa < \infty$. Let $K \subset T_xY$ be the set of zeros of the initial form of $f|_Y$ at x of degree κ, i.e., the tangent cone to $\{f|_Y = 0\}$ at x. Let L be a linear subspace of dimension $k+1$ through x such that $\gamma = L \cap Y$ is one-dimensional and the tangent vector to γ at x does not belong to K. Then the order of $f|_\gamma$ at x is equal to κ. In particular, $f|_\gamma \not\equiv 0$ in the neighborhood of x. Replacing $\mathbf{R}^n$ by L and Y by γ, we can reduce the problem to $n = k+1$.

Renumerating coordinates reduces the problem to the case $(i_1, \ldots, i_k) = (1, \ldots, k)$, and (4) becomes

$$\partial^{m_n}_{\mathbf{h},\mathbf{i},n} f(x) = 0,\ \text{for } 0 \le m_n \le M.$$

The set Y is one-dimensional, and the operator $\partial_{\mathbf{h},\mathbf{i},n}$ is a differentiation along a vector field tangent to Y. Due to (2) this vector field is non-zero at x, and its integral curve passing through x contains a neighborhood of x in the set Y. If f does not vanish identically in the neighborhood of x in Y, the multiplicity at x of the Pfaffian intersection $\{f = h_1 = \ldots = h_k = 0\}$ is greater than M. This contradicts the bound on the multiplicities of the Pfaffian intersections in [4].

Definition 3. Let $f_1, \dots, f_l$ be a family of Pfaffian functions defined in an open domain $G \subset \mathbf{R}^n$. The *number of consistent sign assignments* for this family is the number of all consistent (having a solution in G) systems of equations and strict inequalities of the kind:

$$f_{i_1} = \ldots = f_{i_{l_1}} = 0,\ f_{j_1} > 0, \ldots, f_{j_{l_2}} > 0,\ f_{k_1} < 0, \ldots, f_{k_{l_3}} < 0,$$

where $\{i_1, \ldots, i_{l_1}, j_1, \ldots, j_{l_2}, k_1, \ldots, k_{l_3}\} = \{1, \ldots, l\}$. A (nonempty) set defined by such a system is called a *cell.* Note that any two cells have an empty intersection.

Obviously, the number of all consistent sign assignments does not exceed 3^l. However, if the domain $G = \mathbf{R}^n$, we can obtain a less trivial bound.

Proposition 2 (cf. [13]). *Let $f_1, \ldots, f_l$ be a family of Pfaffian functions of the order r and degrees $(\alpha, \beta_1), \ldots, (\alpha, \beta_l)$ defined in $\mathbf{R}^n$ by the same Pfaffian chain. Then the number of consistent sign assignments for this family does not exceed*

$$\min\{3^l,\ 4^{r^2+n+r} n^r (\alpha + \beta_1 + \cdots + \beta_l)^{n+r}\}.$$

Proof. The bound 3^l is trivial.

Choose one arbitrary point in each cell and obtain a finite set of points $\mathcal{X}$. There exists a positive $\varepsilon \in \mathbf{R}$ such that for every $x \in \mathcal{X}$ and every $i,\ 1 \le i \le l$, the inequality $f_i(x) > 0$ implies $f_i(x) > \varepsilon$, and $f_i(x) < 0$ implies $f_i(x) < -\varepsilon$.

Introduce a Pfaffian function, defined in $\mathbf{R}^n$:

$$g = \prod_{1 \le i \le l} (f_i + \varepsilon)^2 (f_i - \varepsilon)^2$$

of the order r, degree (α, β) where $\beta = 4 \sum_{1 \le i \le l} \beta_i$ (due to the lemma 1).

Let us prove that the points

$$x^{(1)} = \sigma^{(1)} \cap \mathcal{X},\ x^{(2)} = \sigma^{(2)} \cap \mathcal{X},$$

for two different cells, $\sigma^{(1)}$ and $\sigma^{(2)}$, belong to different connected components of $\{g > 0\}$ (according to the definition of ε, neither $x^{(1)}$ nor $x^{(2)}$ belong to $\{g = 0\}$).

Suppose that, contrary to our claim, $x^{(1)}$ and $x^{(2)}$ belong to the same connected component. It follows that there is a connected curve Γ containing $x^{(1)}$ and $x^{(2)}$ and belonging to this connected component.

Since $x^{(1)}$, $x^{(2)}$ belong to different cells, there exists at least one function f_{i_0} $(1 \leq i_0 \leq l)$ having different signs at $x^{(1)}$ and $x^{(2)}$.

Let, for instance, $f_{i_0}(x^{(1)}) > 0$ and $f_{i_0}(x^{(2)}) = 0$. Then $f_{i_0}(x^{(1)}) > \varepsilon$, so there is a point $x \in \Gamma$ such that $f_{i_0}(x) = \varepsilon$. Hence, $g(x) = 0$, which contradicts the definition of Γ.

All other combinations of signs of f_{i_0} at $x^{(1)}$ and $x^{(2)}$ can be treated analogously. Therefore, the number of cells (consistent sign assignments) does not exceed the number K of connected components of $\{g > 0\}$. An estimate

$$K \leq 2^{r^2} \beta^n (r\alpha + n\beta)^r < 4^{r^2} n^r (\alpha + \beta)^{n+r}$$

follows from a more general result of Khovanskii [10].

Definition 4. An *elementary semi-Pfaffian set* is defined by a system of equations and inequalities

$$f_1(x) = \ldots = f_I(x) = 0, \quad g_1(x) > 0, \ldots, g_J(x) > 0 \tag{5}$$

where f_i, g_j are Pfaffian functions with a common Pfaffian chain defined in an open domain $G \subset \mathbf{R}^n$ A *semi-Pfaffian set* is a finite union of elementary semi-Pfaffian sets with a common Pfaffian chain. Thus, a semi-Pfaffian set can be defined by a Boolean formula (in a disjunctive normal form, DNF) with atomic subformulas of the kind $f_i = 0$ and $g_j > 0$. If formula consists of N disjunctions and all functions f_i, g_j $(i = 1, \ldots, I;\ j = 1, \ldots, J)$ are Pfaffian of the order r and degree (α, β), defined by a common Pfaffian chain, then the 6-tuple $(N, I, J, r, \alpha, \beta)$ is called the *format* of the formula. For a system (5), define the format as (I, J, r, α, β).

Definition 5. A *weak stratification* of a semi-Pfaffian set X is a subdivision of X into a disjoint union of smooth, not necessarily connected (or even having a finite number of connected components), possibly empty semi-Pfaffian subsets X_α, called *strata*. A stratification is *elementary* if all strata are elementary semi-Pfaffian sets. The system of equalities and inequalities for each stratum X_α of codimension k includes a set of k Pfaffian functions $h_{\alpha,1}, \ldots, h_{\alpha,k}$ such that $h_{\alpha,j}|_{X_\alpha} \equiv 0$, for $j = 1, \ldots, k$, and $dh_{\alpha,1} \wedge \ldots \wedge dh_{\alpha,k} \neq 0$ at every point of X_α. Note that for the algebraic case exactly the same kind of strata (under the name *algebraic partial manifolds*) were considered by Whitney [19]. We do not require the boundary of a stratum to be a union of some other strata.

3. Algorithm. The following algorithm for a weak stratification of an elementary semi-Pfaffian set is a modification of the algorithm suggested in [3]. It is based on the Whitney [19] approach to stratification of real semi-algebraic sets.

Let $X \subset \mathbf{R}^n$ be an elementary semi-Pfaffian subset (5) of the format (I, J, r, α, β).

Let $M_0 = 1,\ M_1 = M(0, r, \alpha, \beta)$ (see (3) for the definition of $M(k, r, \alpha, \beta, \beta_1, \ldots, \beta_k)$). For $1 \leq k < n$, we define consecutively $\beta_k = \beta + (M_k - 1)[(\alpha - 1)k + \beta_1 + \ldots + \beta_{k-1}]$ and $M_{k+1} = M(k, r, \alpha, \beta_k, \beta_1, \ldots, \beta_k)$.

For $1 \leq k \leq n$, consider a sequence $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \ldots, \mathbf{m}^k)$ where $\mathbf{i} = (i_1, \ldots, i_k)$, $\mathbf{j} = (j_1, \ldots, j_k)$, and $\mathbf{m}^\mu = (m_1^\mu, \ldots, m_{i_\mu}^\mu)$, for $1 \leq \mu \leq k$, with the following properties.

$$1 \leq i_1 < \ldots < i_k \leq n, \quad 1 \leq j_\mu \leq I; \tag{6}$$

$$0 \leq m_1^\mu + \ldots + m_{i_1}^\mu \leq M_1,\ \ldots,\ 0 \leq m_{i_{\mu-1}+1}^\mu + \ldots + m_{i_\mu}^\mu \leq M_\mu;\ 0 < m_{i_\mu}^\mu; \tag{7}$$

$$(m_{i_\nu}^\mu, \ldots, m_1^\mu, j_\mu) \prec (m_{i_\nu}^\nu, \ldots, m_1^\nu, j_\nu),\ \text{for } 1 \leq \nu < \mu. \tag{8}$$

Here $\prec$ is the lexicographic order.

For $1 \leq i \leq i_1$, we define $\hat{\partial}_i = \partial_i$. Let

$$h_1 = \hat{\partial}_{i_1}^{m_{i_1}^1 - 1} \hat{\partial}_{i_1 - 1}^{m_{i_1 - 1}^1} \cdots \hat{\partial}_1^{m_1^1} f_{j_1}, \quad \hat{\partial}_i = \partial_{h_1, i_1, i},\ \text{for } i_1 < i \leq i_2;$$

$$h_2 = \hat{\partial}_{i_2}^{m_{i_2}^2 - 1} \hat{\partial}_{i_2 - 1}^{m_{i_2 - 1}^2} \cdots \hat{\partial}_1^{m_1^2} f_{j_2}, \quad \hat{\partial}_i = \partial_{h_1, h_2, i_1, i_2, i},\ \text{for } i_2 < i \leq i_3;$$

and so on till

$$h_k = \hat\partial_{i_k}^{m^k_{i_k}-1}\,\hat\partial_{i_k-1}^{m^k_{i_k-1}}\cdots\hat\partial_1^{m^k_1} f_{j_k}, \quad \hat\partial_i = \partial_{h_1,\ldots,h_k,i_1,\ldots,i_k,i}, \text{ for } i_k < i \le n.$$

Let $X^0 = \{x\in X;\ \hat\partial_n^{q_n}\cdots\hat\partial_1^{q_1} f_j(x) = 0, \text{ for } 1\le j\le I,\ 0\le q_1+\ldots+q_n\le M_1\}$, and

$$X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k} = \{x\in X,\ \hat\partial_n^{q_n}\cdots\hat\partial_1^{q_1} f_j(x) = 0,$$

$$\text{for } 0\le q_1+\ldots+q_{i_1}\le M_1,\ \ldots,\ 0\le q_{i_{k-1}+1}+\ldots+q_{i_k}\le M_k,$$

$$0\le q_{i_k+1}+\ldots+q_n\le M_{k+1},\ 1\le j\le I,$$

$$(q_{i_1},\ldots,q_1,j)\prec(m^1_{i_1},\ldots,m^1_1,j_1),\ \ldots,\ (q_{i_k},\ldots,q_1,j)\prec(m^k_{i_k},\ldots,m^k_1,j_k),$$

$$\hat\partial_{i_1}h_1(x)\ne 0,\ \ldots,\ \hat\partial_{i_k}h_k\ne 0\},$$

for $1\le k\le n$.

Theorem 1. *Each set $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$ is either empty or non-singular of codimension k, with*

$$\Delta = \det\begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} \end{pmatrix} \ne 0$$

at each point of $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$. The set X is a disjoint union of the sets $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$ over all sequences $(\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k)$ satisfying (6)-(8), *for $0\le k\le n$.*

Proof. The sets $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$ with $\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k$ satisfying the conditions (6)-(8) can be consecutively defined as follows. We consider all the partial derivatives $\partial_n^{q_n}\cdots\partial_1^{q_1} f_j$ with $q_1+\ldots+q_n\le M_1$, ordered lexicographically in $(q_n,\ldots,q_1,j)$. We consider either the set $X^0\subset\mathbf{R}^n$ where all these derivatives vanish, or a set $Z^1_{i_1,j_1,\mathbf{m}^1}$ of those $x\in X$ where all the derivatives in the list preceding $h'_1 = \partial_{i_1}^{m^1_{i_1}}\cdots\partial_1^{m^1_1} f_{j_1}$ vanish, while $h'_1(x)\ne 0$. Obviously, each $x\in X$ belongs either to X^0 or to one of the sets $Z^1_{i_1,j_1,\mathbf{m}^1}$, with $m^1_1+\ldots+m^1_{i_1}\le M_1,\ m^1_{i_1}>0$, and all these sets are disjoint.

In the first case, due to the proposition 1, all the functions f_j are identically zero in a neighborhood of each $x\in X^0$, hence X^0, if non-empty, is a non-singular open set in

$\mathbf{R}^n$. In the second case, we define $h_1 = \partial_{i_1}^{m^1_{i_1}-1} \partial_{i_1-1}^{m^1_{i_1-1}} \cdots \partial_1^{m^1_1} f_{j_1}$, so that $h_1' = \partial_{i_1} h_1$, and consider a non-singular submanifold $Y^1 = \{x \in X,\ h_1(x) = 0,\ h_1'(x) \neq 0\} \supseteq Z^1_{i_1,j_1,\mathbf{m}^1}$. Due to the lemma 4, the formats of all the Pfaffian functions that appear in the equations defining $Z^1_{i_1,j_1,\mathbf{m}^1}$ do not exceed (r, α, β_1). Let us denote the set of all these functions as F^1. Note that, for $i < i_1$, all the functions from F^1, including h_1, do not depend on x_i, due to the proposition 1.

We consider now the partial derivatives

$$\hat\partial_n^{q_n} \cdots \hat\partial_{i_1+1}^{q_{i_1+1}} = \partial_{h_1,i_1,n}^{q_n} \cdots \partial_{h_1,i_1,i_1+1}^{q_{i_1+1}}$$

of the functions $f_\nu \in F^1,\ \nu = (q_{i_1}, \ldots, q_1, j)$, along $Y^1 \cap \{x_i = \text{const},\ \text{for } i < i_1\}$ with $q_{i_1+1} + \ldots + q_n \leq M_2$, ordered lexicographically in $(q_n, \ldots, q_{i_1+1}, \nu)$. We consider either the set $X^1_{i_1,j_1,\mathbf{m}^1}$ where all these derivatives vanish, or a set $Z^2_{i_1,i_2,j_1,j_2,\mathbf{m}^1,\mathbf{m}^2}$ of those $x \in Z^1_{i_1,j_1,\mathbf{m}^1}$ where all the derivatives in the list preceding $h_2' = \hat\partial_{i_2}^{m^2_{i_2}} \cdots \hat\partial_1^{m^2_1} f_{j_2}$ vanish, while $h_2'(x) \neq 0$.

Again, each $x \in X$ belongs either to X^0, or to one of the sets $X^1_{i_1,j_1,\mathbf{m}^1}$, or to one of the sets $Z^2_{i_1,i_2,j_1,j_2,\mathbf{m}^1,\mathbf{m}^2}$, with $m^1_1 + \ldots + m^1_{i_1} \leq M_1,\ m^1_{i_1} > 0$,

$$m^2_{i_1+1} + \ldots + m^2_{i_2} \leq M_2,\ m^2_{i_2} > 0,\ (m^2_{i_1}, \ldots, m^2_1, j_2) \prec (m^1_{i_1}, \ldots, m^1_1, j_1),$$

and all these sets are disjoint.

We apply the proposition 1 to show that $X^1_{i_1,j_1,\mathbf{m}^1} \cap \{x_i = \text{const},\ \text{for } i < i_1\}$ is open in $Y^1 \cap \{x_i = \text{const},\ \text{for } i < i_1\}$. As all the functions in the equations defining $X^1_{i_1,j_1,\mathbf{m}^1}$ and Y^1 do not depend on x_i, for $i < i_1$, this implies that $X^1_{i_1,j_1,\mathbf{m}^1}$ is open in Y^1 in the neighborhood of each $x \in X^1_{i_1,j_1,\mathbf{m}^1}$. Hence $X^1_{i_1,j_1,\mathbf{m}^1}$ is non-singular of the codimension 1.

The same arguments as before show that $Z^2_{i1,i2,j_1,j_2,\mathbf{m}^1,\mathbf{m}^2}$ belongs to

$$Y^2 = \{x \in X,\ h_1(x) = h_2(x) = 0,\ h_2'(x) = \hat\partial_{i_2} h_2(x) =$$

$$= \partial_{i_1} h_1(x) \partial_{i_2} h_2(x) - \partial_{i_2} h_1(x) \partial_{i_1} h_2(x) \neq 0\},$$

with $h_2 = \hat\partial_{i_2}^{m^2_{i_2}-1} \hat\partial_{i_2-1}^{m^2_{i_2-1}} \cdots \hat\partial_1^{m^2_1} f_{j_2}$, which is non-singular of the codimension 2.

The continuation of this procedure leads to the consecutive definition of the sets $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$, for $k = 2, \ldots, n$, with $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \ldots, \mathbf{m}^k)$ satisfying the conditions (6)-(8). The same arguments as above show that all these sets are disjoint, non-singular, and the union of all these sets is equal to X, q.e.d..

The algorithm looks through all the sequences of the kind $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \ldots, \mathbf{m}^k)$ satisfying (6)-(8), and for each of them computes recursively the corresponding functions $\hat\partial_n^{q_n} \cdots \hat\partial_1^{q_1} f_j(x),\ h_\mu$. Each recursion step consists of computing of a determinant of a Jacobian matrix (see Definition 2) whose elements are polynomials in variables $x_1, \cdots, x_n$ and in at most r symbols of function belonging to the Pfaffian chain for the input. This computation can be done effectively by a version of a Gauss algorithm over the ring $\mathbf{R}[x_1, \ldots, x_n, u_1, \ldots u_r]$ (see, e.g., [7]). Thus, the algorithm outputs the system of equations and inequalities defining the set $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\ldots,\mathbf{m}^k}$.

This concludes the description of the algorithm.

4. Complexity of the algorithm.

The computation protocol of the algorithm is a sequence of arithmetic operations over polynomials in variables $x_1, \ldots, x_n$ and in symbols of functions occurring in the Pfaffian chain for $f_1, \ldots, f_I$, actually over real coefficients of these polynomials. An arithmetic operation over two reals, occuring in the sequence, is considered as an *elementary step* of the algorithm.

By the *complexity (running time)* of the algorithm we mean the number of its elementary steps (in the worst case) as a function of the format of the input system of inequalities.

For the complexity estimates we shall need the following lemma.

Lemma 5. *For $0 < k \le n$, the values $M_k = M(k-1, r, \alpha, \beta_k, \beta_1, \ldots, \beta_{k-1})$ and β_k are* *less than*

$$\left(2^{r^2}\beta(\alpha+\beta)\right)^{2^{3k}(2r+3)^k}.$$

Proof. Proceed by induction on k. For $k=1$, due to (3),

$$M(0,r,\alpha,\beta) \le 2^{r(r-1)/2}\beta(\alpha+\beta+1)^r \le 2^{r(r-1)/2}\beta(2\alpha+2\beta)^r \le 2^{r^2}\beta(\alpha+\beta)^r,$$

and, by the definition of symbol β_1, $\beta_1 \le \beta + M(0,r,\alpha,\beta)(\alpha-1) \le \beta + 2^{r^2}\beta(\alpha+\beta)^{r+1}$.

Suppose now, that the bound is proved for β_k, M_k $(1 \le k < n)$.

Observe that according to the definition of β_i, the values β_i increase and $\beta_i \ge i$ $(1 \le i \le k)$.

Because of this property and according to (3),

$$M_{k+1} = M(k,r,\alpha,\beta_k,\beta_1,\dots,\beta_k) =$$

$$= 2^{r(r-1)/2}\beta_k\beta_1\cdots\beta_k(\min\{r,\ k+1\}\alpha+\beta_k+\beta_1+\cdots+\beta_k-k+1)^r \le$$

$$\le 2^{r^2}\beta_k\beta_1\cdots\beta_k((k+1)\alpha+(k+1)\beta_k)^r =$$

$$= 2^{r^2}\beta_k\beta_1\cdots\beta_k(k+1)^r(\alpha+\beta_k)^r \le 2^{r^2}\beta\beta_1\cdots\beta_k(\alpha+\beta_k)^{2r+1}.$$

Hence, due to the definition of symbols β_i,

$$\beta_{k+1} \le \beta + M_{k+1}\big[(\alpha-1)(k+1)+\beta_1+\cdots\beta_k\big] \le$$

$$\le 2^{2r^2}\beta\beta_1\dots\beta_k(\alpha+\beta_k)^{2r+1}(\alpha+\beta_k)(k+1) \le 2^{2r^2}\beta\beta_1\cdots\beta_k(\alpha+\beta_k)^{2r+3}.$$

Using the bounds for β_i $(0 \le i \le k)$ from the inductive hypothesis, we get:

$$M_{k+1} \le \beta_{k+1} \le 2^{2r^2}\left(2^{r^2}\beta(\alpha+\beta)\right)^p\left(\alpha+\left(2^{r^2}\beta(\alpha+\beta)\right)^{2^{3k}(2r+3)^k}\right)^{2r+3},$$

where, according to a formula for the sum of first $k+1$ terms of geometric progression with the multiplier $2^3(2r+3)$, the power

$$p = \frac{2^{3(k+1)}(2r+3)^{k+1}-1}{2^3(2r+3)-1} < \frac{2^{3k+3}(2r+3)^{k+1}}{2^2} = 2^{3k+1}(2r+3)^{k+1}.$$

Hence,

$$M_{k+1} \le \beta_{k+1} \le 2^{2r^2} \left(2^{r^2} \beta(\alpha+\beta)\right)^{2^{3k+1}(2r+3)^k} \left(2^{2r^2} \beta(\alpha+\beta)\right)^{2^{3k}(2r+3)^{k+1}} =$$

$$= 2^{2r^2 + r^2 2^{3k+1}(2r+3)^k + r^2 2^{3k+1}(2r+3)^{k+1}} \beta^{2^{3k+1}(2r+3)^k + 2^{3k}(2r+3)^{k+1}} \times$$

$$\times (\alpha+\beta)^{2^{3k+1}(2r+3)^k + 2^{3k}(2r+3)^{k+1}} \le 2^{r^2 2^{3k+3}(2r+3)^{k+1}} \beta^{2^{3k+3}(2r+3)^{k+1}} (\alpha+\beta)^{2^{3k+3}(2r+3)^{k+1}} =$$

$$= \left(2^{r^2} \beta(\alpha+\beta)\right)^{2^{3(k+1)}(2r+3)^{k+1}}.$$

Lemma is proved.

For an arbitrary $c \in \mathbf{R}$, let

$$B(c) = (\alpha + \beta + 1)^{(r+2)^{cn}}. \tag{9}$$

Lemma 5 implies that $M_n < B(c_1)$, for a positive constant c_1.

The algorithm consideres successively less than $I^n 2^n M_n^{n^2} < I^n B(c_2)$ (for a constant $c_2 > 0$) sequences of the kind $(\mathbf{i}, \mathbf{j}, \mathbf{m}_1, \ldots, \mathbf{m}_k)$, $1 \le k \le n$ for each of which computes the functions $h_1, \ldots, h_k$, and less than $I M_n^n$ functions of the kind $\hat{\partial}_n^{m_n} \cdots \hat{\partial}_1^{m_1} f_j$.

Each of these functions is obtained as a result of a successive application of differential operators of the kind $\partial_{\mathbf{h},\mathbf{p},q}$ (see Definition 2), i.e., of a successive computing the determinants of appropriate Jacobian matrices with elements polynomial in variables $x_1, \ldots, x_n$ and in symbols of functions in the Pfaffian chain for $f_1, \ldots, f_I$. Due to the polynomial-time complexity of the Gauss algorithm from [7] and bounds on the formats of elements of the Jacobian matrices from lemma 5, each function h_μ $\hat{\partial}_n^{m_n} \cdots \hat{\partial}_1^{m_1} f_j$ are computed with the complexity less that $B(c_3)$ for a constant $c_3 > 0$.

It follows that the total complexity of the algorithm is bounded from above by the value $I^{n+1} B(c_4)$ for a constant $c_4 > 0$.

We summarize the results proved in Sections 3 and 4 in the following theorem.

Theorem 2. *There is an algorithm which for an elementary semi-Pfaffian set X produces a finite elementary stratification of X. The number of strata is less than $I^n B(c_2)$. Each* *stratum X_i of codimension k is an elementary semi-Pfaffian set defined by a system of equations and strict inequalities, including the input system (5), a system $h_{i1} = \ldots = h_{ik} = 0$ such that $h_{ij} \equiv 0$ on X_i, for $j = 1, \ldots, k$, $dh_{i1} \wedge \ldots \wedge dh_{ik} \neq 0$ at every point of X_i, and possibly some other Pfaffian equations and inequalities. The format of the system defining X_i is component-wise bounded from above by 5-tuple $(IB(c_4), J + 2^n, r, \alpha, B(c_1))$. All functions of the system have the same Pfaffian chain as the input functions. The running time of the algorithm is less than $I^{n+1}B(c_4)$. Here $c_1, \ldots, c_4$ are positive constants, and $B(c)$ is defined by (9).*

Remark 2. Observe that the algorithm from the Theorem 1 does not involve computations with the functions $g_1, \ldots, g_J$. Thus, the functions need not be Pfaffian or even analytic. Observe also, that we can modify the algorithm by replacing from the start the functions $f_1, \ldots, f_I$ by the sum of their squares. In this case, $I = 1$ and all the bounds in Theorem 2 will not depend on the parameter I.

5. Arbitrary semi-Pfaffian sets.

We can extend the algorithm to an arbitrary semi-Pfaffian set, i.e. finite union of (not necessary disjoint) elementary semi-Pfaffian sets. The idea is to represent the set as a *disjoint* union of elementary sets and then to apply the Theorem 2 to each member of this union.

Corollary 1. *There is an algorithm which, for an arbitrary semi-Pfaffian set Y defined by a Boolean formula in a disjunctive normal form (DNF):*

$$Y = \bigcup_{1 \leq l \leq N} \{f_{l1} = \ldots = f_{lI_l} = 0, \quad g_{l1} > 0, \ldots, g_{lJ_l} > 0\} \tag{10}$$

with $\sum_{1 \leq l \leq N} I_l = I$, $\sum_{1 \leq l \leq N} J_l = J$, with format $(N, I, J, r, \alpha, \beta)$, produces a finite elementary stratification of Y. The number of strata is less than $3^{I+J}(I+J)^n B(c_5)$, the format of each formula defining a stratum is bounded by the 5-tuple

$$\big((I+J)B(c_5), I + J + 2^n, r, \alpha, B(c_1)\big).$$

All functions in a formula have the same Pfaffian chain as the input functions. The running time of the algorithm is less than

$$3^{I+J}(I+J)^{n+1}B(c_5).$$

Here c_5 is a positive constant and $B(c)$ is defined by (9).

Proof. The algorithm considers all (not necessarily consistent) sign assignments for the family of functions $f_{li},\ g_{lj}$, i.e., all 3^{I+J} systems of equations and strict inequalities that can be constructed using these functions.

The set Y is the disjoint union of elementary semi-Pfaffian sets, defined by all sign assignments. For each elementary semi-Pfaffian set the algorithm applies the procedure from Theorem 2.

The number of strata produced by the procedure from the Corollary 1 and its complexity depend on the term 3^{I+J} which did not appear in the bounds of Theorem 2. We can avoid this term by a price of taking the input functions with the whole space $\mathbf{R}^n$ as the domain, and using an *oracle* $\mathcal{O}$ for deciding whether a system of Pfaffian equations and inequalities is consistent.

Oracle is a subroutine which can be used by the algorithm any time it needs to check the consistency. We assume that this subroutine always gives the answer though we do not specify how it actually works. In fact, it is even unclear whether the problem of consistency for an arbitrary Pfaffian system is algorithmically decidable. However for some classes of Pfaffian functions, closed under differentiation and arithmetic operations, the problem is definitely decidable. Apart from polynomials, such a class form, for instance, terms of the kind $P(\mathrm{e}^h, x_1, \dots, x_n)$ where h is a fixed polynomial in $x_1, \dots, x_n$ and P is an arbitrary polynomial in $u, x_1, \dots, x_n$ (see [17]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in a usual sense.

Denote the (possibly unknown) complexity of the oracle $\mathcal{O}$ by D. Thus, D is a function of the parameters of the system to which the oracle is applied, i.e., we assume that each oracle call requires $D(\mathcal{F})$ *elementary oracle steps*, where $\mathcal{F} = (N', I', J', r', \alpha', \beta')$ is a format of a corresponding system of inequalities.

Thus, we assume that an *algorithm with oracle* can have elementary steps of two sorts: arithmetic operations over reals and elementary oracle steps. The complexity of an algorithm with oracle is the number of its elementary steps (in the worst case) as a function of the format of the input Boolean formula.

Lemma 6 (cf. [5]). *Given an oracle $\mathcal{O}$ of complexity D, there is an algorithm which, for an arbitrary Boolean formula in DNF of a format $\mathcal{F} = (N, I, J, r, \alpha, \beta)$, defining a semi-Pfaffian set Y, as in (10), with functions f_{li}, g_{lj} defined in $G = \mathbf{R}^n$, produces another Boolean formula in DNF, defining the same set Y, and such that the disjunction members define a disjoint family of elementary semi-Pfaffian sets. The number of these sets does not exceed*

$$\mathcal{B}(\mathcal{F}) = 4^{r^2+n+r} n^r \left(\alpha + (I+J)\beta\right)^{n+r}.$$

Each elementary set uses the same family of atomic functions as Y and the defining system of equations and strict inequalities with format bounded by a 5-tuple $(I+J, I+J, r, \alpha, \beta)$. The running time of the algorithm is less than

$$(I+J)\mathcal{B}(\mathcal{F})D(I+J, I+J, r, \alpha, \beta).$$

Proof. The algorithm works recursively, building a tree $\mathcal{T}$ of the height not exceeding $I+J$. The vertices of $\mathcal{T}$ are some consistent systems of Pfaffian inequalities and each vertex has a number of sons not exceeding 3.

The root (the vertex of level zero) of $\mathcal{T}$ is identically zero function. Suppose that the algorithm had constructed a system K which is a vertex of $\mathcal{T}$ of level $i < I+J$. The algorithm chooses the $(i+1)$-th function f from the list $f_{11}, \dots, f_{NI_N}, g_{11}, \dots, g_{NJ_N}$ and decides the consistency of systems K & $(f=0)$, K & $(f>0)$ and K & $(f<0)$ with the help of the oracle $\mathcal{O}$. Every consistent among them is a son of K, a vertex of level $i+1$. The process of constructing $\mathcal{T}$ terminates when $i = I+J$.

Observe that the family of sets defined by all terminal vertices of $\mathcal{T}$ coincides with the family of all cells for $f_{11}, \ldots, f_{NI_N}, g_{11}, \ldots, g_{NJ_N}$. Using $\mathcal{O}$ select all the cells contained in Y. Then the disjunction of all selected terminal vertices is a desired output of the algorithm.

¿From the Proposition 2 it follows that the number of cells does not exceed $\mathcal{B}(\mathcal{F})$.

Therefore, the total number of vertices in $\mathcal{T}$ is less than $(I+J)\mathcal{B}(\mathcal{F})$ and the complexity of the algorithm is bounded by

$$(I + J)\mathcal{B}(\mathcal{F})D(I + J, I + J, r, \alpha, \beta).$$

Theorem 3. *Given an oracle $\mathcal{O}$ of complexity D, there is an algorithm which, for an arbitrary semi-Pfaffian set Y defined by a Boolean formula in DNF of a format $(N, I, J, r, \alpha, \beta)$, so that (10) holds, with functions f_{li}, g_{lj} defined in $G = \mathbf{R}^n$, produces a finite elementary stratification of Y. The number of strata is less than $(I + J)^{n+r}B(c_6)$. The format of each formula defining a stratum is bounded by a 5-tuple*

$$\big((I + J)B(c_6), I + J + 2^n, r, \alpha, B(c_6)\big).$$

All functions in a formula have the same Pfaffian chain as the input functions. The running time of the algorithm is less than

$$(I + J)^{n+r}B(c_6)D(I + J, I + J, r, \alpha, \beta).$$

Here c_6 is a positive constant and $B(c)$ is defined by (6).

Proof. First algorithm uses the procedure from the lemma 6 to represent Y as a union of disjoint elementary semi-Pfaffian sets. After that it applies the method from the theorem 2 to stratify each of these sets. The family of all produced strata forms a stratification of Y.

The complexity analysis is straightforward.

6. Fewnomials and exponential polynomials

Generalizing the examples (a) and (e), section 2, we can consider a polynomial $f \in \mathbf{R}[x_1, \dots, x_n]$ as a Pfaffian function of two different formats.

1) (Sparse representation). Each monomial

$$f_{i_1 \dots i_n} = a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

of f with $a_{i_1 \dots i_n} \neq 0$ is a Pfaffian function in the domain $G = \{x_1 \cdots x_n \neq 0\} \subset \mathbf{R}^n$, of the order $n+1$ and the degree (2,1), due to the equations

$$df_{i_1 \dots i_n} = \sum_{1 \le j \le n} i_j f_{i_1 \dots i_n} g_j dx_j$$

$$dg_j = -g_j^2 dx_j,$$

with $g_j = 1/x_j$.

According to lemma 1,a polynomial f is a Pfaffian function in G of degree (2,1) and order $n+m$, where m is the number of all monomials in f with non-zero coefficients.

Let $\mathcal{K}$ be a set of all monomials of f. In sparse setting f is called a *fewnomial* or *sparse polynomial* with support $\mathcal{K}$.

A polynomial $F = P(x_1, \dots, x_n, u_1, \dots, u_m)$ of degree β in variables x_i and monomials $u_j \in \mathcal{K}$ is called a *sparse polynomial of pseudodegree* β with support $\mathcal{K}$. Obviously F is a Pfaffian function of the degree $(2, \beta)$ and of order $n+m$. Note that β may be not equal to the degree d of the polynomial P after substitution of monomials u_j. We shall call d the *degree* of F.

2) (Dense representation). On the other hand (cf. example (a), section 2), polynomial f of the degree d can be considered as "dense," i.e., as a Pfaffian function (in $\mathbf{R}^n$) of order 0 and degree (α, d), where α is arbitrary.

Consider a semi-algebraic set Y defined by a formula (10), where the degrees of the polynomials $f_{li},\ g_{lj} \in \mathbf{R}[x_1, \dots, x_n]$ are less than d and the total number of monomials with non-zero coefficients in the polynomials f_{li} is m.

In the sparse representation (i.e., f_{li}, g_{lj} are considered as fewnomials with a common support $\mathcal{K}$, $\mathrm{card}(\mathcal{K}) = m$) the format of (10) is $(N, I, J, n+m, 2, 1)$. In the dense representation the format can be, e.g., $(N, I, J, 0, 0, d)$.

Note that in the sparse representation the functions f_{li} and g_{lj} are defined only in the domain G.

Applying theorem 3 to the formula (10) in the dense representation, we get an algorithm for a stratification of Y. In this case, we replace the oracle $\mathcal{O}$ by a genuine effective procedure for deciding consistency of semi-algebraic sets [6, 15, 1, 20].

Corollary 2 (dense stratification of semi-algebraic sets). *There is an algorithm which for a semi-algebraic set Y of with format $(N, I, J, 0, 0, d)$ defined by (10) produces a finite elementary stratification for Y. The number of strata is less than $(I+J)^n B'(c_7)$. Each stratum Y_0 is represented by a system of (dense) polynomial equations and strict inequalities of with format bounded by 5-tuple $\big((I+J)B'(c_7), I+J+2^n, 0, 0, B'(c_7)\big)$. The running time of the algorithm is less than*

$$(I+J)^{c_7 n} d^{2^{c_7 n}}.$$

Here c_7 is a positive constant,

$$B'(c) = d^{2^{cn}},$$

for arbitrary $c \in \mathbf{R}$.

Proof. Use the procedures and their complexity bounds from [6,15].

Remark 3. Using the procedures from [6,15] the algorithm can also select all non-empty strata among the produced with the complexity bound $(I+J)^{c_8 n} d^{2^{c_8 n}}$ for a positive c_8. This, appended, algorithm proves a special case of a known theorem (see, e.g., [14, 1, 20]) stating that a *Whitney stratification* of a semi-algebraic set Y can be produced in time $(I+J)^{cn} d^{2^{cn}}$. However, the known proofs of this theorem are specifically algebraic (involving resultants, etc.) and very cumbersome. Also, they do not produce, as the algorithm from the corollary 2, for each stratum, a system of equations having the Jacobian matrix of the maximal rank at every point of the stratum.

Now consider the case of Y defined by (10) in the sparse representation.

Corollary 3 (sparse stratification of semi-algebraic sets). *There is an algorithm which, for a semi-algebraic set Y defined by (10), of the format $(N, I, J, n+m, 2, 1)$, atomic fewnomials with a common support $\mathcal{K}$, $\mathrm{card}(\mathcal{K}) = m$, and degrees less than d, produces a finite elementary stratification for Y. The number of strata is less than $(I+J)^{2n+m} B''(c_8)$. Each stratum Y_0 is represented by a system of sparse polynomial equations and inequalities of pseudodegree $B''(c_8)$ with support $\mathcal{K}$ of a format bounded by $\mathcal{F} = \big((I+J)B''(c_8), I + J + 2^n, n+m, 2, B''(c_8)\big)$. The running time is less than*

$$(I+J)^{c_8 n+m} B''(c_8) d^{c_8 n}.$$

Here c_8 is a positive constant,

$$B''(c) = 2^{(n+m)^{cn}}$$

for arbitrary $c \in \mathbf{R}$.

Proof. The estimates of the number of strata and the format of a stratum are straightforward, taking into the account that a common Pfaffian chain for all f_{li}, g_{lj} has the order $n+m$. Note that the bounds depend only on the format of the input (and do not depend on the degree d). The bound on the running time includes, however, the estimate of the complexity of deciding consistency of systems of polynomial inequalities. According to Theorem 3, the running time is less than $(I+J)^{2n+m} B''(c_9) D(I+J, I+J, r, \alpha, \beta, d)$, where d is an upper bound for the degrees of the input polynomials, considered as dense, and real $c_9 > 0$.

Using the decision procedure and complexity bounds from [6, 15], we can take for $D(I+J, I+J, r, \alpha, \beta, d)$ the value

$$\big((I+J)d\big)^{c_{10} n},$$

for some $c_{10} > 0$. Thus the total running time of the sparse stratification algorithm is less than

$$(I+J)^{c_8 n+m} B''(c_8) d^{c_8 n}$$

for a positive constant c_8, and the corollary is proved.

Remark 4. As in the case of the dense stratification (Remark 3), the algorithm can use the procedures from [6, 15] to select all nonempty strata among the produced with the complexity bound

$$(I+J)^{2n+m} B''(c_{11}) D(\mathcal{F}, \Delta),$$

where Δ is an upper bound for the degrees of the *output* polynomials cosidered as dense. Let us compute Δ.

First observe that for a polynomial f of the degree γ and a set $\mathbf{h} = (h_1, \ldots, h_k)$ of polynomials of the degrees $\gamma_1, \ldots, \gamma_k$ respectively, for a set of distinct indices $\{\mathbf{i} = (i_1, \ldots, i_k), j\}$ and for $m > 0$ the polynomial $\partial^m_{h,\mathbf{i},j}(f)$ is of the degree $\gamma' = \gamma + m(\gamma_1 + \cdots + \gamma_k)$. It follows that the degrees of (dense) polynomials $h_k(x)$, $\hat{\partial}_n^{m_n} \cdots \hat{\partial}_1^{m_1} f_j(x)$, appearing on the recursive steps k of the computation, do not exceed $d_k = d + M_k(d_1 + \cdots + d_{k-1})$ and $d'_{k+1} = d + M_{k+1}(d_1 + \cdots + d_{k-1} + 2d_k)$ respectively where $d_1, \ldots, d_{k-1}$ are upper bounds for degrees of corresponding polynomials $h_1, \ldots, h_{k-1}$ and $M_k = M(k-1, r, \alpha, \beta, \beta_1, \ldots, \beta_{k-1})$. Since the sequence of integers M_i, $1 \le i \le n$, increases (see (3)), we have, by induction,

$$d_k < 2^{k-1} d M_k^{k-2},$$

so, due to lemma 5, Δ can be taken less than

$$2^{n-1} d M_n^{n-2} < d B''(c_{12})$$

for a positive constant c_{12}.

Using the decision procedure and complexity bounds from [6, 15], we can take for $D(\mathcal{F}, \Delta)$ the value

$$\big((I+J)d\big)^{c_{13} n} B''(c_{13}),$$

and the total running time of the appended algorithm will be

$$(I+J)^{c_{14}n+m}B''(c_{14})d^{c_{14}n}.$$

for a positive c_{14}.

Generalizing the case of a semi-algebraic set, consider $Y \subset \mathbf{R}^n$, defined by (10), in which

$$f_{li} = P_{f_{li}}\big(\mathrm{e}^{h(x_1,\ldots,x_n)}, x_1, \ldots, x_n\big),$$

$$g_{lj} = P_{g_{lj}}\big(\mathrm{e}^{h(x_1,\ldots,x_n)}, x_1, \ldots, x_n\big),$$

where $h(x_1, \ldots, x_n),\ P_{f_{li}}(u, x_1, \ldots, x_n),\ P_{g_{lj}}(u, x_1, \ldots, x_n)$ are polynomials of degrees less than d.

Suppose that the number of monomials (with non-zero coefficients) in polynomial h is t and the total number of monomials in polynomials $P_{f_{li}}$ and $P_{g_{lj}}$ is bounded by m. As in the case of polynomials, we can assign at least two different formats to the functions $f_{li},\ g_{lj}$.

Let P be either $P_{f_{li}}$ or $P_{g_{lj}}$.

In the *dense* setting,

$$f = P\big(\mathrm{e}^h, x_1, \ldots, x_n\big) \tag{11}$$

is a Pfaffian function of the order 1 and degree (d, d) in $\mathbf{R}^n$, and the Pfaffian chain consists of the unique function e^h.

On the other hand (*sparse* setting), f is a Pfaffian function of the order $n + t + m$ and the degree (3,1) in $G = \{x_1 \cdots x_n \neq 0\}$. Indeed, let

$$f_{i_0 i_1 \ldots i_n} = a_{i_0 i_1 \ldots i_n} \mathrm{e}^{i_0 h} x_1^{i_1} \cdots x_n^{i_n}$$

be a "monomial" of f. Then

$$\frac{\partial f_{i_0 i_1 \ldots i_n}}{\partial x_j} = i_0 \frac{\partial h}{\partial x_j} f_{i_0 i_1 \ldots i_n} + i_j f_{i_0 i_1 \ldots i_n} g_j,$$

with $g_j = 1/x_j$. Substituting here $\partial h/\partial x_j$ from the equations for the sparse representation of h as a Pfaffian function of the order $n+t$ and degree (2,1) (see the beginning of this section) we represent $\partial f_{i_0 i_1 \dots i_n}/\partial x_j$ as a polynomial of degree 3 in $f_{i_0 i_1 \dots i_n}$ and the $n+t$ elements of the Pfaffian chain for h. Therefore, a Pfaffian chain for f consists of the Pfaffian chain for h, plus the m functions $f_{i_0 i_1 \dots i_n}$. Hence f is a Pfaffian function β with support $\mathcal{K}$.of the order $n+t+m$ and degree (3,1) (lemma 3, though applicable, gives a weaker bound).

Let $\mathcal{K}$ be a set of all monomials of f (i.e., expressions of the kind $f_{i_0 i_1 \dots i_n}$). In sparse setting f *sparse expression with support* $\mathcal{K}$.

The expression $F = Q(x_1, \dots, x_n, u_1, \dots, u_m)$ where Q is a polynomial of degree β in both x_i and $u_j \in \mathcal{K}$ is called *sparse expression of pseudodegree* β with support $\mathcal{K}$.

Thus, F is Pfaffian of degree $(3, \beta)$ and of order $n+m+t$. Here β may be not equal to the degree of Q in $x_1, \dots, x_n, \mathrm{e}^h$ after substitution of monomials u_j.

Corollary 4. 1. (Dense stratification.) *Let Y be a set defined by (10) with f_{ij}, g_{ij} being dense expressions of the form $P(\mathrm{e}^h, x_1, \dots, x_n)$. There is an algorithm, which produces a finite stratification of Y. The number of strata is less than $(I+J)^n B'(c_{15})$. Each stratum Y_0 is represented by a system of equations and strict inequalities with atomic functions of the kind (11) and of the format bounded by $\big((I+J)B'(c_{15}), I+J+2^n, 1, d, B'(c_{15})\big)$ The running time is less than*

$$(I+J)^{c_{15} n} d^{2^{c_{15} n}}.$$

Here c_{15} is a positive constant.

2. (Sparse stratification.) *Let Y be a set defined by (10) with f_{ij}, g_{ij} being sparse expressions of the form $P(\mathrm{e}^h, x_1, \dots, x_n)$. There is an algorithm, which produces a finite stratification of Y. The number of strata is less than $B^{(3)}(c_{16})$. Each stratum Y_0 is represented by a system of equations and strict inequalities of the kind (11) of pseudodegree $B^{(3)}(c_{16})$ with support $\mathcal{K}$ of a format bounded by $\big((I+J)B^{(3)}(c_{16}), I+J+2^n, n+t+m, 3, B^{(3)}(c_{16})\big)$*

The running time is less than

$$(I+J)^{c_{16}n+m}B^{(3)}(c_{16})d^{c_{16}n}.$$

Here c_{16} *is some positive constant,*

$$B^{(3)}(c) = 2^{(n+t+m)^{cn}}$$

for arbitrary $c \in \mathbf{R}$.

Proof is analogous to proofs of the corollaries 2, 3 (and statements from the remarks 3, 4), except that here we use the procedure for deciding consistency of systems of exponential polynomial inequalities and its complexity estimate from [17].

Remark 5. By *coefficients* of a Pfaffian function f we mean the coefficients of all polynomials g_{ij} and P from the definition 1. Let, for polynomials or exponential functions $f_{li},\ g_{lj}$ in (10), their coefficient be integral with absolute values less than 2^M for a positive integer M. Then a straightforward computation shows that the bounds on the bit sizes of coefficients of atomic functions in formulas, defining smooth strata, from the theorems 2, 3, corollaries 2, 3, 4, and remarks 3, 4, depend polynomially on M. Taking into the account the size of coefficients, it is natural to take bit operations over integers as elementary steps of the algorithm (see the beginning of section 4). In these terms the complexities of the algorithms from the corollaries 2, 3, 4 (and remarks 3, 4) depend polynomially on M.

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