SURFACE SINGULARITIES IN $\mathbb{R}^4$: FIRST STEPS TOWARDS LIPSCHITZ KNOT THEORY

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Abstract. A link of an isolated singularity of a two-dimensional semialgebraic surface in $\mathbb{R}^4$ is a knot (or a link) in S^3. Thus the ambient Lipschitz classification of surface singularities in $\mathbb{R}^4$ can be interpreted as a bi-Lipschitz refinement of the topological classification of knots (or links) in S^3. We show that, given a knot K in S^3, there are infinitely many distinct ambient Lipschitz equivalence classes of outer metric Lipschitz equivalent singularities in $\mathbb{R}^4$ with the links topologically equivalent to K.

1. Introduction

There are three kinds of equivalence relations in Lipschitz Geometry of Singularities. One equivalence relation is bi-Lipschitz homeomorphism (of the germs at the origin) of singular sets with respect to the inner metric, where the distance between two points of a set X is defined as infimum of the lengths of paths inside X connecting the two points. The second equivalence relation, outer Lipschitz equivalence, is bi-Lipschitz homeomorphism with respect to the outer metric, where the distance is defined as the distance between the points in the ambient space. A set X is called normally embedded if its inner and outer metrics are equivalent.

In [2], we considered the third equivalence relation, Lipschitz ambient equivalence. Two germs X and Y of semialgebraic sets at the origin of $\mathbb{R}^n$ are called Lipschitz ambient equivalent if there exists a germ of a bi-Lipschitz homeomorphism h of $(\mathbb{R}^n, 0)$ such that $Y = h(X)$. In particular, such sets X and Y are outer Lipschitz equivalent. Two outer Lipschitz equivalent sets are always inner Lipschitz equivalent, but can be ambient topologically non-equivalent (see Neumann-Pichon [4]).

Let X and Y be two semialgebraic surface singularities (two-dimensional germs at the origin) in $\mathbb{R}^n$ which are outer Lipschitz equivalent. Suppose also that X and Y are topologically ambient equivalent. Does

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it imply that the sets X and Y are Lipschitz ambient equivalent? It seems plausible that the answer is "yes" when $n \geq 5$, or when X and Y are normally embedded. However, examples in [2] show that the answer may be "no" when $n = 3$ or $n = 4$.

One class of examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ is based on the theorem of Sampaio [6]: Lipschitz ambient equivalence of two sets implies Lipschitz ambient equivalence of their tangent cones. Thus any two sets with topologically ambient non-equivalent tangent cones cannot be Lipschitz ambient equivalent.

The case $n = 4$ is especially interesting, as in that case the link of a two-dimensional germ X in $\mathbb{R}^4$ is a knot (or a link) in S^4, and the arguments are based on the knot theory. For a given surface $X \subset \mathbb{R}^3$ there are finitely many distinct Lipschitz ambient equivalence classes of the surfaces which are topologically ambient equivalent and outer Lipschitz equivalent to X. However, there may be infinitely many such Lipschitz ambient equivalence classes for a surface in $\mathbb{R}^4$. Moreover, a more delicate argument, based on the "bridge construction" below, provides infinitely many distinct Lipschitz ambient equivalence classes of surfaces which are topologically ambient equivalent to a given surface $X \subset \mathbb{R}^4$ and belong to the same outer metric Lipschitz equivalency class, even when each of these surfaces has a tangent cone consisting of a single ray.

2. Examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ based on Sampaio's theorem

Example 1 (see [2]). Let X_1 and X_2 be two surfaces in $\mathbb{R}^3$ with the links at the origin shown in Fig. 1a and Fig. 1b, and the links of their tangent cones at the origin shown in Fig. 1a' and Fig. 1b'. The tangency exponent of the arcs γ_+ and γ_- in Fig. 1a and Fig. 1b is $\alpha > 1$, and the tangency exponent of the arcs ζ_+ and ζ_- is $\beta > 1$. Thus the arcs γ_+ and γ_- correspond to a single ray γ of the tangent cone, and the arcs ζ_+ and ζ_- correspond to a single ray ζ. One can define X_1 and X_2 by explicit semialgebraic formulas. Both surfaces X_1 and X_2 are ambient topologically equivalent to a cone over a circle and bi-Lipschitz equivalent with respect to the outer metric, but not ambient Lipschitz equivalent by Sampaio's theorem, since their tangent cones are not ambient topologically equivalent.

Example 2 (see [2]). Let X_1 and X_2 be two surfaces in $\mathbb{R}^4$ with the links at the origin shown in Fig. 2a and Fig. 2b, and the links of their tangent cones at the origin shown in Fig. 2a' and Fig. 2b'. The tangency exponent of the arcs γ_+ and γ_- in Fig. 2a and Fig. 2b is $\alpha > 1$, thus the arcs γ_+ and γ_- correspond to a single ray γ of the tangent cone.

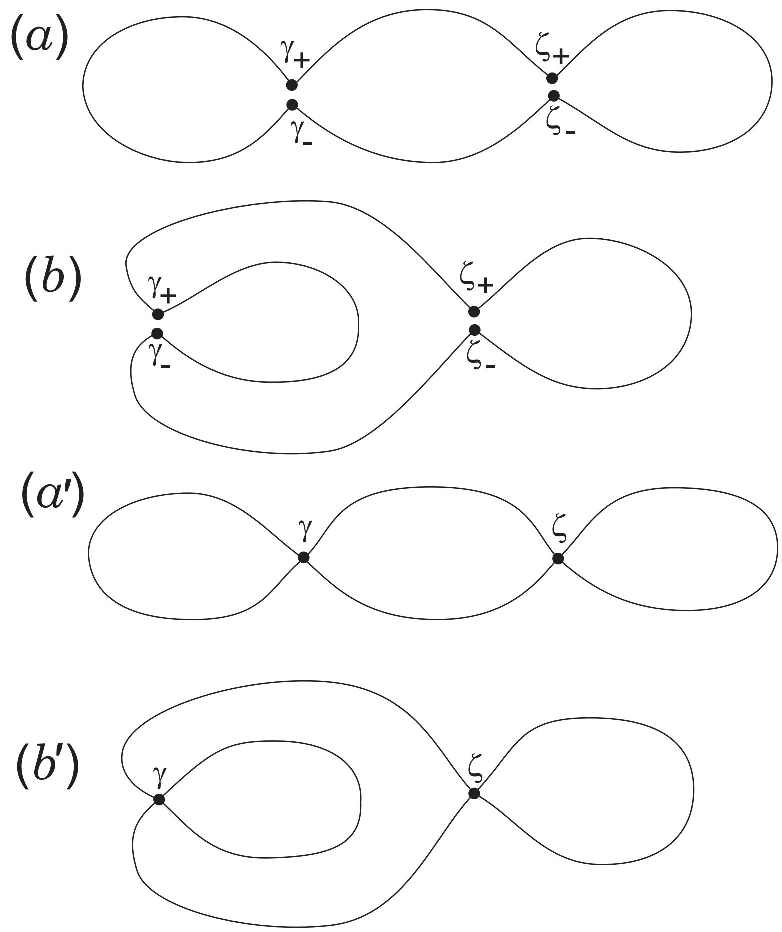

FIGURE 1. The links of the surfaces X_1 and X_2 in Example 1, and of their tangent cones.

One can define X_1 and X_2 by explicit semialgebraic formulas. Both surfaces X_1 and X_2 are ambient topologically equivalent to a cone over a circle and bi-Lipschitz equivalent with respect to the outer metric, but not ambient Lipschitz equivalent by Sampaio's theorem, since their tangent cones are not ambient topologically equivalent.

3. Bridge construction

A (q,β)-*bridge* is the set $A_{q,\beta} = T_+ \cup T_- \subset \mathbb{R}^4$ where $1 < \beta < q$ and

$$T_\pm = \left\{ 0 \le t \le 1,\ -t^\beta \le x \le t^\beta,\ y = \pm t^q,\ z = 0 \right\}.$$

Its link is shown in Fig. 3 (left). A *broken* (q,β)-*bridge* $B_{q,\beta}$ is obtained from $A_{q,\beta}$ by the *saddle operation*, removing from $T_\pm$ two p-Hölder

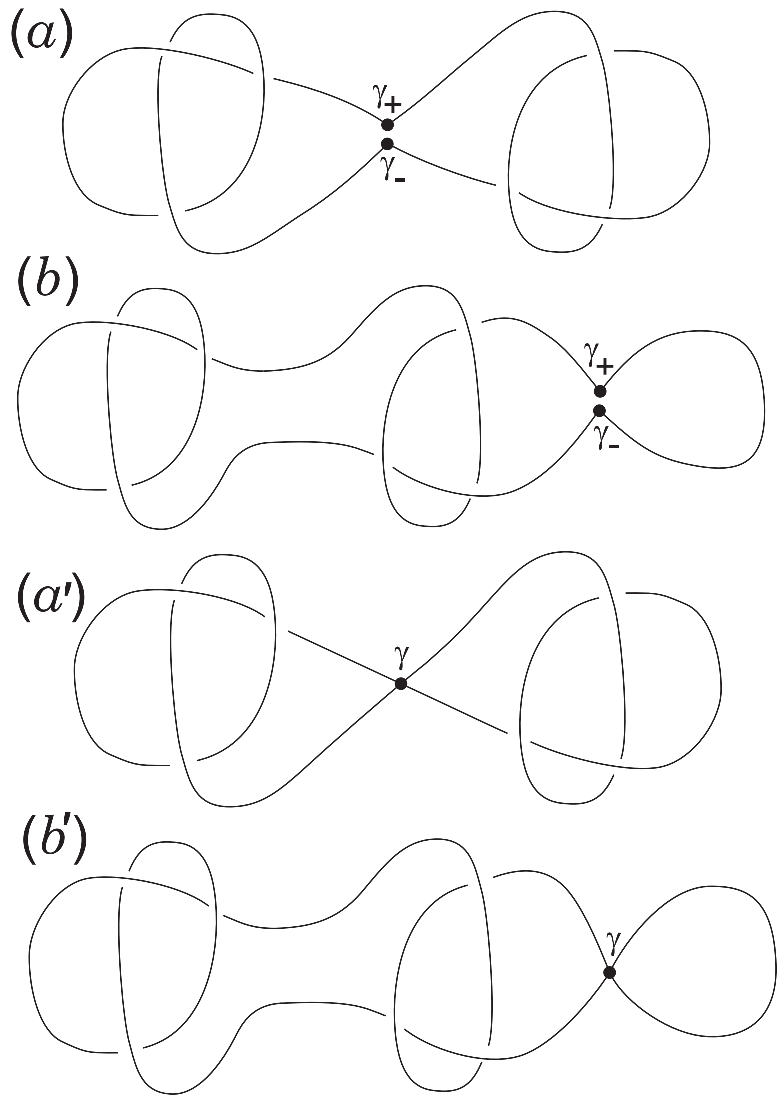

FIGURE 2. The links of the surfaces X_1 and X_2 in Example 2, and of their tangent cones.

triangles $\{t \geq 0,\ |x| \leq t^p,\ y = \pm t^q,\ z = 0\}$ where $p > q$, and replacing them by two q-Hölder triangles $\{0 \leq t \leq 1,\ x = \pm t^p,\ |y| \leq t^q,\ z = 0\}$. Its link is shown in Fig. 3 (right). We call (q, β)-bridge any surface outer Lipschitz equivalent to $A_{q,\beta}$. It was shown in [2] that ambient Lipschitz equivalence $h : X \to Y$ of two surfaces in $\mathbb{R}^4$ maps a (q, β)-bridge in X to a $(q, beta)$-bridge in Y, and that the two surfaces remain ambient Lipschitz equivalent when their (q, β)-bridges are replaced by the broken (q, β)-bridges.

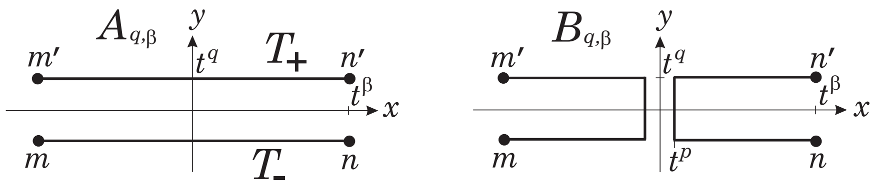

FIGURE 3. The links of a (q, β)-bridge $A_{q,\beta}$ and a broken (q, β)-bridge $B_{q,\beta}$.

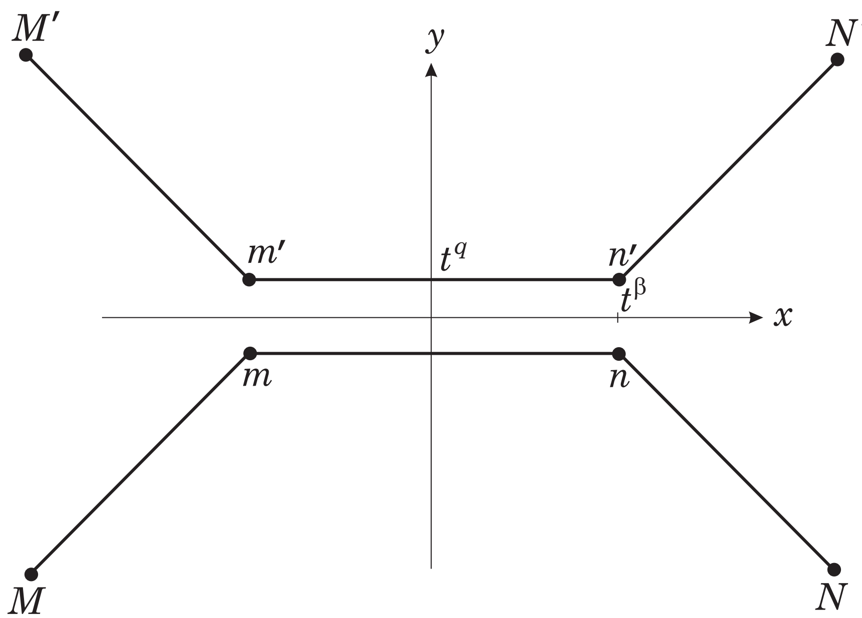

FIGURE 4. The link of the surface G in Example 3.

Remark 3.1. Our definition of a broken bridge is slightly different from the definition in Example 4 of [2], where it was defined with $p < q$. Condition $p > q$ makes the "broken bridge" operation invertible: *two surface germs with the same (q, β)-bridge are ambient Lipschitz equivalent if and only if they are ambient Lipschitz equivalent after the bridge is broken (with the same $p > q$).* Note that this invertibility is never used here or in [2].

Example 3 (see [2]). The common boundary of $A_{q,\beta}$ and $B_{q,\beta}$ consists of the four arcs $\left\{0 \le t \le 1,\ x = \pm t^\beta,\ y = \pm t^q,\ z = 0\right\}$ shown as m, n, m', n' in Fig. 3. Let $G \subset \mathbb{R}^4$ be a semialgebraic surface containing $A_{q,\beta}$ and bounded by the four straight line segments $\{0 \le t \le 1,\ \pm x = \pm y = t,\ z = 0\}$ (see Fig. 4 where the boundary arcs of G are

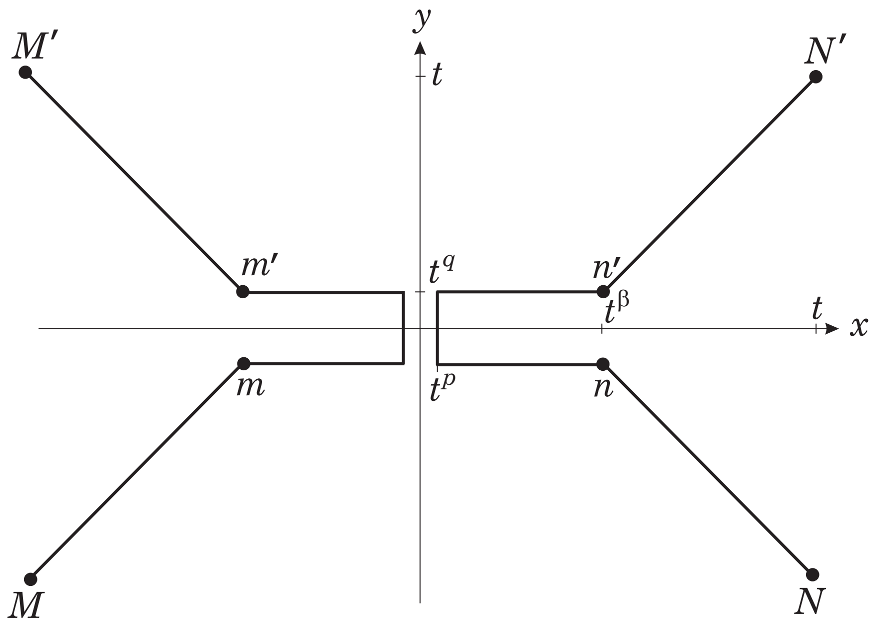

FIGURE 5. The link of the surface H in Example 3.

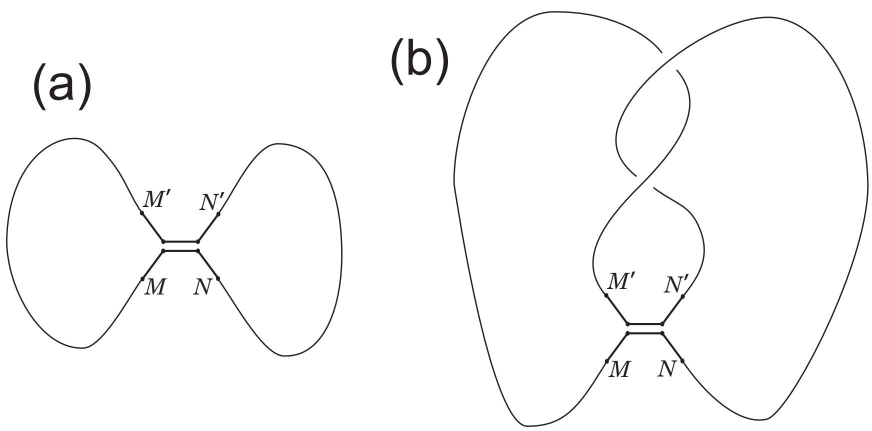

FIGURE 6. The links of the surfaces X and Y in Example 3.

shown as M, N, M', N'). Let H be the surface obtained from G by replacing the bridge $A_{\beta,q}$ by the broken bridge $B_{\beta,q}$ (see Fig. 5).

Consider two topologically trivial knots K and L in the hyperplane $\{t = 1\} \subset \mathbb{R}^4_{x,y,z,t}$ as shown in Fig. 6a and Fig. 6b. Each of these two knots contains the curve $g = G \cap \{t = 1\}$, We define the surface X as the union of G and a straight cone over $K \setminus g$, and the surface Y as the union of G and a straight cone over $L \setminus g$.

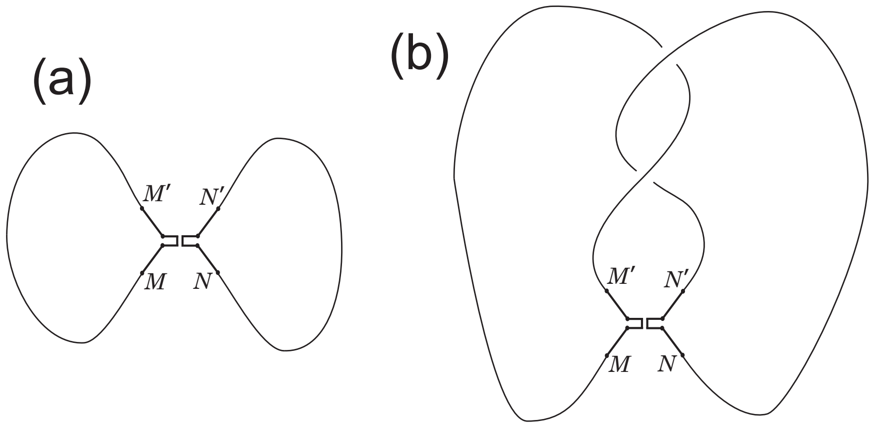

FIGURE 7. The links of the surfaces X' and Y' in Example 3.

Theorem 3.2. *(see [2] Theorem 3.2.) The germs of the surfaces X and Y at the origin are outer Lipschitz equivalent, topologically ambient equivalent, but not Lipschitz ambient equivalent.*

This is proved by replacing the (q, β)-bridges in X and Y by the broken (q, β)-bridges, resulting in the new surfaces X' and Y', shown in Fig. 7. The link of X' consists of two unlinked circles while the two circles in the link of Y' are linked. Thus X' and Y' are not topologically ambient equivalent, which implies that X and Y are not Lipschitz ambient equivalent.

Remark 3.3. Notice that the tangent cones of both surfaces X and Y in Example 3 are ambient topologically equivalent to a cone over two unknotted circles, pinched at one point. Thus Sampaio's theorem does not apply, and we need the bridge construction in this example. Notice also that the bridge construction employed in this example allows one to construct examples of outer bi-Lipschitz equivalent, topologically ambient equivalent but Lipschitz ambient non-equivalent surface singularities in $\mathbb{R}^4$ with the tangent cones as small as a single ray.

The surfaces X and Y in Example 3 differ by a "twist" of the (q, β)-bridge, which can be extended to a homeomorphism of the ambient space, but not to a bi-Lipschitz homeomorphism. One can iterate such a twist to obtain infinitely many Lipschitz ambient non-equivalent surfaces. On can also attach an additional knot to the links of both surfaces X and Y (see Fig. 8). This yields the following "universality" result (see [2] Theorem 4.1).

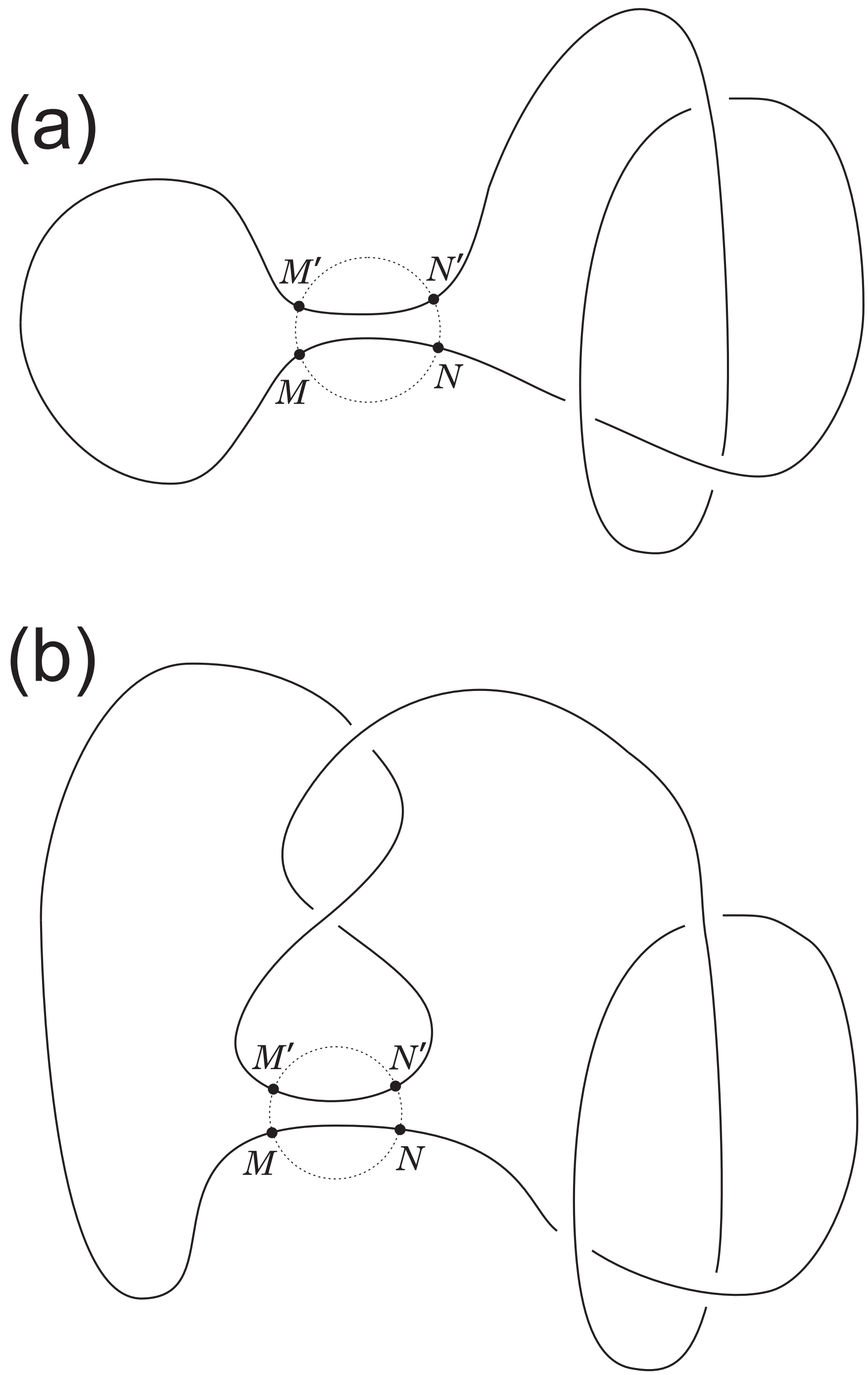

FIGURE 8. The links of the surfaces X and Y with an extra knot attached.

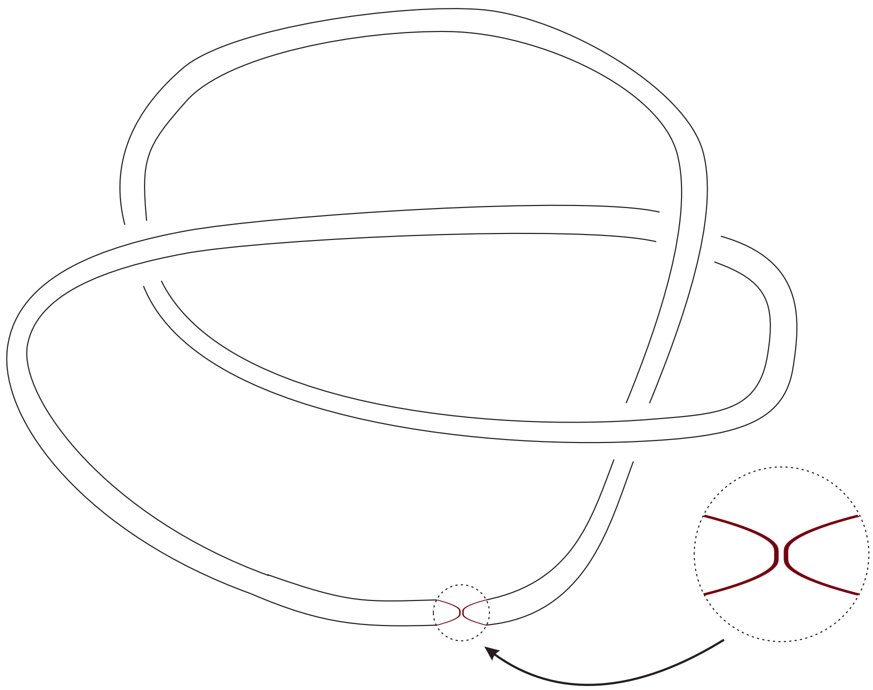

FIGURE 9. The link of the surface X in Theorem 3.5.

Theorem 3.4. *For any semialgebraic surface germ $S \subset \mathbb{R}^4$ there exist infinitely many semialgebraic surface germs $X_i \subset \mathbb{R}^4$ such that*

1) *For all i, the germs X_i are topologically ambient equivalent to S;*
2) *All germs X_i are outer Lipschitz equivalent;*
3) *The tangent cones of all germs X_i at the origin are ambient topologically equivalent;*
4) *For $i \neq j$ the germs X_i and X_j are not Lipschitz ambient equivalent.*

Other versions of universality can be formulated. In particular, the following theorem is proved in [1].

Theorem 3.5. *For each knot K there exists a semialgebraic surface $X \subset \mathbb{R}^4$ with a singular point at the origin, such that*

1) *The germ of X at the origin has one (q, β)-bridge.*
2) *The link of X at the origin is a trivial knot.*
3) *The link of $S(X)$ at zero is a two component link, such that each component of this link is the knot K.*

This result is illustrated in Fig. 9. Notice that the tangent cones of the surface X in Fig. 9 are not equivalent for non-equivalent knots K, so this result could be also obtained by Sampaio's theorem.

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