PROJECTIONS OF SEMI-ANALYTIC SETS

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INTRODUCTION

In this paper we shall study sets which are the images of real semi-analytic sets under analytic mappings.

It is known that the image of a complex analytic set under a proper mapping is an analytic set.

On the other hand, according to the Tarski-Seidenberg principle, the image of a real semi-algebraic set under any algebraic mapping is a semi-algebraic set.

An analogous assertion is not true for the images of real semi-analytic sets, even relative to compacta.* However, for some problems in analysis, it is necessary to have information about such sets (called, in this paper, $\mathcal{P}$ sets).

The fundamental result of this paper is, essentially, that the class of $\mathcal{P}$ sets is closed with respect to set-theoretical operations (specifically, with respect to taking the complement; the others are trivial). On the other hand, it is known that each $\mathcal{P}$ set on the plane is a semi-analytic set (see [1]). This makes it possible to study the metric properties of $\mathcal{P}$ sets by using the Pyuize expansion. In particular, it is possible to establish regular expandability, the Whitney property (near points of a closed $\mathcal{P}$ set can be joined by short curves), etc.

Another use of the basic theorem is the proof of the finitude of the number of connected components of a $\mathcal{P}$ set, depending on a parameter.

In this paper, we also give a local description of a $\mathcal{P}$ set, similar to the local descriptions of analytic sets.

For our proof, we make essential use of the results of Lojasiewicz [1] as they apply to semi-analytic sets.

1. Definitions, Notation, Results

Set L in n-dimensional real space $\mathbf{R}^n$, is said to be semi-analytic if, in the neighborhood of each point $x_0 \in \mathbf{R}^n$ it is the finite union of sets of the form $\{f_i(x) = 0, g_j(x) > 0, i = 1, \ldots, i_0, j = 1, \ldots, j_0\}$ where f_i and g_j are analytic functions in the neighborhood of point x_0. We give the name $\mathcal{P}$ set to a set in n-dimensional space of the form $\{x \in \mathbf{R}^n | \exists y \in \mathbf{R}^m: (x, y) \in L\}$, where L is a relatively compact semi-analytic set in $\mathbf{R}^{n+m}$.

Note. It is obvious that the image of a relatively compact semi-analytic set under any analytic mapping is a $\mathcal{P}$ set.

The dimension of a $\mathcal{P}$ set in n-dimensional space is the number k such that the image of this set upon projection onto some k-dimensional subspace is dense somewhere, whereas the image upon projection upon any (k + 1)-dimensional subspace is a nowhere dense set.

In the sequel, we shall consider only semi-analytic sets of the form $\{x \in I^n \mid f_i(x) = 0, g_j(x) > 0\}$, where $I^n = \{x \in \mathbf{R}^n \mid |x_i - x_i^0| \leqslant \varepsilon\}$ is some cube in $\mathbf{R}^n$. This limitation is unessential, since any relatively compact semi-analytic set can be represented in the form of a finite union of sets of the specified form. In this case, the corresponding $\mathcal{P}$ sets have the form $\{x \in I^n | \exists y \in I^m: f_i(x, y) = 0, g_j(x, y) > 0\}$.

*It is simple to obtain counterexamples from the remarks to Theorem 10 of Chapter 4 of [2].

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If M is a $\mathscr{P}$ set in cube I^n, we then denote by ${}^cM$ (the complement to M) the set $I^n \setminus M$. If $I^n = I^k \times I^{n-k}$, where $x' = x_1, \ldots, x_k$ is the coordinate in I^k and $x'' = x_{k+1}, \ldots, x_n$ is the coordinate in I^{n-k}, we then denote by $(x'')_{x_0'}$, $x_0' \in I^k$, the subset $\{x_0'\} \times I^{n-k}$ in I^n. If M is some set in I^n, we then denote by $M_{x_0'}(x'')$ the set $M \cap (x'')_{x_0'}$.

In the sequel we shall frequently not specify explicitly the cube, I^n, in which some set or another is contained. In such cases, when we say, for example, that set M is dense in $\mathbf{R}^n$, we understand that M is dense in the corresponding cube I^n.

The image of a set under the projection $\mathbf{R}^{m+n} \to \mathbf{R}^n$ will usually be called simply the projection of this set.

We shall need the following properties of semi-analytic sets [1]:

(1) Unions and intersections of semi-analytic sets are semi-analytic sets.

(2) The closure of a semi-analytic set is a semi-analytic set.

(3) Each semi-analytic set is the locally finite union of connected semi-analytic sets (its connected components).

(4) Any two points, a_1 and a_2, of a connected semi-analytic set $L \subset \mathbf{R}^n$ can be joined by a semi-analytic curve. This means that there exists some embedding $f\colon [0, 1] \to L$ whose image is a semi-analytic set in $\mathbf{R}^n$, with $f(0) = a_1$ and $f(1) = a_2$. We note that a semi-analytic curve is analytic everywhere with the exception of a finite number of points.

(5) The image of a relative compact one-dimensional semi-analytic set under an analytic mapping is a semi-analytic set.

(6) The image of a relatively compact semi-analytic set under projection onto a two-dimensional plane is a semi-analytic set.

(7) A closed semi-analytic set in the neighborhood of each point has the form $L = \bigcup_{k=0}^{k_0} L_k$, where $L_k = \{x \mid f_{ik}(x) = 0,\ g_{jk}(x) \geq 0\}$.

From these enumerated properties there immediately flow the following properties of $\mathscr{P}$ sets, which we shall henceforth utilize without explicit citation:

(1) Unions and intersections of $\mathscr{P}$ sets are $\mathscr{P}$ sets.

(2) The closure of a $\mathscr{P}$ set is a $\mathscr{P}$ set.

(3) Each $\mathscr{P}$ set is the union of a finite number of connected $\mathscr{P}$ sets (its connected components).

(4) Any two points of a connected $\mathscr{P}$ set can be joined by a semi-analytic curve. (This follows from properties (4) and (5) of semi-analytic sets.)

(5) Each $\mathscr{P}$ set in $\mathbf{R}^2$ is a semi-analytic set.

(6) A closed $\mathscr{P}$ set is the projection of a compact analytic set (since a semi-analytic set of the form $\{f_i = 0, g_j \geq 0\}$ is the projection of the set $\{f_i(x) = 0, g_j(x) = \xi_j^2\}$).

In this paper, we shall prove the following assertions:

THEOREM 1. The complement of a $\mathscr{P}$ set is a $\mathscr{P}$ set.

COROLLARY 1. Let $M(x, \xi)$ be a $\mathscr{P}$ set in $\mathbf{R}^{n+k}$, $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_k)$. Let $N(\xi)$ be the number of connected components of set $M_\xi(x)$. Then, $N(\xi) < N$, where N is some constant which does not depend on ξ.

THEOREM 2. In cube I^n of space $\mathbf{R}^n$ let there be chosen a basis $\{x' = (x_1, \ldots, x_k), x'' = (x_{k+1}, \ldots, x_n)\}$, and let M be a k-dimensional $\mathscr{P}$ set in I^n. There then exist open $\mathscr{P}$ sets Ω_p $(p = 0, \ldots, N)$ in space (x') such that:

(1) $\bigcup_p \Omega_p$ is everywhere dense in I^k;

(2) set $M \cap \{\Omega_0 \times X(x'')\}$ is vacuous;

(3) set $\bar{M} \cap \{\Omega_p \times (x'')\}$ ($p = 1, \ldots, N$) coincides with the set $\{x' \in \Omega_p, x_{k+j} = \bar{f}_{pj}(x'), j = 1, \ldots, n-k\}$ where $\bar{f}_{pj}$ is a p-valued analytic function in Ω_p.

COROLLARY 2. Each $\mathcal{P}$ set is the union of a finite number of locally analytic $\mathcal{P}$ sets (i.e., of sets which are analytic in the neighborhood of each of their points).

To prove these assertions, we require the following auxiliary assertions.

LEMMA 1. In an (n+1)-dimensional space, let there be chosen a basis $\{x = (x_1, \ldots, x_n), t\}$, and let M be a $\mathcal{P}$ set in the semi-space $t > 0$, nowhere dense in the neighborhood of the space $t = 0$. (This means that, in each neighborhood of each point of $(x, 0)$, we can find a point (x^1, t), $t > 0$, some neighborhood of which does not contain a point of set M.) Then, set $\bar{M} \cap \{t = 0\}$ is nowhere dense in the space $t = 0$.

LEMMA 2. A $\mathcal{P}$ set whose complement is everywhere dense is itself a set which is nowhere dense.

LEMMA 3. Let $L = \{x, z \mid f(x, z) = 0\}$ ($x = x_1, \ldots, x_n$; $z = z_1, \ldots, z_m$) be an analytic set, and let the projection of set L onto space (x) be a set which is dense somewhere. Then, one of the two following statements holds:

(a) there exists a proper analytic subset, L', of set L such that the projection of $L \setminus L'$ on (x) is a nowhere dense set.

(b) It is possible so to specify set L by analytic functions f_i, $i = 1, \ldots, k$, that the analytic set $L'' = \left\{(x, z) \in L \mid \operatorname{rk}\begin{pmatrix}\operatorname{grad} f_i \\ x_j\end{pmatrix} < k - n\right\}$ ($i = 1, \ldots, k$; $j = 1, \ldots, n$) will be a proper subset of set L.

ASSERTION W_n. Let M be a closed, nowhere dense, $\mathcal{P}$ set in n-dimensional space. In this space we single out variable x_n, denoting the remaining variables by (x'). Then, there exists a constant N, as well as a $\mathcal{P}$ set Ω, everywhere dense in space (x'), such that each set $M_{x'}(x_n)$, $x' \in \Omega$, contains not more than N points.

ASSERTION P_n. The complement to a closed and nowhere dense $\mathcal{P}$ set in n-dimensional space contains a $\mathcal{P}$ set which is everywhere dense in this space.

ASSERTION T_n. The complement to a closed $\mathcal{P}$ set in n-dimensional space is a $\mathcal{P}$ set.

2. Proofs

LEMMA 1. We pursue the proof by induction on n. For $n = 0$, the assertion is obvious (since a $\mathcal{P}$ set consists of a finite number of connected components). We assume the assertion to be true for n-1. Let M be a $\mathcal{P}$ set in the (n+1)-dimensional space (x, t) which satisfies the conditions of the Lemma, its closure being everywhere dense in the space $t = 0$. (In case $\bar{M} \cap \{t = 0\}$ is everywhere dense only in some open set $\Omega' \subset (x)$, we replace I^n by some cube which is contained in Ω'.) We denote by x' the variables $x_1, \ldots, x_{n-1}$. We produce all the possible subspaces $(x', t)_{x_n}$, and close each of them from the sets $M_{x_n}(x', t)$. We denote by M' the intersection of the set thus obtained with the space $t = 0$. In space $t = 0$, we produce all the possible lines $(x_n)_{x'}$. Assume that, on some line $(x_n)_{x'_0}$, there is a countable number of points not lying in M'. Consider then the set $N = \{x_n, \varepsilon \mid \exists\, x', t\colon (x', x_n, t) \in M, |(x', t) - (x'_0, t)| < \varepsilon\}$. Set N is a $\mathcal{P}$ set on the plane and, consequently, is a semi-analytic set. By assumption, the set ${}^cN \cap \{\varepsilon > 0\}$ (which will also be semi-analytic) contains a countable number of points with different coordinates x_n. Consequently, there exist two points, (x_n^1, ε^1) and (x_n^2, ε^2), $(x_n^1 \neq x_n^2)$, lying in one connected component of this set. We join these points by a curve in ${}^cN \cap \{\varepsilon > 0\}$. Then, the space under this curve does not contain points of set $\bar{N}$ and, consequently, segment $[x_n^1, x_n^2]$ on the line $x' = x'_0$, $t = 0$ does not contain points of set $\bar{M}$, contradicting the assumption that $\bar{M} \cap \{t = 0\}$ is everywhere dense.

We may thus assume that, on each line $(x_n)_{x'}$, there exists only a finite number of points which are not in set M'. Let us number the rational points of space (x') in the sequence $\{a_k\}$. Let $P'_k = \{x_n \mid (a_k, x_n) \in {}^cM_{a_k}(x_n)\}$, $P_k = \bigcup_{i=1}^{k} P'_i$. Each set P_k consists of a finite number of points. Consider some space $(x')_{x_n}$, where $x_n \in {}^c\left(\bigcup_{k=1}^{\infty} P_k\right)$. All the rational points of this space lie in M', so that set $M_{x_n}(x', t)$ does not satisfy the conditions of Lemma 1, and we can find, in space $(x')_{x_n}$, an everywhere dense set, C, of points in some neighborhood of each of which in the semi-space $\{x', t | t > 0\}$ set $Mx_n(x', t)$ is dense. (If set C is not dense somewhere, Lemma 1 can be applied to a neighborhood not containing points of set C.) Since set C is open, its complement in nowehere dense. We now can find point $x_n^* \in {}^c\left(\bigcup_{k=1}^{\infty} P_k\right)$ such that, in space $(x')x_n^*$, set ${}^cCx_n^*(x')$ will be everywhere dense, which leads us to a contradiction.

In space $t = 0$, consider the set C_k of points (x'^0, x_n^0) for which $Mx_n^0(x')$ is dense in the neighborhoods $\{x', t | t > 0, |(x', t) - (x_0', 0)| < 1/k\}$. Each of sets C_k is nowhere dense in space $t = 0$. Indeed, if C_k were dense in some open set, then M would be dense over each point of this set, contradicting the assumption that M satisfies the conditions of Lemma 1.

We now choose point x_n^* in the following way. Initially, we find a point $x_n^1 \in {}^cP_1$ such that, in a 1-neighborhood of point a_1 in space $(x')x_n^1$, there exists a point b_{11} in some close neighborhood of which in space $t = 0$ no points of set C_1 are contained. By moving point x_n^1 about a bit, we obtain a closed segment on the x_n axis whose points satisfy the same conditions as x_n^1.

On the k-th step, let the closed segment of set ${}^cP_k$ we have found be such that the following conditions hold for each point, x_n^k, of this segment; in space $(x')x_n^k$, there exist points $b_{ij}^k (i = 1, \ldots, k;\ j = 1, \ldots, k)$ such that $|b_{ij}^k - a_j| < 1/i$, and, in some closed neighborhood of each of the points b_{ij}^k in space (x) no points of set C_k are contained. It is obvious that one can find, within an already found segment on the x_n axis, a segment contained in ${}^cP_{k+1}$ on which the same conditions will hold with k replaced by k + 1, whereby, for $i, j = 1, \ldots, k$, the neighborhoods of points b_{ij}^{k+1} will be contained in the previously found neighborhoods of points b_{ij}^k.

The point x_n^*, common to all the segments constructed, will also be the point we are seeking.

LEMMA 2. Let M be a $\mathcal{P}$ set such that ${}^cM$ is an everywhere dense set. We can assume that $M = \{x | \exists\, y: f_i(x, y) = 0, g_j(x, y) > 0\}$. Consider the set $M' = \{x, \varepsilon | \exists y: f_i(x, y) = 0, g_j(x, y) \geq \varepsilon, \varepsilon > 0\}$. It is clear that, if $x \in {}^cM$, $\varepsilon > 0$, then some neighborhood of point (x, ε) is contained in ${}^cM'$. Consequently, M' satisfies the conditions of Lemma 1, and $\bar{M}' \cap \{\varepsilon = 0\}$ is a nowhere dense set. But $\bar{M}' \cap \{\varepsilon = 0\} = \bar{M}$, and the assertion is proven.

LEMMA 3. We define set L by the functions $f_i (i = 1, \ldots, k)$ such that the vectors grad f_i will be linearly dependent only on some proper subset $L_0 \subset L$. We can assume that the projection of set $L \setminus L_0$ on (x) is a set which is dense somewhere, otherwise we could choose $L' = L_0$.

Let $A_p = \begin{pmatrix}\text{grad } f_i \\ x_j\end{pmatrix}$ $(i = 1, \ldots, k;\ j = 1, \ldots, p)$ be an $(m+n) \times (k+p)$ matrix, $L_p = \{x, z | \text{rk } A_p < k + p\}$. If $L'' = L_n$ is a proper subset, the assertion is proven.

Let $L_n = L$. We then select p such that the projection of set $L \setminus L_p$ on (x) is a set which is somewhere dense, while the projection of $L \setminus L_{p+1}$ on (x) is a nowhere dense set. We now show then that L_{p+1} is a proper subset, so that we can choose $L' = L_{p+1}$. Indeed, let $L_{p+1} = L$. Then, since the projection of set $L \setminus L_p$ on space (x) is dense somewhere, we can find, by virtue of Lemma 2, on some line $(x_{p+1})x_1^0, \ldots, x_p^0$ a segment parallel to the x_{p+1} axis, lying in the projection of $L \setminus L_p$ on the space $(x_1, \ldots, x_{p+1})$. Consequently, we can find two different points, a_1 and a_2, on this segment lying in the projection of singly connected component, $\mathcal{M}$, of the set $(L \setminus L_p)_{x_1^0, \ldots, x_p^0}(x_{p+1}, \ldots, x_n, z)$. We select one point each in the preimages of points a_1 and a_2 in $\mathcal{M}$, and join them by a semi-analytic curve, λ, in $\mathcal{M}$. Let l be the unit tangent vector to this curve. Since the vectors grad $f_i, x_1, \ldots, x_p$ are linearly dependent at all points of our curve, the vector x_{p+1} can be expressed in terms of the vectors grad $f_i, x_1, \ldots, x_p$ and, since vector l, lying in the tangent space to $L_{x_1^0, \ldots, x_p^0}(x_{p+1}, \ldots, x_n, z)$, is orthogonal to all these vectors, then $(l, x_{p+1}) = 0$ at each point of the curve. But $\left|\int_\lambda (l, x_{p+1})\, d\lambda\right| = |a_1 - a_2| \neq 0$, which leads to a contradiction.

Assertions W_n, P_n, and T_n. The proofs are by induction on n in the following manner: $T_{n-1}, W_{n-1} \Rightarrow W_n$, $T_{n-1}, W_n \Rightarrow P_n$, $P_n \Rightarrow T_n$ (for n = 1, all three Assertions are obvious).

$T_{n-1}, W_{n-1} \Rightarrow W_n$. Since M is closed, we can consider it to be the projection of some analytic set, L, from the (n+m)-dimensional space (x, y).

We first single out the $\mathscr{P}$ set Ω_1 in space (x') in the following way: ${}^c\Omega_1 = \overline{N \cap \{\varepsilon > 0\}} \cap \{\varepsilon = 0\}$, where

$$N = \{x', \ \varepsilon \mid \exists\, x_n^1, x_n^2: (x', x_n^i) \in M, |x_n^1 - x_n^2| = \varepsilon\}.$$

The set of points x' for which set $M_{x'}(x_n)$ contains only a finite number of points is everywhere dense in space (x') (it is not hard to prove that , otherwise, M would not be nowhere dense). Consequently, set ${}^cN$ is everywhere dense, while N, by virtue of Lemma 2, is nowhere dense, and set $N \cap \{\varepsilon > 0\}$ satisfies the conditions of Lemma 1, from which it follows that ${}^c\Omega_1$ is nowhere dense, while Ω_1 is a $\mathscr{P}$ set which is everywhere dense in (x').

Each set $M_{x'}(x_n)$, $x \in \Omega_1$, contains no more than a finite number of points. Assume that Assertion W_n does not hold for set M, i.e., that for each K the set

$$M^K = \{x' \in \Omega_1 \mid \exists\, x_n^1, \ldots, x_n^K: x_n^i \neq x_n^j, (x', x_n^i) \in M\}$$

is non-vacuous and is dense somewhere in space (x').

In space (x'), we single out variable x_{n-1}, we denote the variables $x_1, \ldots, x_{n-2}$ by x'', and the variables x_{n-1}, x_n by x'''. In space (x''), we choose the everywhere dense $\mathscr{P}$ set Ω_2 such that, for all $x'' \in \Omega_2$, the set $({}^c\Omega_1)_{x''}(x_{n-1})$ contains only a finite number of points (the construction is analogous to that of set Ω_1).

Each set $M^1_{x''}(x''') = (M \cap \{\Omega_1 \times (x_n)\})_{x''}(x''')$ consists of a finite number of connected components which can be either isolated points or semi-analytic curves, subject to a one-to-one single valued projection into $(x_{n-1})_{x''}$. Consider the ends of these curves. They may lie, either in set M' (closed ends), or in set ${}^c\Omega_1 \times (x_n)$ (unclosed ends). Let $\Omega_{x''}(x''')$ be the set of isolated points and closed ends in $M^1_{x''}(x''')$. If, for some K_0, the set $Q_{K_0} = \{x'' \mid$ more than K_0 points are contained in $Q_{x''}(x''')\}$ is nowhere dense then, for each K, there exists a set, Q^1_K, somewhere dense in (x''), such that, for $x'' \in Q^1_K$, there exist in set $M^1_{x''}(x''')$ more than K semi-analytic curves projected onto some interval $(x^1_{n-1}, x^2_{n-1})_{x''}$:

$$x^1_{n-1}, x^2_{n-1} \in ({}^c\Omega_1)_{x''}(x_{n-1}).$$

Let

$$M'' = \left\{x'', x_n \mid x'' \in \Omega_2, \ \exists\, x_{n-1}: (x'', x_{n-1}, x_n) \in M', \ \exists\, x^1_{n-1}, x^2_{n-1} \in {}^c\Omega_1: \frac{x^1_{n-1} + x^2_{n-1}}{2} = x_{n-1}\right\}.$$

Set M'' is nowhere dense in (x'', x_n) (since ${}^cM''$ is everywhere dense) and, for each K, there exists a set Ω^1_K, somewhere dense in (x''), such that $\overline{M}''_{x''}(x_n)$ $(x'' \in \Omega^1_K)$ contains more than K points, which leads to a contradiction to Assertion W_{n-1}.

Hence, we assume that, for all K, the set Q_k is dense somewhere in (x''). Now, we construct a proper analytic subset, L^*, in L for which each set Q^*_K (defined analogously to Q_K) will be dense somewhere in (x'') (or we arrive at a contradiction with W_{n-1}). The Assertion will be thereby proven, since we cannot continue to decrease set L indefinitely.

We now make use of Lemma 3 for $x = x''$, $z = (x''', y)$. If (a) holds, we can set $L^* = L'$. Let (b) hold. It can be assumed that the projection, S, of the set $L \setminus L'$ on space (x') is dense somewhere. Indeed, let set S be dense nowhere. Then, we can either set $L^* = L''$ or, for each K, the set $\{x'' \mid ((S \times x_n) \cap M')_{x''}(x''')$ contains more than K points$\}$ will be dense somewhere in (x''), after which one can easily proceed to a contradiction to W_{n-1}. Consequently, it can be assumed that S is dense somewhere in (x'), and that there exists a point x''_0 such that curve λ_0 is contained in the projection of set $(L \setminus L'')_{x''_0}(x''', y)$ onto space (x''').

Consider the set

$$L''' = \left\{(x, y) \in L \mid \operatorname{rk}\begin{pmatrix}\operatorname{grad} f_i \\ x''_j \\ x_{n-1}\end{pmatrix} < k+n-1, \ \operatorname{rk}\begin{pmatrix}\operatorname{grad} f_i \\ x''_j \\ x_n\end{pmatrix} < k+n-1\right\}$$

$$(i = 1, \ldots, k; \ j = 1, \ldots, n-2).$$

In the projection, S', of this set onto space (x), set Q is contained. Indeed, set $L \setminus L'''$ contains only those points of set L at which the tangent space, T, to L exists and for which $T_{x''}(x''', y)$ does not traverse the point upon projection onto $(x''')_{x''}$. Consequently, the projection of set $(L \setminus L''')_{x''}(x''', y)$ onto space $(x''')_{x''}$ does not intersect $Q_{x''}(x''')$ and, even more so, $Q_{x''}(x''')$ is contained in the projection of set $L'''_{x''}(x''')$.

We now show that L''' is a proper subset of L. Let $L''' = L$. On curve λ_0 we choose two different points, a_1 and a_2, lying in the projection of one singly connected component, $\mathcal{M}$, of set $(L \setminus L'')_{x_0''}(x''', y)$. We choose one point in each of the preimages of points a_1 and a_2 in $\mathcal{M}$, and we join these selected points by the semi-analytic curve λ in $\mathcal{M}$. Let l be the unit tangent vector to λ. Then, on the one hand, $l_{n-1} = l_n = 0$ at each point of curve λ and, on the other hand, $\left|\int_\lambda (l_{n-1}, l_n)\, d\lambda\right| = |a_1 - a_2| \neq 0$. We have arrived at a contradiction.

$T_{n-1}, W_n \Rightarrow P_n$. Let M be a closed, nowhere dense, $\mathcal{P}$ set in n-dimensional space. In the basis of this space we single out coordinate x_n, denoting the remaining coordinates by (x'). Let Ω be an open, everywhere dense, $\mathcal{P}$ set in (x') such that, when $x' \in \Omega$, the set $M_{x'}(x_n)$ contains no more than K points, where K does not depend on x'. Let A_l' $(l = 1, \ldots, K)$ be the subset in Ω defined by the formula

$$A_l' = \{x' \in \Omega \mid \exists\, x_n^1, \ldots, x_n^l : (x', x_n^i) \in M,\ x_n^1 > \ldots > x_n^l\}.$$

We set $A_0 = {}^c(A_1') \cap \Omega$, $A_l = A_l' \cap {}^c(A_{l+1}')$. Let

$$B_{00} = \{A_0 \times x_n\},\ B_{lj} = \{x', x_n \mid x' \in A_l,\ \exists\, x_n^1, \ldots, x_n^l : (x', x_n^i) \in M,$$
$$x_n^1 < \ldots < x_n^j < x_n < x_n^{j+1} < \ldots < x_n^l\} \quad (l = 1, \ldots, K;\ j = 0, \ldots, l).$$

Then, the $\mathcal{P}$ set $\bigcup_{l,j} B_{lj}$ is contained in ${}^cM$ and is everywhere dense in space (x).

$P_n \Rightarrow T_n$. Let M be a closed $\mathcal{P}$ set in n-dimensional space. We may suppose that $M = \{x \mid \exists\, y \in I^m : (x, y) \in L\}$, where $L = \{x, y \mid f(x, y) = 0\}$ is some analytic set. Then, ${}^cM = \{x \mid \min_{y \in I^m} f^2(x, y) > 0\}$ and it suffices to show that the set $N = \{x, t \mid \min_{y \in I^m} f^2(x, y) = t\}$ is a $\mathcal{P}$ set.

Let $L_1 = \left\{x, y, t \mid f^2(x, y) = t,\ \frac{\partial f^2(x, y)}{\partial y_i} = 0,\ i = 1, \ldots, m\right\}$ be the set of "singular y points" of the function f^2 (more precisely, the set of corresponding points of the diagram of function f^2). It is possible to construct the analogous set, L_j, $(j = 2, \ldots, 3^m)$ for each face of the cube I^m. Let N_j' be the projection of set L_j onto space (x), and $N' = \bigcup N_j'$. Then, set N' contains N and, for each x_0, the set $N'_{x_0}(t)$ contains only a finite number of points. Indeed, on each connected component of each of the sets $(L_j)_{x_0}(y, t)$, function f^2 maintains a constant value, so that, consequently, each set $(N_j')_{x_0}(t)$ contains only a finite number of points. Since, for each x_0, set $N'_{x_0}(t)$ contains a finite number of points, set $N''_{x_0}(\varepsilon)$, where

$$N'' = \{x, \varepsilon \mid \exists\, t^1, t^2 : (x, t^i) \in N',\ |t_1 - t_2| < \varepsilon\},$$

also contains only a finite number of points, so that ${}^cN''$ is everywhere dense, while N' is nowhere dense in space (x, ε). But then, $N'' \cap \{\varepsilon > 0\}$ satisfies the conditions of Lemma 1, and the set $P = \overline{N'' \cap \{\varepsilon > 0\}} \cap \{\varepsilon = 0\}$ is nowhere dense in space (x). Consequently, ${}^cP$ contains the $\mathcal{P}$ set P' which is everywhere dense in space (x). Each connected component of set $N''' = \{P' \times (t)\} \cap N'$ may be either completely contained in N, or have a null intersection with N (this readily follows from the continuity of the function $\min f^2$). But, since set P' is everywhere dense in space (x), and the diagram of function $\min f^2$ is closed, the closure of the union of the connected components of set N''', which is contained in N, is equal to N. Consequently, N is a $\mathcal{P}$ set, q.e.d.

THEOREM 1. The proof of this Theorem is by induction on the dimensionality of the $\mathcal{P}$ set M. Let M be a k-dimensional $\mathcal{P}$ set in n-dimensional space. We represent ${}^cM$ in the form

$${}^cM = {}^c\overline{M} \cup (\overline{M} \setminus M) = {}^c\overline{M} \cup ((\overline{\overline{M} \setminus M}) \cap {}^c((\overline{\overline{M} \setminus M}) \cap M)).$$

It follows from T_n that ${}^c\overline{M}$ is a $\mathcal{P}$ set. Therefore, it suffices to prove that $\overline{\overline{M} \setminus M}$ is a $\mathcal{P}$ set, and that $\dim (\overline{M} \setminus M) < \dim M$.

Set $\overline{\overline{M} \setminus M}$ is a $\mathcal{P}$ set since, if $M = \{x \mid \exists y : f_i(x, y) = 0,\ g_j(x, y) > 0\}$, then

$$\overline{\overline{M} \setminus M} = \overline{({}^c\{x, \varepsilon \mid \exists\, y : f_i(x, y) = 0,\ g_j(x, y) \geqslant \varepsilon\}) \cap \{\overline{M} \times (\varepsilon)\} \cap \{\varepsilon > 0\})} \cap \{\varepsilon = 0\}.$$

For the proof that $\dim \overline{\overline{M} \setminus M} < \dim M$, it suffices to show that the projection of set $\overline{M} \setminus M$ on any k-dimensional subspace is a nowhere dense set. Let $x' = x_1, \ldots, x_k$; $x'' = x_{k+1}, \ldots, x_n$. We assume that the projection of set $\overline{M} \setminus M$ on space (x') is dense somewhere in this space. We show that we can then find a vector, $z \in (x'')$, such that the projection of set $\overline{S_z} \setminus S_z$ on space (x') is dense somewhere (S_z is the projection of set M on the space (x', z)).

Since, for each j, the projection, $\overline{S_j}$, of set $\overline{M}$ on space (x', x_j) is a nowhere dense set, it follows from W_{k+1} that there exists an open set, Ω_j, everywhere dense in (x'), such that, for each $x' \in \Omega_j$, the set $(\overline{S}_j)_{x'}(x_j)$ contains not more than K_j points, where K_j does not depend on x'. But then, for each $x' \in \Omega = \bigcap_j \Omega_j$, the set $M_{x'}(x'')$ contains not more than $K = \prod_j K_j$ points. We choose a countable set of points $(x_i', x_i'') \in \overline{M} \setminus M$, $x_i \in \Omega$, whose projection on space (x') is dense somewhere.

For each i, the set Q_i of vectors in $(x'')_{x_i'}$ which are such that the hyperplane in (x'') passed through x_i'' is orthogonal to $x'' \in Q_i$, has a nowhere dense intersection with $M_{x_i'}(x'') \setminus \{x_i''\}$. Consequently, we can find a vector $z \in {}^c(\bigcup_i Q_i)$. Obviously, this is the vector we have been seeking.

Let $A_q' = \{x' \in \Omega \mid \exists z^1, \ldots, z^q: (x', z^i) \in \overline{S}_z, z^i > z^{i+1}\}$.

We set $A_0 = {}^c\overline{A}_1$; $A_q = A_q' \setminus A_{q+1}'$. Let

$$B_{lq} = \{(x', z) \in \overline{S}_z \mid x' \in \Omega, \exists z^1, \ldots, z^{q-1}: (x', z^i) \in \overline{S}_z, z^1 > \ldots > z^{l-1} > z > z^l > \ldots > z^{q-1}\}$$

be a "slice" of set $\overline{S}_z$ over A_q. Then, for some l, q, the projection of set $B_{lq} \setminus (B_{lq} \cap S_z)$ onto (x') will be dense somewhere. Since B_{lq} and S_z are $\mathcal{P}$ sets, it follows from Lemma 2 that there exist an open set, $U_1 \subset \overline{A}_q$, such that each point $(x', z) \in B_{lq}$, $x' \in U_1$ lies in set $B_{lq} \setminus (B_{lq} \cap S_z)$. But each such point lies in set $\bigcup_{l' \neq l} \overline{B_{l'q}}$. Therefore, we can assume that the projection of set $B_{lq} \cap \left(\bigcup_{l' > l} \overline{B_{l'q}}\right)$ is dense somewhere in (x') (when $l' < l$, the proof is carried through analogously). Consequently, there exists an open set $U_2 \subset U_1$ such that each point $(x', z) \in B_{lq}$, $x' \in U_2$ lies in the set $B_{lq} \cap \left(\bigcup_{l' > l} \overline{B_{l'q}}\right)$.

Consider the set

$$N = \{x', \varepsilon \mid x' \in U_2, \exists z^1, z^2: (x', z^1) \in B_{lq}, (x', z^2) \in \bigcup_{l' > l} B_{l'q}, z^1 - z^2 = \varepsilon\}.$$

This set satisfies the conditions of Lemma 1, so that $\overline{N} \cap \{\varepsilon = 0\}$ is nowhere dense in (x'). This means that we can find an open set $U_3 \subset U_2$ such that it will follow from $(x', \varepsilon) \in N$, $x' \in U_3$ that $\varepsilon > \varepsilon_0$, where ε_0 is some positive number. Let $(x', z) \in B_{lq}$, $x' \in U_3$. This point is the limit of a sequence of points of $\bigcup_{l' > l} B_{l'q}$. Consequently, there exists a point $(x'^1, z^1) \in \bigcup_{l' > l} B_{l'q}$, $x'^1 \in U_3$, $|z - z^1| < \varepsilon_0/2$. But then, there exists the point $(x'^1, z^2) \in B_{lq}$, $z^2 - z > \varepsilon_0/2$. Since this reasoning applies to each point of B_{lq}, we arrive at a contradiction with the boundedness of set B_{lq}.

COROLLARY 1. Initially, we reduce the problem to the case when, for each ξ, the set $M_\xi(x)$ is closed. For this, we consider, for fixed ξ, the function on $M_\xi(x)$ $\rho_\xi(x) = d(x, (\overline{M}_\xi(x) \setminus M_\xi(x)))$, after which we consider the set $M_\xi^*(x)$ of local maxima of this function (if $\overline{M}_\xi(x) \setminus M_\xi(x)$ is vacuous, we then set $M_\xi^* = M_\xi(x)$). The set $M^* = \bigcup_\xi M_\xi^*(x)$ is a $\mathcal{P}$ set, as demonstrated by the following calculations.

Let

$$M_1 = \{x, \xi, \varepsilon \mid \exists x^1: (x^1, \xi) \in M, |x - x^1| < \varepsilon\},$$

$$M_2 = \{x, \xi \mid \exists \varepsilon > 0: (x, \xi, \varepsilon) \in {}^cM_1\},$$

$M' = {}^cM_2$ (the closure of M with respect to variable x);

$$M_3 = \{x, \xi, \rho \mid (x, \xi) \in M, \exists x^1: (x^1, \xi) \in M' \setminus M, |x - x^1| < \rho\},$$

$$M_4 = \{x, \xi, \rho, \varepsilon \mid (x, \xi, \rho) \in {}^cM_3, \exists \rho^1: (x, \xi, \rho^1) \in M_3, |\rho - \rho^1| < \varepsilon\},$$

$$M_5 = \{x, \xi, \rho \mid (x, \xi, \rho) \in {}^cM_3, \exists \varepsilon > 0: (x, \xi, \rho, \varepsilon) \in {}^cM_4\},$$

$M'' = {}^c(M_3 \cup M_5)$ (the diagram of the function $\rho_\xi(x)$ cited in the previous paragraph);

$$M_6 = \{x, \xi, \varepsilon \mid \exists \rho, \rho^1, x^1: (x, \xi, \rho) \in M'', (x^1, \xi, \rho^1) \in M'', |x - x^1| < \varepsilon, \rho^1 > \rho\},$$

$$M^* = \{x, \xi \mid (x, \xi) \in M, \exists \varepsilon > 0: (x, \xi, \varepsilon) \in {}^cM_6\}.$$

Then, M^* is the set we seek. Consider the set $N = \bigcup_\xi (\overline{M_\xi^*(x)})$. It is not difficult to prove that N is also a $\mathcal{P}$ set. We show that, for all ξ, the number of connected components of set $N_\xi(x)$ is not less than the corresponding number for set $M_\xi(x)$. For this, it suffices to prove that a point of set $N_\xi(x)$ is contained in each component of set $M_\xi(x)$, and that each connected component of set $N_\xi(x)$ is contained in set

$M_\xi(x)$. If $\mathcal{M}$ is some connected component of set $M_\xi(x)$ then, obviously, the point $(x, \xi) \in \mathcal{M}$, at which function $\rho_\xi(x)$ has a maximum on $\mathcal{M}$, lies in $N_\xi(x)$. Now, let $\mathcal{N}$ be a connected component of set $N_\xi(x)$. It is obvious that function $\rho_\xi(x)$ has a constant positive value on $\mathcal{N} \cap M_\xi(x)$. Therefore, any given point of set $\mathcal{N} \subset M_\xi(x)$ cannot lie in $M_\xi(x) \setminus M_\xi(x)$, so that it must lie in $M_\xi(x)$.

Now, let $M_\xi(x)$ be closed for each ξ. We define the sets $M_i (i = 0, \ldots, n)$ as follows:

$$M_0 = M,$$

$$M_i' = \{x, \xi, \varepsilon \mid \exists x^1 \colon (x^1, \xi) \in M_{i-1}, \ |x^1 - x| < \varepsilon, \ x_i^1 > x_i\} \ (i = 1, \ldots, n),$$

$$M_i = \{x, \xi \mid (x, \xi) \in M_{i-1}, \ \exists x^1, \varepsilon \colon (x^1, \xi) \in M_{i-1}, \ (x^1, \xi, \varepsilon) \in {}^cM_i', \ \varepsilon > 0, \ x_i^1 = x_i\}.$$

Contained in set $(M_i)_\xi(x)$ are $(x, \xi) \in M_{i-1}$ for which there exists x^1 such that $x_i^1 = x_i$, $(x^1, \xi) \in M_{i-1}$ while, at the point x^1, ξ, function x_i has a local maximum on $(M_{i-1})_\xi(x)$. It is obvious that $M_{i+1} \subset M_i$ and that, for fixed ξ, the coordinates, $x_1, \ldots, x_i$, of point $x \in (M_i)_\xi(x)$ assume only a finite number of values, that each set $(M_i)_\xi(x)$ is closed and, finally, that each connected component of set $(M_{i-1})_\xi(x)$ contains a point of set $(M_i)_\xi(x)$. It follows from all this that, for each ξ, the set $(M_n)_\xi(x)$ contains only a finite number of points, the number of these points being no less than the number of connected components of set $M_\xi(x)$.

Our problem then comes to the following: if, for each ξ, the number, $N(\xi)$, of points of set $M_\xi(x)$ is finite, it is then bounded by some constant, N, which does not depend on ξ.

Let M be a $\mathcal{P}$ set such that, for each ξ, set $M_\xi(x)$ is no more than finite and, for each K, the set

$$M^K = \{\xi \mid \exists x^1, \ldots, x^K \colon x^i \neq x^j, \ (\xi, x^i) \in M\}$$

is nonvacuous. It can be assumed that x is a single variable since if, for $j = 1, \ldots, n$, the number of points of set $(M_j)_\xi(x)$, where M_j denotes the projection of set M on (ξ, x_j), is bounded by constant K_j, not depending on ξ, then the number of points in set $M_\xi(x)$ is bounded by the constant $K = \prod_j K_j$.

Let $\dim' M^K$ denote the greatest k such that, for some $\bar{p} = p_1, \ldots, p_k$, the projection, $M_{\bar{p}}^K$, of set M^K on space $(\xi_{\bar{p}}) = (\xi_{p_1}, \ldots, \xi_{p_k})$ is dense somewhere in $(\xi_{\bar{p}})$. There exists K_0 such that $\dim' M^K = \dim' M^{K_0}$ for all $K > K_0$. Consequently, we can assume that $\dim' M^K = k$ for each $K \geq 1$ (replacing M by $\{M^{K_0} \times (x)\} \cap M$). Obviously, there exists $\bar{p} = (p_1, \ldots, p_k)$ such that the projection, $M_{\bar{p}}^K$, of set M^K on $(\xi_{\bar{p}})$ is dense somewhere for each K. But, it is not difficult to deduce, from the projection of set M^1 on each of the spaces $(\xi_{\bar{p}}, \xi_j)$ $(j \neq p_1, \ldots, p_k)$ being nowhere dense, that the projection, M_p, of set M on $(\xi_{\bar{p}}, x)$ is nowhere dense, and that, for each K, the set $M_{\bar{p}}^K$ is dense somewhere in $(\xi_{\bar{p}})$, which contradicts W_{k+1}.

<u>THEOREM 2.</u> Since the projection, S_j, of set M on each of the spaces (x', x_j) $(j = k+1, \ldots, n)$ is nowhere dense, it then follows from W_{k+1} that there exists an open $\mathcal{P}$ set, Ω_j^1, everywhere dense in (x'), such that, for $x' \in \Omega_j$, the set $(S_j)_{x'}(x_j)$ contains no more than K_j points, where K_j does not depend on x'. For $p = 1, \ldots, K_j$, we set

$$A_{p,j}'' = \{x' \in \Omega_j \mid \exists x_j^1, \ldots, x_j^p \colon (x', x_j^i) \in S_j, \ x_j^i > x_j^{i+1}\},$$

$$A_{0,j}' = {}^c(\overline{A_{1,j}''}), \quad A_{p,j}' = \operatorname{int}(A_{p,j}'' \setminus A_{p+1,j}'').$$

Let $\bar{p} = (p_{k+1}, \ldots, p_n)$. Let $A_{\bar{p}}' = \bigcap_j A_{p_j, j}'$. Obviously, set $A_{\bar{p}}'$ is open, and $\bigcup_{\bar{p}} A_{\bar{p}}'$ is everywhere dense in (x').

Let

$$B_{lpj} \{(x', x_j) \in S_j \mid x' \in A_{p,j}', \ \exists x_j^1, \ldots, x_j^{p-1} \colon (x', x_j^i) \in S_j,$$
$$x_j^1 > \ldots > x_j^{l-1} > x_j > x_j^l > \ldots > x_j^{p-1}\}.$$

For $l\bar{p} = (l_{k+1}, \ldots, l_n, p_{k+1}, \ldots, p_n)$ we set $B_{l\bar{p}} = \{x', x_{k+1}, \ldots, x_n \mid x' \in A_{\bar{p}}', (x', x_j)\ B_{lp_jj}\}$. Each nonvacuous set $B_{l\bar{p}}$ has the form $\{x' \in A_{\bar{p}}', x_j = f_j(x') \ (j = k+1, \ldots, n)\}$. If we can show that, on each open $\mathcal{P}$ set $\Omega_{l\bar{p}}$ which is everywhere dense in $A_{\bar{p}}'$, functions f_j are analytic, the theorem will then be proven.

In the proof of Theorem 1, it was shown that $\dim (\overline{M} \setminus M) < \dim M$ for each $\mathcal{P}$ set M. Therefore, there exists a $\mathcal{P}$ set, $A_{\bar{p}}$, everywhere dense in $A_{\bar{p}}'$, such that function f_j is continuous on $A_{\bar{p}}$. For simplicity, we shall write A instead of $A_{\bar{p}}$, and B instead of $B_{l\bar{p}}$. It can be assumed that $\bar{B} = \{x \mid \exists y: (x, y) \in L\}$, where L is an analytic set.

We first show that we can find analytic subsets, L_i, in L, and open $\mathcal{F}$ sets $A_i \subset A$, such that $A \setminus \bigcup_i A_i$ is nowhere dense and, for each $x' \in A_i$, the set $(L_i)_{x'}(x'', y)$ is nonvacuous and finite.

For this, we apply Lemma 3 to L. If (a) holds (case 1), we set $L^* = L'$, $A^* = \text{int } S'$ (S' is the projection of set L' onto space (x')). Suppose (b) holds. Then,

$$L = \{x, y \mid f_j(x, y) = 0 \quad (j = 1, \ldots, j_0)\}$$

and the set

$$L'' = \left\{(x, y) \in L \mid \text{rk}\begin{pmatrix}\text{grad} f_j \\ x_i'\end{pmatrix} < j_0 + k\right\} \qquad (j = 1, \ldots, j_0;\ i = 1, \ldots, k)$$

is a proper subset of set L. Let S'' be the projection of set L'' onto space (x'); we set $S''' = A \setminus S''$. If int $S''' = \emptyset$ (case 2), we set $L^* = L''$, $A^* = \text{int } S''$. If int $S''' \neq \emptyset$ and, for each $x' \in \text{int } S'''$, set $L_{x'}(x'', y)$ is finite (case 3), we set $L_1 = L$, $A_1 = \text{int } S'''$, $L^* = L''$, and $A^* = \text{int } S''$. If int $S''' \neq \emptyset$, and there exists a point $x_0' \in \text{int } S'''$ such that $L_{x_0'}(x'', y)$ contains an analytic curve λ (case 4), we choose a vector y_q in the basis of space (y) such that, on curve λ, there exist two points, (x^1, y^1) and (x^2, y^2), such that $y_q^1 \neq y_q^2$. Consider the set

$$L''' = \left\{(x, y) \in L \mid \text{rk}\begin{pmatrix}\text{grad} f_j \\ x_i' \\ y_q\end{pmatrix} < j_0 + k + 1\right\} \qquad (j = 1, \ldots, j_0;\ i = 1, \ldots, k).$$

It is a proper subset of set L since, were it not, the tangent vector to curve λ at any point would be orthogonal to vector y_q. Consequently, there cannot be two points with different coordinates y_q, contrary to our assumption. Moreover, the projection of set L''' onto (x') coincides with the projection of set L, since $L''' \supset L''$, while in each set $L_{x_0'}(x'', y)$, $x'_0 \in \bar{A} \setminus S''$, the point (x'', y) having the maximal coordinate y_q lies in L'''. We set $L^* = L'''$ and $A^* = A$.

By applying the same reasoning to set L^* (to which one of the cases considered will apply), we find sets L^{**} and A^{**} and (in case 3) sets L_2, A_2.

Repeating these constructions, we wind up with the sets L_i and A_i which we have been seeking.

Now, let $B_i = B \cap \{A_i \times (x'')\}$. It is obvious that

$$B_i = \{x', x'' \mid x' \in A_i,\ (x', x'') \in B,\ \exists y: (x', x'', y) \in L_i\}.$$

We now apply Lemma 3 to set L_i. If (a) holds (case 1), we set $L_i^* = L_i'$, $A_i^* = A_i \cap \text{int } S_i'$, where S_i' is the projection of set L_i' onto (x'). If (b) holds, then $L_i = \{x, y \mid f_{ij}(x, y) = 0 \ (j = 1, \ldots, j_i)\}$, where

$$L_i'' = \left\{(x, y) \in L_i \mid \text{rk}\begin{pmatrix}\text{grad} f_{ij} \\ x_l'\end{pmatrix} < j_i + k\right\} \qquad (j = 1, \ldots, j_i;\ l = 1, \ldots, k)$$

is a proper subset of L_i.

If the projection S_i'', of set L_i'' onto (x') is everywhere dense in A_i (case 2), we set $L_i^* = L_i''$, $A_i^* = A_i \cap \text{int } S_i''$.

If $S_i''' = \text{int } (A_i \setminus S_i'') \neq \emptyset$ (case 3), we set $L_i^* = L_i''$, $A_i^* = \text{int } S_i''$, $L_{i1} = L_i$, $A_{i1} = S_i'''$. Arriving at the pair (A_i^*, L_i^*) (arising in one of the cases considered), just as with (A_i, L_i), and iterating, we obtain the sets A_{ih} and L_{ih} such that $\bigcup_h A_{ih}$ is everywhere dense in A_i, and each set L_{ih} is so given by the functions $f_{ihj}(j = 1, \ldots, j_{ih})$ that $\text{rk}\begin{pmatrix}\text{grad} f_{ihj} \\ x_l'\end{pmatrix} = j_{ih} + k$ for $x' \in A_{ih}$. Moreover, since for $x' \in A_{ih}$ the set $(L_{ih})_{x'}(x'', y)$ is zero-dimensional, it follows from the implicit function theorem that, in the neighborhood of point $(x', x'', y) \in L_{ih}$, $x' \in A_{ih}$, set L_{ih} has the form $\{(x'', y) = F(x')\}$, where $F = (f_1, \ldots, f_{m+n-k})$ is an analytic vector function in A_{ih}. Consequently,

$$B \cap \{A_{ih} \times (x'')\} = \{x' \in A_{ih}: x_q'' = f_q(x')\},$$

and, for the proof of the Theorem, it remains only to set $\Omega_{lp} = \bigcup_{i,h} A_{ih}$.

COROLLARY 2. Let M be a $\mathcal{P}$ set in the space $(x) = (x_1, \ldots, x_n)$. We use induction on dim' M, this being the greatest k such that the projection of set M on some k-dimensional subspace $(x_{p_1}, \ldots, x_{p_k})$ is dense somewhere.

Let $p = (p_1, \ldots, p_k)$, $(x'_p) = (x_{p_1}, \ldots, x_{p_k})$, (x''_p) be the orthogonal complement to (x'_p). We order the vectors p in the sequence $\{\bar{p}_i\}$

It follows from Theorem 2 that, in space $(x'_{\bar{p}_1})$, there exists an open, everywhere dense, set, $\Omega_{\bar{p}_1}$, such that $M_1 = M \cap \{\Omega_{\bar{p}_1} \times (x''_{\bar{p}_1})\}$ is a locally analytic set. If $\dim'(M \setminus M_1)$ is less than k, then all is proven. If, however, $\dim'(M \setminus M_1) = k$, by applying Theorem 2 to $M \setminus M_1$ instead of to M, and to $\bar{p}_2$ instead of to $\bar{p}_1$, and iterating, we obtain a collection of locally analytic sets, M_i, such that the projection of set $M^* = M \setminus \bigcup M_i$ on any space (x_p) is nowhere dense. It immediately follows that $\dim' M^* < k$, and our assertion is proven.

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