Asymmetric Abelian Avalanches and Sandpiles

Andrei Gabrielov

Mathematical Sciences Institute

Cornell University, Ithaca, NY 14853 †

E-mail: andrei@math.toronto.edu

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ABSTRACT: We consider two classes of threshold failure models, Abelian avalanches and sandpiles, with the redistribution matrices satisfying natural conditions guaranteeing absence of infinite avalanches. We investigate combinatorial structure of the set of recurrent configurations for these models and the corresponding statistical properties of the distribution of avalanches. We introduce reduction operator for redistribution matrices and show that the dynamics of a model with a non-reduced matrix is completely determined by the dynamics of the corresponding model with a reduced matrix. Finally, we show that the stationary distributions of avalanches in the two classes of models: discrete, stochastic Abelian sandpiles and continuous, deterministic Abelian avalanches, are identical.

Introduction. Different cellular automaton models of failure (sandpiles, avalanches, forest fires, etc.), starting with Bak, Tang and Wiesenfeld [BTW1, BTW2] sandpile model, were introduced in connection with the concept of self-organized criticality. Traditionally, all of these models are considered on uniform cubic lattices of different dimensions.

Recently Dhar [D1] suggested a generalization of the sandpile model with a general (modulo some natural sign restrictions) integer matrix Δ of redistribution of accumulated particles during an avalanche. An important property of this Abelian sandpile (ASP) model is the presence of an Abelian group governing its dynamics.

Dhar introduced the set of recurrent configurations for an Abelian sandpile model, the principal geometric object governing its dynamics in the stationary state. The burning algorithm introduced in [MD] allows to recognize, for a symmetric sandpile model, when

† Currently at Department of Mathematics, University of Toronto, Toronto M5S 1A1, Canada

a stable configuration is recurrent. A more sophisticated script algorithm suggested in [S] plays the same role for asymmetric models. Both algorithms provide, in fact, certain information on the combinatorial structure of the set of recurrent configurations.

Abelian sandpiles were also studied in [WTM, D2, DM, GM, Cr, Go, CC, G1, G2, S, DMan]. In a non-dissipative case ($\sum_j \Delta_{ij} = 0,$ for all i) an avalanche in the ASP model coincides with a chip-firing game on a directed graph [BLS, BL] where $-\Delta$ is the Laplace matrix of the underlying digraph.

Another class of lattice models of failure, slider block models introduced by Burridge and Knopoff [BK] and studied in [CaL, Ca, N, MT, LKM], as well as models [FF, D-G, OFC, CO, Z, PTZ, GNK] which are equivalent to quasistatic block models, have continuous time and continuous quantity at the lattice sites which accumulates in time and is redistributed during avalanches. This quantity is called the slope, height, stress or energy by different authors. In slider block models it corresponds to force [OFC]. We use the term *height* as in [D1].

Gabrielov [G1], introduced Abelian avalanche (AA) models, deterministic lattice models with continuous time and height values at the sites of the lattice, and with an arbitrary redistribution matrix. For a symmetric matrix, these models are equivalent to arbitrarily interconnected slider block systems. In the case of a uniform lattice, these models were studied in [FF, D-G] and in [GNK] (as series case a).

The stationary behavior of the AA model is periodic or quasiperiodic, depending on the loading rate vector. At the same time, the distribution of avalanches for a discrete, stochastic ASP model is *identical* to the distribution of avalanches for an arbitrary quasiperiodic trajectory (or to its average over all periodic trajectories) of a continuous, deterministic AA model with the same redistribution matrix and loading rate [G1].

In this paper, we introduce general conditions on redistribution matrices that are equivalent to the absence of infinite avalanches in the model. The models satisfying these conditions include, in particular, the models with non-negative dissipation or codissipation considered before. We allow spatially inhomogeneous loading rate and show that the set of recurrent configurations does not depend on the loading rate vector, as long as a natural condition guaranteeing absence of non-loaded components in the model is satisfied.

We continue the study of the combinatorial structure of the set of recurrent configurations started in [D1, DM, MD, S] and introduce a new description of this set, essentially improving the script algorithm suggested in [S] for asymmetric redistribution matrices,

Additional possibility for the study of the models with asymmetric matrices arises from the matrix reduction operations. These operations act on the redistribution matrices in the same way as the topple operations act on unstable configurations, and satisfy the same property of the independence of the resulting reduced matrix on the possible change of the order of reductions. Each reduction operation simplifies the redistribution matrix, and replaces the original model by a simpler reduced model such that the combinatorics of the set of recurrent configurations for the reduced model completely determines the combinatorics for the original non-reduced model.

In the first section, we define configurations, redistribution matrices and avalanche operators. We show, following a construction implicitly present in [S], that a legal sequence of topples satisfies certain minimality condition among all (possibly illegal) sequences of topples with the same final stable configuration (lemma 1.1). This minimality condition provides, in particular, a new proof of the principal Abelian property (theorem 1.2). We introduce the class of avalanche-finite redistribution matrices satisfying eight equivalent conditions and check these conditions for the matrices with non-negative dissipations and codissipations.

In the second section, we define, following [G1], the AA model as a sequence of loading periods and avalanches and describe the dynamics of the model on its attracting set of recurrent configurations. The arguments here are similar to the arguments of Dhar [D1] for the ASP models.

In the third section, we study the combinatorial structure of the set of recurrent configurations of the AA model. The principal result here, the theorem 3.8, describes this set as the complement in the set of all stable configurations to the union of negative octants with the vertices in a finite set $\mathcal{N}$. An explicit constructive description of this set $\mathcal{N}$ is given in the theorem 3.11.

In the fourth section, we show the possibilities to extract the information on the dynamics of an AA model from the dynamics of another model with a simplified redistribution matrix. We introduce the total reduction operator for redistribution matrices, similar to the avalanche operator for configurations. We show that the stationary dynamics of an AA model with a redistribution matrix Δ is completely determined by the dynamics of the corresponding model with a reduced redistribution matrix, the total reduction of Δ.

In the fifth section, we introduce marginally stable configurations and derive formulas for the mean number of avalanches. The arguments here are again similar to those of Dhar [D1], modified for the more general situation considered here.

In the sixth section, we establish the identity between the distributions of avalanches for AA and ASP models with the same redistribution matrices.

Some of the results of this paper were announced in [G2].

1. Redistribution matrices and avalanches. Let V be a finite set of N elements (sites), and let Δ be a $N \times N$ real matrix with indices in V. We call Δ a *redistribution matrix* when

$$\Delta_{ii} > 0, \text{ for all } i; \quad \Delta_{ij} \le 0, \text{ for all } i \ne j. \tag{1}$$

A real vector $\mathbf{h} = \{h_i,\ i \in V\}$ is called a *configuration*. The value h_i is called the *height* at the site i. For every site i, a threshold H_i is defined, and a site i with $h_i < H_i$ is called *stable*. A configuration is stable when all the sites are stable.

For $i \in V$, a *topple operator* $\mathcal{T}_i$ is defined as

$$\mathcal{T}_i(\mathbf{h}) = \mathbf{h} - \delta_i \tag{2}$$

where $\delta_i = (\Delta_{i1}, \ldots, \Delta_{iN})$ is the i-th row vector of Δ. Obviously, every two topple operators commute. The topple $\mathcal{T}_i(\mathbf{h})$ is *legal* if $h_i \geq H_i$, i.e. if the site i is unstable. No topples are legal for a stable configuration. A sequence of consecutive legal topples is called an *avalanche* if it is either infinite or terminates at a stable configuration. In the latter case, the integer vector $\mathbf{n} = \{n_i,\ i \in V\}$ where n_i is the number of topples at a site i during the avalanche is called its *script*, and the total number $\sum_i n_i$ of topples in the avalanche is called its *size*.

The following lemma shows that the avalanches are "extremal" among all the sequences of (possibly, illegal) consecutive topples with the same endpoints. It allows, in particular, to give an alternative proof of the principal property of avalanches — the script and the final stable configuration depend only on the starting configuration, not on the possible choice in the sequence of topples (theorem 1.2 below).

Lemma 1.1. *Let* $\mathbf{h}$ *be an arbitrary configuration and let* $\mathbf{m}$ *be an integer vector with non-negative components* m_i *such that* $\mathbf{g} = \mathbf{h} - \sum m_i\delta_i$ *is a stable configuration. For any finite sequence of consecutive legal topples started at* $\mathbf{h}$*, with* n_i *topples at a site* i*, we have* $m_i \geq n_i$.

Proof. The arguments appear implicitly in [S]. We use induction on the size $n = \sum n_i$ of the sequence of legal topples. For $n = 0$, the statement is trivial. Let it be true, i.e. $m_i \geq n_i$, for a sequence with n_i topples at a site i. If a site j is unstable for a configuration $\mathbf{f} = \mathbf{h} - \sum n_i\delta_i$ then $g_j < f_j$. Due to (1), this implies $m_j > n_j$, hence the statement remains true when we add a topple at the site j to the sequence.

Theorem 1.2. (Sf. [D1], [BLS], [BL].) *Every two avalanches starting at the same configuration* $\mathbf{h}$ *are either both infinite or both finite. In the latter case, the scripts of both avalanches coincide. In particular, both avalanches terminate at the same stable configuration and have the same size.*

Proof. The statement follows easily from the lemma 1.1.

Remark 1.3. If we consider a configuration as an initial state of a game, and every legal topple as a legal move, an avalanche becomes a (solitary) game. The theorem 1.2 means that this game is strongly convergent in the definition of [E].

Lemma 1.4. *For every site i that toppled at least once during an avalanche, $h_i \geq H_i - \Delta_{ii}$ till the end of the avalanche.*

The statement follows from (1) and (2).

Let $\mathbf{R}_+^V = \{h_i \geq 0, \text{ for all } i\}$ and $\mathbf{R}_-^V = -\mathbf{R}_+^V$ denote positive and negative closed octants in $\mathbf{R}^V$, and let $\dot{\mathbf{R}}_+^V = \{h_i > 0, \text{ for all } i\}$ be an open positive octant. Let Δ' be the transpose of the matrix Δ.

Theorem 1.5. *For a redistribution matrix Δ, the following properties are equivalent.*

i. *Every avalanche for Δ is finite.*

ii. $\Delta(\mathbf{R}_+^V \setminus \{0\}) \cap \mathbf{R}_-^V = \emptyset$.

iii. $\Delta(\mathbf{R}_+^V) \supseteq \mathbf{R}_+^V$, *i.e.* Δ^{-1} *exists and all its elements are non-negative.*

iv. $\Delta(\mathbf{R}_+^V) \cap \dot{\mathbf{R}}_+^V \neq \emptyset$.

i$'$. *Every avalanche for Δ' is finite.*

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iii$'$. $\Delta'(\mathbf{R}_+^V) \supseteq \mathbf{R}_+^V$, *i.e.* Δ'^{-1} *exists and all its elements are non-negative.*

iv$'$. $\Delta'(\mathbf{R}_+^V) \cap \dot{\mathbf{R}}_+^V \neq \emptyset$.

Proof. (**ii$'$**) $\Rightarrow$ (**i**). Let us show that for Δ satisfying (ii$'$), every avalanche is finite. If there exists an infinite avalanche started at a configuration $\mathbf{h}$, let $\mathbf{r}(k) = \{k_i/k, \ i \in V\}$ where k_i is the number of topples at a site i after a total number of topples k. According to (2), the configuration after k topples is $\mathbf{h}(k) = \mathbf{h} - k\Delta'\mathbf{r}(k)$. Let $\mathbf{r} \in \mathbf{R}^V \setminus \{0\}$ be an accumulation point for $\mathbf{r}(k)$ (it exists because all these vectors have unit length) and $\mathbf{p} = -\Delta'\mathbf{r}$ an accumulation point for $(\mathbf{h}(k) - \mathbf{h})/k$. According to lemma 1.4, components of $\mathbf{h}(k) - \mathbf{h}$ are bounded from below. Hence all components of $\mathbf{p}$ are non-negative, and Δ does not satisfy (ii$'$).

(i) $\Rightarrow$ **(iii′)**. Let $\mathbf{h}$ be a configuration in $\mathbf{R}_+^V$, and let a finite avalanche starting at $k\mathbf{h}$ terminates at a stable configuration $\mathbf{h}(k)$. Let $\mathbf{r}(k) = \{k_i/k,\ i \in V\}$ where k_i is the number of topples at a site i during this avalanche. We have $\Delta'\mathbf{r}(k) = \mathbf{h} - \mathbf{h}(k)/k$. Let $\mathbf{r}$ be an accumulation point for $\mathbf{r}(k)$, as $k \to \infty$. Then $\mathbf{r} \in \mathbf{R}_+^V$ and $\Delta'\mathbf{r} = \mathbf{h}$, because $\mathbf{h}(k)$ remains bounded as $k \to \infty$.

(iv) $\Rightarrow$ **(ii′)**. Suppose that Δ does not satisfy (ii′). This means that there exists a linear form $l \neq 0$ with non-negative coefficients such that $l(\delta_i) \leq 0$, for all i. Hence $l(\sum c_i\delta_i) \leq 0$ for any combination of the vectors δ_i with non-negative coefficients c_i. At the same time, $l(\delta) > 0$, for every $\delta \in \dot{\mathbf{R}}_+^V$. This means that $\Delta(\mathbf{R}_+^V) \cap \dot{\mathbf{R}}_+^V = \emptyset$ and Δ does not satisfy (iv).

(iii′) $\Rightarrow$ **(iv′)**. The implication is obvious.

Combining the four implications, we have (iv) $\Rightarrow$ (ii′) $\Rightarrow$ (i) $\Rightarrow$ (iii′) $\Rightarrow$ (iv′). The same arguments applied to Δ' instead of Δ imply (iv′) $\Rightarrow$ (ii) $\Rightarrow$ (i′) $\Rightarrow$ (iii) $\Rightarrow$ (iv). This completes the proof.

Definition 1.6. A redistribution matrix satisfying the conditions of the theorem 1.5 is called *avalanche-finite.*

Remark 1.7. Let Δ be an avalanche-finite matrix, and let $\mathbf{t} \in \dot{\mathbf{R}}_+^V$, $\Delta\mathbf{t} \in \dot{\mathbf{R}}_+^V$. Such a vector $\mathbf{t}$ always exists due to the property (iii) or (iv). Let $|\mathbf{h}|_{\mathbf{t}} = (\mathbf{h}, \mathbf{t})$ be the $\mathbf{t}$-weighted length of a configuration $\mathbf{h}$. Then $|\mathcal{T}_i(\mathbf{h})|_{\mathbf{t}} < |\mathbf{h}|_{\mathbf{t}}$, for all $i \in V$, i.e. every topple operator dissipates the $\mathbf{t}$-weighted length. This can be also used to prove the implications (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i).

Definition 1.8. The value $s_i = \sum_j \Delta_{ij}$ is called the *dissipation* at the site i, and the value $s'_j = \sum_i \Delta_{ij}$ is called the *codissipation* at the site j. A site i is called *dissipative* (*non-dissipative*) if $s_i > 0$ ($s_i = 0$). A site j is called *codissipative* (*non-codissipative*) if $s'_j > 0$ ($s'_j = 0$).

An *underlying digraph* $\Gamma = \Gamma(\Delta)$ of a redistribution matrix Δ is defined by the vertex set $V(\Gamma) = V$ and an edge from a site i to a site j drawn iff $\Delta_{ij} < 0$.

Let $\mathbf{s}'$ be a diagonal matrix with $\mathbf{s}'_{ii} = s'_i$, and let

$$\Delta_0 = \Delta - \mathbf{s}' \tag{3}$$

be the non-codissipative part of Δ. The matrix Δ_0 coincides with the Kirchhoff matrix of Γ, with conductance of an edge $\overrightarrow{ij}$ defined as $-\Delta_{ij}$ [T, p.138].

A subset W of V is called a *sink* in Γ if there are no edges from sites in W to sites outside W, and a *source* if there are no edges from sites outside W to sites in W.

A matrix Δ is called *weakly dissipative* if all the dissipation values s_i are non-negative and the digraph $\Gamma(\Delta)$ has no non-dissipative sinks, i.e. from every site there exists a directed path in $\Gamma(\Delta)$ to a dissipative site.

Proposition 1.9. *A matrix with non-negative dissipation values is avalanche-finite if and only if it is weakly dissipative.*

Proof. If the graph $\Gamma(\Delta)$ has a non-dissipative sink $W \subseteq V$ then $\sum_{i \in W} h_i$ does not decrease during an avalanche, hence the avalanche started at a configuration with large enough values of h_i, $i \in W$, cannot be finite.

Suppose now that Δ is weakly dissipative. It follows from the definition 1.8 that $\sigma = \sum_i h_i$ does not increase at any topple and decreases when a dissipative site topples. Suppose that there exists an infinite avalanche, and let $W \subset V$ be the subset of sites that topple infinite number of times in this avalanche. Then all the sites in W are non-dissipative, otherwise σ would decrease indefinitely, in contradiction to the lemma 1.4. At the same time, W is a sink of Γ, otherwise h_j would increase indefinitely at any site $j \notin W$ such that $\Delta_{ij} < 0$, for some $i \in W$. This contradicts the definition 1.8.

Definition 1.10. A matrix Δ is called *weakly codissipative* if all the codissipation values s'_j are non-negative and the digraph $\Gamma(\Delta)$ has no non-codissipative sources, i.e. to every site there exists a directed path in $\Gamma(\Delta)$ from a codissipative site.

Proposition 1.11. *A matrix with non-negative codissipation values is avalanche-finite if and only if it is weakly codissipative.*

Proof. The statement follows from the theorem 1.5 and proposition 1.9, because the transpose of a weakly codissipative matrix is weakly dissipative.

Proposition 1.12. *For every avalanche-finite matrix* Δ, $\det(\Delta) > 0$.

Proof. Let a redistribution matrix Δ satisfy the condition (ii) of the theorem 1.5. For $t \in [0, 1]$, all the matrices $\Delta_t = t\Delta + (1-t)E$ from the segment connecting with the unit matrix E satisfy (ii). Hence all these matrices are avalanche-finite. Due to the condition (iii), all the matrices in this segment are non-singular, hence their determinants have the same (positive) sign.

In the following, we consider only avalanche-finite redistribution matrices.

Definition 1.13. For a configuration $\mathbf{h}$, the *avalanche operator* $\mathcal{A}\mathbf{h}$ is defined as the stable configuration that terminates an avalanche initiated at $\mathbf{h}$. Due to the theorem 1.2, this stable configuration is unique. If $\mathbf{h}$ is stable, $\mathcal{A}\mathbf{h} = \mathbf{h}$.

Example 1.14. The sandpile model introduced in [BTW], $n \times n$ square lattice with the nearest neighbor interaction and particles dropping from the boundary, is defined by a symmetric redistribution matrix Δ of the size $n^2 \times n^2$. The rows and columns of Δ are specified by a vector index $\mathbf{i} = (i_1, i_2)$ with $1 \le i_\nu \le N$, for $\nu = 1, 2$, $\quad \Delta_{\mathbf{i},\mathbf{i}} = 4$, $\quad \Delta_{\mathbf{i},\mathbf{j}} = -1$, for $i_1 = j_1,\ i_2 = j_2 \pm 1$, and for $i_1 = j_1 \pm 1,\ i_2 = j_2$, $\quad \Delta_{\mathbf{i},\mathbf{j}} = 0$ otherwise. This matrix is weakly (co-) dissipative, hence avalanche-finite.

Example 1.15. The 1-dimensional model with the failure depending on the local slope, introduced in [BTW] (for $m = 1$) and studied in [KNWZ, LLT, LT, S, CFKKP], is defined as follows. At every site i, $1 \le i \le N$, we place k_i particles, and set $k_{N+1} = 0$. The site i topples when $k_i - k_{i+1} \ge m$. The topple operator removes m particles from the site i and adds one particle to each site $j = i+1, \ldots, i+m$ as soon as $j \le N$. After the transformation $h_i = k_i - k_{i+1}$, for $1 \le i \le N$, this model can be defined by a redistribution matrix Δ with $\Delta_{i,i} = m + 1$, for $i < N$, $\quad \Delta_{N,N} = m \quad \Delta_{i,i-1} = -m$, for $i > 1$, $\quad \Delta_{i,\nu} = -1$, for $i < N$

and $\nu = \max(i+m, N)$ (sf. [S]). This matrix is not symmetric when $m > 1$. It is weakly (co-) dissipative, hence avalanche-finite.

Example 1.16. A chip-firing game introduced in [BLS, BL] is defined by a (directed) graph Γ with a certain number of chips placed at each of its vertices, and a sequence of legal moves (fires), when one particle is allowed to be moved from a vertex i to the end of each edge directed from i, in case the total number of chips at the vertex i is not less than the total number of the edges directed from i. The corresponding redistribution matrix is, after a sign change, the Laplace matrix of Γ. It is always degenerate (all the dissipations are equal 0) hence not avalanche-finite.

Example 1.17. For $N \le 3$, a redistribution matrix Δ is avalanche-finite iff $\det(\Delta) > 0$. However, for $N \ge 4$ there exist redistribution matrices with positive determinant which are not avalanche-finite. Consider, for example, an 4×4-matrix

$$\Delta = \begin{pmatrix} 1 & -3 & -1 & 0 \\ -3 & 1 & -1 & 0 \\ -1 & -1 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

We have $\det(\Delta) = 16 > 0$. At the same time, the matrix Δ is not avalanche-finite, because $\Delta(1,1,0,0) = (-2,-2,-2,0)$, in contradiction to the condition (ii) of the theorem 1.5.

2. Abelian avalanche model. In this section, we define the Abelian avalanche model as a sequence of slow loading periods and fast redistribution events (avalanches). Many of the statements in this section are similar to the corresponding statements of Dhar [D1] for the ASP models. We present these statements with short proofs to make the paper self-contained. Also, the class of the redistribution matrices and loading vectors considered here is more general than in [D1].

Let $\mathbf{v} = \{v_i,\ i \in V\}$ be a non-zero vector with non-negative components. For an (avalanche-finite) redistribution matrix Δ, an *Abelian avalanche* (AA) model [G1] with a *loading rate* vector $\mathbf{v}$ is defined as follows.

For every stable configuration, every height h_i increases in time with the constant rate v_i, until a height h_i equals or exceeds a threshold value H_i at some site i. Then the site i topples according to (2) starting an avalanche which terminates at a stable configuration. After this the loading resumes, and the process continues indefinitely.

Definition 2.1. An AA model is called *properly loaded* if the digraph $\Gamma(\Delta)$ does not contain non-loaded sources, e.g. to every site there exists a directed path in $\Gamma(\Delta)$ from a loaded site. Here a site i is called *loaded* if $v_i > 0$ and *non-loaded* if $v_i = 0$.

If the model is not properly loaded, some parts of the system do not evolve in time. For a properly loaded model, it is easy to show that the rate of topples at every site is positive.

The dynamics of the model does not change if we replace the values H_i by some other values, and add the difference to all configuration vectors. For convenience we take $H_i = \Delta_{ii}$. In this case, $h_i(t) \geq 0$ for any trajectory $\mathbf{h}(t)$ of the model when the i-th element has been toppled at least once. Hence, for a properly loaded model, only configurations $\mathbf{h} \in \mathbf{R}_+^V$ are relevant for the long-term dynamics. Let $\mathcal{S} = \{0 \leq h_i < \Delta_{ii}\}$ be the set of all stable configurations in $\mathbf{R}_+^V$.

Remark 2.2. In the case of a symmetric matrix Δ and $v_i = s_i$, for all i, the AA model is equivalent to a system of blocks where i-th block is connected to j-th block by a coil spring of rigidity $-\Delta_{ij}$, if $\Delta_{ij} < 0$, and to a slab moving with a unit rate by a leaf spring of rigidity s_i. For every block, a static friction force H_i is defined, and a block is allowed to move by one unit of space when the total force h_i applied to this block from other blocks and the moving slab, equals or exceeds H_i. The weak dissipation, weak codissipation and proper loading conditions coincide in this case, and are satisfied when the system of blocks (including the moving slab) is connected.

Proposition 2.3. (Sf. Dhar [D1].) *Let* $\mathbf{r} = \{r_i\}$ *be defined by* $\Delta'\mathbf{r} = \mathbf{v}$. *Then* r_i *is equal to the average, per unit time, rate of topples at a site* i, *independent of the initial* *configuration.*

Proof. Let $\mathbf{r}(t) = \{k_i(t)/t,\ i \in V\}$ where $k_i(t)$ is the number of topples at a site i during a time interval t. We have $\mathbf{r}(t) \to \mathbf{r}$ as $t \to \infty$. Let $\mathbf{h}$ be an initial stable configuration at $t = 0$. According to (2), the stable configuration at the time t is $\mathbf{h}(t) = \mathbf{h} + \mathbf{v}t - t\Delta'\mathbf{r}(t)$. As $t \to \infty$, the stable configuration $\mathbf{h}(t)$ remains bounded according to lemma 1.4, hence $\Delta'\mathbf{r}(t) \to \mathbf{v}$.

Definition 2.4. For $\mathbf{u} \in \mathbf{R}_+^V$, a *load-avalanche* operator $\mathcal{C}_\mathbf{u}$ is defined as $\mathcal{C}_\mathbf{u}\mathbf{h} = \mathcal{A}(\mathbf{h}+\mathbf{u})$.

Theorem 2.5. (Sf. Dhar [D1].) *For $\mathbf{u}, \mathbf{v} \in \mathbf{R}_+^V$, we have $\mathcal{C}_\mathbf{u} \circ \mathcal{C}_\mathbf{v} = \mathcal{C}_{\mathbf{u}+\mathbf{v}}$. In particular, every two load-avalanche operators commute.*

Proof. For every configuration $\mathbf{h}$ with an unstable site i, the site i remains unstable when we add a vector with non-negative components to $\mathbf{h}$. Hence in any sequence of legal topples and loading periods all the topples remain legal if we do all the loading first.

Remark 2.6. Due to the theorem 2.5, the load-avalanche operators for an AA model constitute an Abelian group, which is why the model is called “Abelian”.

Definition 2.7. A configuration $\mathbf{h}$ is called *reachable* if there exists an avalanche started at a configuration with arbitrarily large components passing through $\mathbf{h}$. A configuration is *recurrent* if it is reachable and stable. It will be shown below that only recurrent configurations can (and do) appear in the stationary state of any properly loaded AA model, independent of the loading vector.

Definition 2.8. Let, as before, $\delta_i = (\Delta_{i1}, \ldots, \Delta_{iN})$ be the i-th row vector of the matrix Δ. Integer combinations of vectors δ_i generate a lattice $\mathcal{L} = \mathcal{L}(\Delta)$ in $\mathbf{R}^V$. Two configurations $\mathbf{h}$ and $\mathbf{h}'$, are called *equivalent* if $\mathbf{h}' - \mathbf{h}$ belongs to $\mathcal{L}$. A subset $U \subset \mathbf{R}^V$ is a *fundamental domain* for $\mathcal{L}$ if, for every $\mathbf{h} \in \mathbf{R}^V$, there exists precisely one configuration in U equivalent to $\mathbf{h}$ modulo $\mathcal{L}$.

Theorem 2.9. (Sf. Dhar [D1].) *The set $\mathcal{R}$ of recurrent configurations is a fundamental domain for the lattice $\mathcal{L}$.*

Proof. For every $\mathbf{h} \in \mathbf{R}_+^V$, configuration $\mathcal{A}\mathbf{h}$ is equivalent to $\mathbf{h}$ and belongs to $\mathcal{S}$. Hence $\mathcal{S}$ contains a fundamental domain for $\mathcal{L}$. For $\mathbf{u} \in \mathbf{R}_+^V$, the set $\mathcal{S} + \mathbf{u}$ contains a fundamental domain for $\mathcal{L}$ because this property is translation-invariant. Hence $\mathcal{A}(\mathcal{S} + \mathbf{u}) = \mathcal{C}_\mathbf{u}(\mathcal{S})$ contains a fundamental domain for $\mathcal{L}$. The following lemma completes the proof.

Lemma 2.10. *$\mathcal{C}_\mathbf{u}(\mathcal{S})$ does not depend on $\mathbf{u}$ and is a fundamental domain for $\mathcal{L}$ when*

$$u_i \geq \Delta_{ii}, \text{ for all } i. \tag{4}$$

Proof. It is enough to show that, if (4) holds, $\mathcal{A}\mathbf{h} = \mathcal{A}\mathbf{h}'$, for every configuration $\mathbf{h} \in \mathcal{S}+\mathbf{u}$, and for any configuration $\mathbf{h}'$ with large enough components equivalent to $\mathbf{h}$ modulo $\mathcal{L}$.

Due to the condition (iii)$'$ of the theorem 1.5, (4) implies that $\mathbf{h} - \mathcal{A}\mathbf{h}' = \sum n_i \delta_i$ with non-negative integer n_i. Because the components of $\mathbf{h}$ and $\mathcal{A}\mathbf{h}'$ and the values of n_i remain bounded, while the components of $\mathbf{h}'$ are arbitrarily large, we can suppose that there exists a sequence of legal topples, with n_i topples at a site i, starting at $\mathbf{h}'' = \mathbf{h}' + \mathbf{h} - \mathcal{A}\mathbf{h}'$ (this is the exact meaning of "large enough" components of $\mathbf{h}'$). This sequence of topples transforms $\mathbf{h}''$ into $\mathbf{h}'$. Hence $\mathcal{A}\mathbf{h}'' = \mathcal{A}\mathbf{h}'$. At the same time, an avalanche from $\mathbf{h}'$ to $\mathcal{A}\mathbf{h}'$ shifted by $\mathbf{h} - \mathcal{A}\mathbf{h}'$ (due to the condition (4), all components of $\mathbf{h} - \mathcal{A}\mathbf{h}'$ are non-negative) transforms $\mathbf{h}''$ into $\mathbf{h}$. Hence $\mathcal{A}\mathbf{h}'' = \mathcal{A}\mathbf{h}$. This implies $\mathcal{A}\mathbf{h}' = \mathcal{A}\mathbf{h}$.

Remark 2.11. The lemma 2.10 contains in fact a stronger statement than the theorem 2.9. In particular, the "configuration with arbitrarily large components" in the definition 2.7 can be replaced by the "configuration $\mathbf{g}$ with $g_i \geq \Delta_{ii}$."

Let $d\mathbf{h} = dh_1 \cdots dh_N$ be the uniform measure in $\mathbf{R}^V$. For a subset $D \subset \mathbf{R}^V$, the volume $\mathrm{Vol}(D)$ is defined as $\int_D d\mathbf{h}$. It follows from the theorem 2.9 and proposition 1.12 that $\mathrm{Vol}(\mathcal{R}) = \det(\Delta)$, because every fundamental domain for $\mathcal{L}$ has volume $|\det(\Delta)|$.

Theorem 2.12. (Sf. Dhar [D1].) *For a properly loaded AA model, the set $\mathcal{R}$, with the uniform measure $d\mathbf{h}$, is invariant under the dynamics of the model. Every trajectory starting outside $\mathcal{R}$ arrives in $\mathcal{R}$ after a finite time period, hence $\mathcal{R}$ is a global attractor.*

Proof. The loading applied to a recurrent configuration obviously produces a reachable configuration (we can shift the whole sequence of topples by the loading vector). Due to the definition 2.7, $\mathcal{A}\mathbf{h} \in \mathcal{R}$, for every reachable configuration $\mathbf{h}$. Hence $\mathcal{R}$ is invariant. The measure $d\mathbf{h}$ is obviously invariant under both loading and avalanche operators.

Let us show that $\mathcal{R}$ is a global attractor. If all the components of the loading vector $\mathbf{v}$ of the model are positive then $\mathbf{h}+\mathbf{v}t$ has arbitrarily large components, for any configuration $\mathbf{h}$, if t is large enough. It follows then from the theorem 2.5 that the trajectory starting at $\mathbf{h}$ arrives in $\mathcal{R}$. In case some components of $\mathbf{v}$ are zero, it follows from the definition 2.1 that, for t sufficiently large, there exists an avalanche starting at $\mathbf{h} + \mathbf{v}t$ and passing through a configuration with arbitrarily large components. This completes the proof.

Proposition 2.13. *For an AA model with a loading vector $\mathbf{v}$, let $\mathbf{r} = \Delta'^{-1}\mathbf{v}$ be the topple rate vector. If, for some $T > 0$, $T\mathbf{r} = \mathbf{n}$ is an integer vector, then every trajectory in $\mathcal{R}$ is periodic, with a period T, and a site i topples n_i times during a period T, for every periodic trajectory. Otherwise, every trajectory in $\mathcal{R}$ is quasiperiodic, with a rotation vector $\mathbf{r}$.*

Proof. If $T\mathbf{r} = \mathbf{n}$ then $\Delta'\mathbf{n} = \mathbf{v}T$. Hence $\mathbf{h} + \mathbf{v}T$ is equivalent to $\mathbf{h}$ modulo $\mathcal{L}$, for any configuration $\mathbf{h}$. If $\mathbf{h} \in \mathcal{R}$, due to the theorems 2.5 and 2.9, the stable configuration at the trajectory started from $\mathbf{h}$ after time T coincides with $\mathbf{h}$, hence T is a period. In general, due to the theorems 2.9 and 2.12, the dynamics of the AA model on $\mathcal{R}$ is equivalent to the flow on the n-torus $\mathbf{R}^V/\mathcal{L}$ with a constant rate $\mathbf{v}$ or, taking δ_i as a new basis, to the flow on the torus $\mathbf{R}^V/\mathbf{Z}^V$ with a constant rate $\mathbf{r}$.

Example 2.14. The model from the example 1.14, with any non-zero loading vector with non-negative components, is a properly loaded AA model. The set of recurrent configurations of this symmetric model is described in [D, MD] in terms of forbidden subconfigurations. See also [S] and the remark 3.13 below.

Example 2.15. The model from the example 1.15, with any non-zero loading vector with non-negative components, is a properly loaded AA model. In particular, the loading vector $(v, 0, \ldots, 0)$, corresponds in the original sandpile model to adding particles to the first site [LLT, LT, S]. However, adding particles to any other site produces a loading vector with a negative component. This leads to non-commuting load-avalanche operators and essentially more complicated dynamics [KNWZ, CFKKP].

Example 2.16. Let $N = 2$, and let

$$\Delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } a > 0,\ d > 0,\ b \leq 0,\ c \leq 0,\ ad > bc.$$

Then Δ is avalanche-finite, and $\mathrm{Vol}(\mathcal{R}) = ad - bc$.

If $-b \leq d$ and $-c \leq a$, i.e. the matrix Δ is weakly codissipative, then the set $\mathcal{R}$ consists of all stable configurations (h_1, h_2) $(0 \leq h_1 < a,\ 0 \leq h_2 < d)$ outside the open negative quadrant $\{h_1 < -c,\ h_2 < -b\}$. Otherwise, let $-b > d$ (the case $-c > a$ is similar). Let $k = [-b/d]$ be the integer part of $-b/d$, and

$$\Delta_1 = \begin{pmatrix} a + kc & b + kd \\ c & d \end{pmatrix}.$$

It will be shown in section 4 that the set $\mathcal{R}(\Delta)$ shifted by $(kc, 0)$ coincides with the set $\mathcal{R}(\Delta_1)$. Iterating if necessary this procedure, we eventually reduce the matrix Δ to a matrix with $-b \leq d$ and $-c \leq a$ and with the set of recurrent configuration coincident with $\mathcal{R}(\Delta)$, after an appropriate shift. In particular, $\mathcal{R}(\Delta)$ is either a rectangle with a vertex at (a, d) and the sides parallel to the coordinate axes, or a difference of such a rectangle and an open negative quadrant with a vertex inside it.

In particular, for $(a, b, c, d) = (8, -7, -5, 5)$, the set $\mathcal{R}(\Delta)$ consists of (h_1, h_2) with $5 \leq h_1 < 8,\ 4 \leq h_2 < 5$ or $7 \leq h_1 < 8,\ 2 \leq h_2 < 4$.

3. Recurrent configurations. We want to investigate the combinatorial structure of the set $\mathcal{R}$ of recurrent configurations for the AA model. We show that this set can be defined by subtracting certain open negative octants from the set $\mathcal{S}$ of all stable configurations. Our goal is to reduce as much as possible the number of the necessary octants. This would allow to estimate the combinatorial complexity of the set $\mathcal{R}$, and to reduce the number of operations necessary to check whether a given configuration is recurrent.

For two vectors $\mathbf{u}$ and $\mathbf{v}$, we write $\mathbf{u} > \mathbf{v}$ if $u_i > v_i$, for all i, $\mathbf{u} \geq \mathbf{v}$ if $u_i \geq v_i$, for all i, $\mathbf{u} \succ \mathbf{v}$ if $\mathbf{u} \geq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$. For a subset X of V, let $\mathbf{1}_X$ be the vector with the i-th component equal 1 when $i \in X$ and 0 otherwise.

Let $\mathbf{Q} = \mathrm{diag}(\Delta) = (\Delta_{11}, \ldots, \Delta_{NN})$. For an integer vector $\mathbf{n}$, let $\mathbf{Q_n} = \mathbf{Q} - \Delta'\mathbf{n} = \mathbf{Q} - \sum n_i \delta_i$ and let $\mathcal{V}_\mathbf{n} = \{\mathbf{h} < \mathbf{Q_n}\}$ be an open negative octant with a vertex at $\mathbf{Q_n}$. In particular, $\mathcal{V}_\mathbf{0} = \{\mathbf{h} < \mathbf{Q}\}$ coincides with the set of all stable configurations. For $\mathbf{n} = \mathbf{1}_X$, we write $\mathbf{Q}_X$ and $\mathcal{V}_X$ instead of $\mathbf{Q_n}$ and $\mathcal{V}_\mathbf{n}$.

Proposition 3.1. *If* $\mathbf{n}$ *has at least one positive component, the set* $\mathcal{V}_\mathbf{n}$ *contains no reachable configurations. In particular,*

$$\mathcal{R} = \mathcal{V}_\mathbf{0} \setminus \bigcup \mathcal{V}_\mathbf{n}, \tag{5}$$

the union taken over integer vectors $\mathbf{n}$ *with at least one positive component.*

Proof. If there exists a reachable configuration $\mathbf{g} \in \mathcal{V}_\mathbf{n}$, then all configurations in $\mathcal{V}_\mathbf{n}$ close to $\mathbf{Q_n}$ are reachable, because they can be obtained from $\mathbf{g}$ by non-negative loading. There exists a configuration $\mathbf{h}$ in $\mathcal{R}$ arbitrarily close to $\mathbf{Q}$. The configuration $\mathbf{h_n} = \mathbf{h} - \Delta'\mathbf{n}$ is equivalent to $\mathbf{h}$, belongs to $\mathcal{V}_\mathbf{n}$ and is close to $\mathbf{Q_n}$. Any avalanche starting at a configuration with large enough components that passes through $\mathbf{h_n}$ should terminate at $\mathbf{h} \in \mathcal{R}$. But this is possible only if all the components of $\mathbf{n}$ are non-positive.

We want to show that the set of the integer vectors $\mathbf{n}$ in (5) necessary to define all recurrent configurations can be significantly reduced.

Definition 3.2. Let Δ be a redistribution matrix, not necessarily avalanche-finite. Let $X \subseteq V$. For a configuration $\mathbf{h}$, a site i is called *X-stable* if

$$h_i < \Delta_{ii}, \text{ for } i \in X, \quad h_i < 2\Delta_{ii}, \text{ for } i \notin X. \tag{6}$$

A configuration is X-stable when all the sites are X-stable. In particular, the V-stable configurations are the ordinary stable configurations. A topple $\mathcal{T}_i(\mathbf{h})$ is *X-legal* if the site i is X-unstable. A sequence of consecutive X-legal topples is called an *X-avalanche* if it is either infinite or terminates at an X-stable configuration.

A site or a configuration is *marginally X-stable* if "$<$" can be replaced by "$\leq$" at both places in (6), and a topple at a site which is not marginally X-stable is called *marginally X-legal*. A sequence of consecutive marginally X-legal topples is called a *marginal X-avalanche* if it is either infinite or terminates at a marginally X-stable configuration.

The lemma 1.1 and theorem 1.2 can be reformulated for the (marginal) X-avalanches as follows.

Proposition 3.3. *Let $\mathbf{h}$ be an arbitrary configuration and let $\mathbf{m}$ be an integer vector with non-negative components m_i such that $\mathbf{g} = \mathbf{h} - \sum m_i \delta_i$ is a (marginally) X-stable configuration. For any finite sequence of consecutive (marginally) X-legal topples started at $\mathbf{h}$, with n_i topples at a site i, we have $m_i \geq n_i$.*

Every two (marginal) X-avalanches starting at the same configuration $\mathbf{h}$ are either both infinite or both finite. In the latter case, the number of topples at every site is the same for both avalanches and the two avalanches terminate at the same (marginally) X-stable configuration.

Obviously, for an avalanche-finite matrix, every (marginal) X-avalanche is finite. In the following, we consider only avalanche-finite matrices.

Definition 3.4. We define the *X-avalanche operator* $\mathbf{h} \mapsto \mathcal{A}_X \mathbf{h}$ where $\mathcal{A}_X \mathbf{h}$ is the unique X-stable configuration terminating any X-avalanche started at the configuration $\mathbf{h}$. The *marginal X-avalanche operator* $\mathbf{h} \mapsto \overline{\mathcal{A}}_X \mathbf{h}$ is defined by the unique marginally X-stable configuration terminating any marginal X-avalanche started at a configuration $\mathbf{h}$.

For $\mathbf{h} = \mathbf{Q}$, we have $\mathcal{A}_X \mathbf{Q} = \mathbf{Q} - \Delta' \mathbf{N}_X$ where $\mathbf{N}_X$ is a non-negative integer vector. It is easy to show that $\mathcal{A}_X \mathbf{Q} = \mathcal{A}_X \mathbf{Q}_X$, hence $\mathbf{N}_X \geq \mathbf{1}_X$. The vector $\mathbf{N}_X$ is called the *X-script*. For $X = \{i\}$, we write $\mathbf{N}_i$ instead of $\mathbf{N}_X$. The *marginal X-script* $\overline{\mathbf{N}}_X$ is defined by $\overline{\mathcal{A}}_X \mathbf{Q}_X = \mathbf{Q} - \Delta' \overline{\mathbf{N}}_X$. By definition, $\overline{\mathbf{N}}_X \geq \mathbf{1}_X$.

Lemma 3.5. *We have*

$$0 < (\Delta' \mathbf{N}_X)_i \leq \Delta_{ii}, \text{ for } i \in X, \quad -\Delta_{ii} < (\Delta' \mathbf{N}_X)_i \leq 0, \text{ for } i \notin X; \tag{7}$$

$$0 \leq (\Delta' \overline{\mathbf{N}}_X)_i \leq \Delta_{ii}, \text{ for } i \in X, \quad -\Delta_{ii} \leq (\Delta' \overline{\mathbf{N}}_X)_i \leq 0, \text{ for } i \notin X. \tag{8}$$

The statement follows easily from the definitions 3.2 and 3.4.

Remark 3.6. If the graph $\Gamma(\Delta)$ is strongly connected, i.e. from every site i there exists a directed path in $\Gamma(\Delta)$ to every site $j \neq i$, the marginal V-script $\overline{\mathbf{N}}_V$ coincides with the script $\mathbf{N}$ defined in [S].

Theorem 3.7. (Sf. Speer [S].) *A stable configuration is recurrent if and only if it does not belong to any octant $\mathcal{V}_{\mathbf{n}}$, for $\mathbf{0} \prec \mathbf{n} \leq \overline{\mathbf{N}}_V$.*

This theorem reduces the number of the negative octants $\mathcal{V}_{\mathbf{n}}$ required in the proposition 3.1 for the description of $\mathcal{R}$. The following stronger statement provides a further reduction.

Theorem 3.8. *For an integer vector $\mathbf{n}$, let $Z(\mathbf{n}) = \{i \in V : \ (\mathbf{Q}_{\mathbf{n}})_i < \Delta_{ii}\}$. Let $\mathcal{N}$ be the set of all integer vectors $\mathbf{n}$ such that $Z(\mathbf{n}) \neq \emptyset$, $\mathbf{N}_{Z(\mathbf{n})} \leq \mathbf{n} \leq \overline{\mathbf{N}}_V$, and $0 \leq (\mathbf{Q}_{\mathbf{n}})_i < 2\Delta_{ii}$, for all i. A stable configuration is recurrent if and only if, for any $\mathbf{n} \in \mathcal{N}$, it does not belong to $\mathcal{V}_{\mathbf{n}}$.*

Proof. Let $\mathbf{n}_0$ be an integer vector with at least one non-negative component. We want to find an integer vector $\mathbf{n} \in \mathcal{N}$ such that $\mathcal{V}_{\mathbf{n}_0} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{n}} \cap \mathcal{V}_{\mathbf{0}}$. The claim of the theorem 3.8 follows then from the proposition 3.1.

Let $\mathbf{P} = \mathbf{Q}_{\mathbf{n}_0}$. If $\mathbf{P}$ has a negative component P_i, then $\mathcal{V}_{\mathbf{n}_0} \cap \mathcal{V}_{\mathbf{0}} \subset \mathcal{V}_{\{i\}} \cap \mathcal{V}_{\mathbf{0}}$, so it is enough to consider the case when all the components of $\mathbf{P}$ are non-negative.

The set $Z(\mathbf{n}_0)$ is not empty, otherwise $\mathcal{V}_{\mathbf{n}_0}$ contains $\mathcal{V}_{\mathbf{0}}$ and there are no recurrent configurations at all. If $P_i \geq 2\Delta_{ii}$, for some i, then $\mathcal{V}_{\mathbf{n}_0} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{n}_0+\mathbf{1}_i} \cap \mathcal{V}_{\mathbf{0}}$. Applying the same arguments to $\mathbf{n}_0 + \mathbf{1}_i$, we can finally find an integer vector $\mathbf{n}_1$ such that $0 \leq (\mathbf{Q}_{\mathbf{n}_1})_i < 2\Delta_{ii}$, for all i, and $\mathcal{V}_{\mathbf{n}_0} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{n}_1} \cap \mathcal{V}_{\mathbf{0}}$ (the process cannot continue indefinitely because every step is a legal topple and the matrix Δ is avalanche-finite). Note that $Z(\mathbf{n}_1) \subseteq Z(\mathbf{n}_0)$. Due to the proposition 3.3, $\mathbf{n}_1 \geq \mathbf{N}_{Z(\mathbf{n}_1)}$. If $\mathbf{n}_1 \not\leq \overline{\mathbf{N}}_V$ then $\mathbf{n}_2 = \mathbf{n}_1 - \mathbf{N}_V$ has at least one non-negative component and $\mathcal{V}_{\mathbf{n}_1} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{n}_2} \cap \mathcal{V}_{\mathbf{0}}$. Again, $Z(\mathbf{n}_2) \subseteq Z(\mathbf{n}_1)$. If $(\mathbf{Q}_{\mathbf{n}_2})_i \geq 2\Delta_{ii}$, for some i, we can proceed as before and find a vector $\mathbf{n}_3$ such that $0 \leq (\mathbf{Q}_{\mathbf{n}_3})_i$, for all i, and $\mathcal{V}_{\mathbf{n}_2} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{n}_3} \cap \mathcal{V}_{\mathbf{0}}$, and so on.

Let us show that the sequence of vectors $\mathbf{n}_k$ cannot be infinite. First, the sequence $Z(\mathbf{n}_k)$ is non-increasing, and $Z = \cap Z(\mathbf{n}_k) \neq \emptyset$. Second, for any $i \in Z$, the i-th component of $\mathbf{n}_k$ decreases when we subtract $\overline{\mathbf{N}}_V$ and does not increase when we add $\mathbf{1}_j$, $j \notin Z$. Finally, for an avalanche-finite matrix there are only finitely many integer vectors $\mathbf{m}$ such that $\mathbf{Q}_{\mathbf{m}}$ is Z-stable and $\mathcal{V}_{\mathbf{n}_0} \cap \mathcal{V}_{\mathbf{0}} \subseteq \mathcal{V}_{\mathbf{m}}$. Hence the sequence $\{\mathbf{n}_k\}$ terminates at a vector $\mathbf{n}$ which necessarily belongs to $\mathcal{N}$.

Example 3.9. Let $N = 2$, and let the matrix Δ be defined as in the example 2.16.

If $-b < d$ and $-c < a$ then $\mathbf{N}_1 = (1,0)$, $\mathbf{N}_2 = (0,1)$, $\bar{\mathbf{N}}_V = (1,1)$, with $\mathbf{Q}_{(1,0)} = (0, d+b)$, $\mathbf{Q}_{(0,1)} = (a+c, 0)$, $\mathbf{Q}_{(1,1)} = (-c, -b)$.

If $kd \leq -b < (k+1)d$, for an integer $k \geq 1$, then $\mathbf{N}_1 = (1,k)$, $\mathbf{N}_2 = (0,1)$. The script $\overline{\mathbf{N}}_V$ can be found by the reduction procedure described in the example 2.16.

In particular, for $(a,b,c,d) = (8,-7,-5,5)$, we have $\mathbf{N}_1 = (1,1)$, $\mathbf{N}_2 = (0,1)$, $\bar{\mathbf{N}}_V = (2,3)$, with $\mathbf{Q}_{(1,1)} = (5,7)$, $\mathbf{Q}_{(0,1)} = (13,0)$, $\mathbf{Q}_{(2,3)} = (7,4)$. The set $\mathcal{N}$ includes also $(1,2)$ and $(2,2)$, with $\mathbf{Q}_{(1,2)} = (10,2)$ and $\mathbf{Q}_{(2,2)} = (2,9)$. Only three of the five vectors $\mathbf{n} \in \mathcal{N}$ are sufficient to define $\mathcal{R}$, those with $\mathbf{Q}_{\mathbf{n}} = (5,7)$, $(7,4)$, and $(10,2)$.

Example 3.10. Let $N = 3$, and

$$\Delta = \begin{pmatrix} 6 & -2 & -2 \\ -3 & 9 & -5 \\ -2 & -2 & 6 \end{pmatrix}.$$

The matrix Δ has positive dissipations, and $\mathrm{Vol}(\mathcal{R}) = \det(\Delta) = 160$. We have $\mathbf{N}_X = \mathbf{1}_X$, for all subsets $X \subseteq V$, except $\mathbf{N}_{\{1,1\}} = (1,1,1)$ and $\mathbf{N}_V = \overline{\mathbf{N}}_V = (2,1,2)$. The corresponding vertices $\mathbf{Q_n}$ are $(0,11,8)$, $(9,0,11)$, $(8,11,0)$, $(5,4,7)$, $(2,13,2)$, $(11,2,5)$, $(1,8,3)$. The set $\mathcal{N}$ contains, except these 7 vectors, $\mathbf{n} = (1,1,2)$, with $\mathbf{Q_n} = (7,6,1)$. It is easy to check that all 8 vertices $\mathbf{Q_n}$, $\mathbf{n} \in \mathcal{N}$, are necessary to define $\mathcal{R}$. In particular, the set $\mathcal{R}$ is different from a set of recurrent configurations for any weakly codissipative 3×3-matrix, where 7 vertices are always enough.

Theorem 3.11. *The set $\mathcal{N}$ defined in the theorem 3.8 coincides with the minimal set of integer vectors containing*

(a) *for every nonempty subset X of V, the script $\mathbf{N}_X$;*

(b) *for any vector $\mathbf{m} \in \mathcal{N}$ with $Z(\mathbf{m}) = X$, a vector $\mathbf{n} = \mathbf{m} + \mathbf{N}_Y$ when $\mathbf{n} \leq \overline{\mathbf{N}}_V$, $Z(\mathbf{n}) = Y$, and $0 \leq (\mathbf{Q_n})_i < 2\Delta_{ii}$, for all i.*

Proof. Let $\mathbf{n} \in \mathcal{N}$, i.e. $\mathbf{N}_{Z(\mathbf{n})} \leq \mathbf{n} \leq \overline{\mathbf{N}}_V$ and $0 \leq (\mathbf{Q_n})_i < 2\Delta_{ii}$, for all i. Let $\mathbf{P} = \mathbf{Q_n}$ and $Y = Z(\mathbf{n}) = \{i : P_i < \Delta_{ii}\}$. Due to the proposition 3.3, $\mathbf{n} \geq \mathbf{N}_Y$. If $\mathbf{n} = \mathbf{N}_Y$, the claim of the theorem is true. Otherwise, let $\mathbf{P}' = \mathbf{P} + \Delta' \mathbf{N}_Y$. Due to the lemma 3.5, $0 < P'_i < 2\Delta_{ii}$, for all i. Let $\mathbf{m} = \mathbf{n} - \mathbf{N}_Y$ and $X = \{i : P'_i < \Delta_{ii}\}$. Then $\mathbf{m} \succ 0$, $\mathbf{P}' = \mathbf{Q_m}$, and $X = Z(\mathbf{m})$. Due to the proposition 3.3, $\mathbf{m} \geq \mathbf{N}_X$. Hence $\mathbf{m} \in \mathcal{N}$ and the condition (b) of the theorem 3.11 is valid for $\mathbf{m}$ and $\mathbf{n}$.

Remark 3.12. The theorem 3.11 provides a simple algorithm to find a small (not always the smallest) set of conditions defining $\mathcal{R}$. First, we have to find the scripts $\mathbf{N}_X$, for $\emptyset \neq X \subset V$, and the script $\overline{\mathbf{N}}_V$. Then, we have to check all the scripts $\mathbf{N} = \mathbf{N}_X + \mathbf{N}_Y$, for $X \neq Y$, and add to the set those of them for which $\mathbf{N} \prec \overline{\mathbf{N}}_V$, $Z = Z(\mathbf{Q_N})$ is either X or Y, and $0 \leq (\mathbf{Q_N})_i < 2\Delta_{ii}$, for all i. For every new script $\mathbf{N}$ added at this step, we check the scripts $\mathbf{N}' = \mathbf{N} + \mathbf{N}_X,\ \emptyset \neq X \subset V$, and add to the set those of them for which $\mathbf{N}' \prec \overline{\mathbf{N}}_V,\ Z(\mathbf{Q}_{\mathbf{N}'}) = X$, and $0 \leq (\mathbf{Q}_{\mathbf{N}'})_i < 2\Delta_{ii}$, for all i, and so on, until there will be nothing to add.

Remark 3.13. If $\mathbf{n} = \mathbf{1}_X$, the set $\mathcal{S} \cap \mathcal{V}_{\mathbf{n}}$ coincides with the *stable forbidden subconfigurations* [D1]

$$\mathcal{F}_X = \{\mathbf{h} \in \mathcal{S},\ h_j < -\sum_{i \in X,\, i \neq j} \Delta_{ij},\ \text{for } j \in X\}. \tag{9}$$

For a weakly codissipative matrix, $\overline{\mathbf{N}}_V = (1, \ldots, 1)$ and the theorem 3.8 implies the description of $\mathcal{R}$ in terms of forbidden subconfigurations [D1, MD, S]. A direct combinatorial proof of this description is given in [G1]. It is essentially equivalent to the following nontrivial theorem of linear algebra (see [G1]).

Theorem 3.14. *For any matrix* Δ,

$$\det(\Delta) = \sum_{l \geq 0} (-1)^{|V|-|X_l|} \sum_{X_0 \subset X_1 \subset \ldots \subset X_l} \prod_{j \notin X_l} \Delta_{jj} \prod_{i=1}^{l} \prod_{j \in X_i \setminus X_{i-1}} \sum_{\nu \in X_i, \nu \neq j} \Delta_{\nu j}.$$

4. Matrix reduction. In this section, we show that the dynamics of the AA model with a non-reduced (see the definition 4.2 below) redistribution matrix Δ can be completely determined by the dynamics of a simpler AA model with a reduced redistribution matrix Θ, such that the rows of Θ are combinations of the rows of Δ with non-negative integer coefficients. This implies that each topple operator of the second model is a combination of the topple operators of the first model. At the same time, the sets of recurent configurations and the avalanche operators for both models in the stationary state are essentially the same, i.e. the topple operators in any recurrent avalanche of the original model appear necessarily in the combinations required to constitute the topple operators of the second model (this is why the second model is simpler — its avalanches are shorter).

Theorem 4.1. *Let* Δ *be an avalanche-finite redistribution matrix, and let* $K = (k_{ij})$ *be a matrix with non-negative integer elements with* $\det(K) = 1$. *Let* $\Theta = K\Delta$. *If* Θ *is also a redistribution matrix then*

(a) Θ *is avalanche-finite.*

(b) *The sets* $\mathcal{R}(\Delta)$ *and* $\mathcal{R}(\Theta)$ *of recurrent configurations for the AA models with matrices* Δ *and* Θ *satisfy* $\mathcal{R}(\Delta) - \operatorname{diag}(\Delta) = \mathcal{R}(\Theta) - \operatorname{diag}(\Theta)$.

(c) *For every avalanche for the AA model with the matrix* Δ *starting at* $\mathbf{h}$ *and terminating at* $\mathcal{A}_\Delta \mathbf{h} \in \mathcal{R}(\Delta)$, *with a script* $\mathbf{N}$, *there exists an avalanche for the AA model with the matrix* Θ *starting at* $\mathbf{h}' = \mathbf{h} + \operatorname{diag}(\Theta) - \operatorname{diag}(\Delta)$ *and terminating at* $\mathcal{A}_\Theta \mathbf{h}' = \mathcal{A}_\Delta \mathbf{h} + \operatorname{diag}(\Theta) - \operatorname{diag}(\Delta) \in \mathcal{R}(\Theta)$, *with a script* $\mathbf{n}$ *such that* $\mathbf{N} = K'\mathbf{n}$.

Proof. (a) Every row vector θ_i of the matrix Θ is a non-zero combination of vectors δ_j with non-negative coefficients,

$$\theta_i = \sum_j k_{ij} \delta_j. \tag{10}$$

As Δ is avalanche-finite, a non-zero combination of vectors δ_i with non-negative coefficients cannot belong to $\mathbf{R}^V_-$, Due to the property (ii)$'$ of the theorem 1.5. Hence the same property (ii)$'$ of the theorem 1.5 is valid for Θ.

(b) Due to (10) the set $\mathcal{R}(\Theta) - \operatorname{diag}(\Theta) + \operatorname{diag}(\Delta)$ coincides with $\mathcal{V}_0 \setminus \cup \mathcal{V}_{\mathbf{n}}$ where the union is taken over all non-zero integer combinations of vectors $\mathbf{k}_i = (k_{i1}, \ldots, k_{iN})$. This set contains $\mathcal{R}(\Delta)$ due to (5). As $\det(\Theta) = \det(\Delta)$, the two sets coincide.

(c) Let $\mathbf{h}' = \mathbf{h} + \operatorname{diag}(\Theta) - \operatorname{diag}(\Delta)$, and let $\mathbf{h}'' = \mathcal{A}_\Delta \mathbf{h} + \operatorname{diag}(\Theta) - \operatorname{diag}(\Delta)$. As $\mathcal{A}_\Delta \mathbf{h}$ is equivalent to $\mathbf{h}$ modulo the lattice $\mathcal{L}$ generated by the integer combinations of θ_i, we have $\mathbf{h}''$ is equivalent to $\mathbf{h}'$ modulo the same lattice $\mathcal{L}$ generated by the integer combinations of δ_i. (The two lattices coincide because $\det(K) = 1$.) As $\mathcal{A}_\Delta \mathbf{h} \in \mathcal{R}(\Delta)$, we have $\mathbf{h}'' \in \mathcal{R}(\Theta)$ due to (b). As $\mathcal{R}(\Theta)$ is a fundamental domain for the lattice $\mathcal{L}$, we have $\mathbf{h}'' = \mathcal{A}_\Theta \mathbf{h}'$. The formula for the scripts follows from (10).

Definition 4.2. Redistribution matrix Δ is *reduced* if $-\Delta_{ij} < \Delta_{jj}$, for all $i \neq j$.

We describe here two methods that allow to replace a model with a non-reduced redistribution matrix by an equivalent, in the sense of the theorem 4.1, model with a reduced rredistribution matrix. First, we use the results of the previous section to construct a reduced redistribution matrix from a given non-reduced matrix. Next, we introduce reduction operations on redistribution matrices, similar to the topple operations on configurations. Although these operations do not always commute, the reduced matrix terminating a sequence of consecutive reduction operations does not depend on the possible choice of the sequence, similarly to the Abelian property of the topple operations. At the same time, each reduction operation replaces a redistribution matrix by an equivalent one.

Lemma 4.3. *Let $\mathbf{N}_X$ be the X-script defined in 3.4. For any two subsets X and Y of V with $X \cap Y = \emptyset$, we have $\mathbf{N}_{X \cup Y} \geq \mathbf{N}_X + \mathbf{N}_Y$.*

Proof. By definition, every X-legal topple is also $(X \cup Y)$-legal. Hence every topple in the X-avalanche started at $\mathbf{Q}$ and terminated at $\mathcal{A}_X \mathbf{Q} = \mathbf{Q}_{\mathbf{N}_X}$ is $(X \cup Y)$-legal.

Let us now shift by $\mathbf{Q}_{\mathbf{N}_X} - \mathbf{Q}$ the Y-avalanche started at $\mathbf{Q}$ and terminated at $\mathcal{A}_Y \mathbf{Q}_Y = \mathbf{Q}_{\mathbf{N}_Y}$. Due to the lemma 3.5, all the topples in this sequence started at $\mathbf{Q}_{\mathbf{N}_X}$ and terminated at $\mathbf{P} = \mathbf{Q}_{\mathbf{N}_X} + \mathbf{Q}_{\mathbf{N}_Y} - \mathbf{Q} = \mathbf{Q}_{\mathbf{N}_X + \mathbf{N}_Y}$ are $X \cup Y$-legal. Hence there exists an $X \cup Y$-avalanche started at $\mathbf{Q}$ passing through $\mathbf{P}$, and $\mathbf{N}_{X \cup Y} \geq \mathbf{N}_X + \mathbf{N}_Y$.

Lemma 4.4. *For every $\mathbf{n} \geq \mathbf{0}$ with $(\mathbf{Q}_\mathbf{n})_i < 2\Delta_{ii}$, for all i, we have $\mathbf{n} = \sum k_i \mathbf{N}_i$, with non-negative integer k_i.*

Proof. Let $Z = Z(\mathbf{n})$. The case $\mathbf{n} = \mathbf{0}$ is trivial, and for $\mathbf{n} \succ \mathbf{0}$ the set Z is nonempty, due to the property (ii′) of the theorem 1.5. Let $i \in Z$. As $\mathbf{n}$ is Z-stable, we have $\mathbf{n} \geq \mathbf{N}_Z$ due to the proposition 3.3, hence $\mathbf{n} \geq \mathbf{N}_i$ due to the lemma 4.3. If $\mathbf{n} = \mathbf{N}_i$ there is nothing to prove. Otherwise, let $\mathbf{n}' = \mathbf{n} - \mathbf{N}_i \succ \mathbf{0}$ and $P = \mathbf{Q}_{\mathbf{n}'}$. From the lemma 3.5, $(\Delta' \mathbf{N}_i)_i \leq \Delta_{ii}$ and $(\Delta' \mathbf{N}_i)_j \leq 0$, for $j \neq i$. Hence $\mathbf{P}_i < 2\Delta_{ii}$, for all i, and the statement is reduced to the same statement for $\mathbf{n}' \prec \mathbf{n}$.

Theorem 4.5. *Let $\theta_i = \Delta' \mathbf{N}_i$ and let Θ be the matrix with θ_i as an i-th row. Then Θ is an avalanche-finite redistribution matrix with* $\det(\Theta) = \det(\Delta)$. *The matrices Δ and Θ satisfy the conditions of the theorem 4.1, for the matrix K with the rows $\mathbf{N}_i$. We have $\Theta = \Delta$ if and only if Δ is reduced.*

Proof. Due to the definition 3.4 and lemma 3.5, all components of θ_i are non-positive, except the i-th component which is positive. Hence Θ is a redistribution matrix. As Δ is avalanche-finite, a non-zero combination of vectors δ_i with non-negative coefficients cannot belong to $\mathbf{R}^V_-$, due to the property (ii)$'$ of the theorem 1.5. Hence the same property (ii)$'$ of the theorem 1.5 is valid for Θ.

As the row vectors of Θ are linear combinations of the row vectors of Δ, we have $\mathcal{L}(\Theta) \subseteq \mathcal{L}(\Delta)$. The opposite inclusion follows from the lemma 4.4. In particular, $\det(\Theta) = \det(\Delta)$. By definition, $\Theta = K\Delta$, hence $\det(K) = 1$.

Finally, $\Theta = \Delta$ means that $\mathbf{N}_i = \mathbf{1}_i$, for all i, which is equivalent to the condition of 4.2.

Now we want to investigate another method of matrix reduction, using the reduction operations.

Definition 4.6. For $i \neq j$, an operation $\mathcal{T}_{ij}$ replacing Δ_{ik} by $\Delta_{ik} + \Delta_{jk}$, for all k, is called a *reduction*. A reduction is *legal* if $-\Delta_{ij} \geq \Delta_{jj}$. A sequence of consecutive legal reductions is called a *total reduction* if it terminates at a reduced matrix.

Lemma 4.7. *For an avalanche-finite redistribution matrix Δ, the result of a legal reduction $\mathcal{T}_{ij}(\Delta)$ is an avalanche-finite redistribution matrix.*

Proof. Due to the property (ii) of the theorem 1.5, $-\Delta_{ij} \geq \Delta_{jj}$ implies $-\Delta_{ji} < \Delta_{ii}$, otherwise $\Delta(\mathbf{1}_i + \mathbf{1}_j)$ would have all non-positive components. Hence $\mathcal{T}_{ij}(\Delta)$ is a redistribution matrix, for a legal reduction $\mathcal{T}_{ij}$. As the property (iv) of the theorem 1.5 is valid for Δ, it is valid also for $\mathcal{T}_{ij}(\Delta)$. Hence $\mathcal{T}_{ij}(\Delta)$ is avalanche-finite.

Lemma 4.8. *For an avalanche-finite redistribution matrix, every sequence of consecutive legal reductions is finite.*

Proof. Due to the lemma 4.7, the result of any sequence of consecutive legal reductions is an avalanche-finite matrix.

Theorem 4.9. *Every two total reductions terminate at the same reduced matrix $\mathcal{T}\Delta$, the total reduction of Δ.*

Proof. The proof is based on a non-commutative variant of the "diamond lemma" [B]. The reduction operations $\mathcal{T}_{ij}$ defined for $j \neq i$ satisfy the following commutation relations.

$$\mathcal{T}_{ij}\mathcal{T}_{kl} = \mathcal{T}_{kl}\mathcal{T}_{ij}, \text{ for } j \neq k, l \neq i; \quad \mathcal{T}_{ki}\mathcal{T}_{ij} = \mathcal{T}_{ij}\mathcal{T}_{kj}\mathcal{T}_{ki}.$$

Suppose that there exists a matrix Δ with two different total reductions, i.e. with two sequences of consecutive legal reductions terminated at two different reduced matrices Θ' and Θ''. Let $\mathcal{T}_{ij}$ and $\mathcal{T}_{kl}$ be the first reductions in these two sequences.

We have $(k,l) \neq (j,i)$, otherwise we would have $-\Delta_{ij} \geq \Delta_{jj}$ and $-\Delta_{ji} \geq \Delta_{ii}$, which is impossible for an avalanche-finite redistribution matrix.

If $j \neq k$ and $l \neq i$, it is easy to check that the reduction $\mathcal{T}_{kl}$ is legal for $\mathcal{T}_{ij}\Delta$ and the reduction $\mathcal{T}_{ij}$ is legal for $\mathcal{T}_{kl}\Delta$. Let $\Delta' = \mathcal{T}_{kl}\mathcal{T}_{ij}\Delta = \mathcal{T}_{ij}\mathcal{T}_{kl}\Delta$. A sequence of consecutive legal reductions started at Δ' terminates at a reduced matrix which is different form at least one of the matrices Θ' and Θ''. Hence at least one of the two matrices $\mathcal{T}_{ij}\Delta$ and $\mathcal{T}_{kl}\Delta$ allows two different total reductions.

Finally, if $l = i, j \neq k$ then the operation $\mathcal{T}_{ki}$ is legal for $\mathcal{T}_{ij}\Delta$, the operation $\mathcal{T}_{kj}$ is legal for $\mathcal{T}_{ki}\Delta$, and the operation $\mathcal{T}_{ij}$ is legal for $\mathcal{T}_{kj}\mathcal{T}_{ki}\Delta$ Let $\Delta' = \mathcal{T}_{ki}\mathcal{T}_{ij}\Delta = \mathcal{T}_{ij}\mathcal{T}_{kj}\mathcal{T}_{ki}\Delta$. A sequence of consecutive legal reductions started at Δ' terminates at a reduced matrix which is different form at least one of the matrices Θ' and Θ''. Hence at least one of the two matrices $\mathcal{T}_{ij}\Delta$ and $\mathcal{T}_{ki}\Delta$ allows two different total reductions.

Continuing this process, we can find a sequence of consecutive legal reductions started at Δ such that the resulting matrix at any step allows two different total reductions. This sequence terminates at a reduced matrix, which has obviously the only one total reduction (itself). This brings us to a contradiction.

Remark 4.10. The matrix reduction procedure described in this section can be also considered as a game on matrices, with a legal reduction corresponding to a legal move.

The statement of the theorem 4.9 means that this game, for an avalanche-finite matrix, always terminates at the same state, the total reduction, independent of the possible choice of the legal moves. However, this is not a strongly convergent game, in the definition of [E], since the number of steps in the game depends on the choice of moves.

5. Marginally stable recurrent configurations. Let

$$\mathcal{R}_i = \overline{\mathcal{R}} \cap \{h_i = \Delta_{ii}\} \tag{11}$$

be the set of *marginally stable* configurations where the recurrent avalanches with a first topple at i start. Here $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$. Let $d\mathbf{h}_i = dh_1 \cdots dh_{i-1} dh_{i+1} \cdots dh_N$ be the uniform measure on $\mathcal{R}_i$. For a subset $D \subset \mathcal{R}_i$, we define $\mathrm{Vol}(D) = \int_D d\mathbf{h}_i$

Proposition 5.1. *Let D be an open domain in $\mathcal{R}_i$. For any quasiperiodic trajectory of an AA model, the mean (per unit time) frequency $p_i(D)$ of the intersections with D is equal to*

$$p_i(D) = v_i \mathrm{Vol}(D)/\mathrm{Vol}(\mathcal{R}) = v_i \mathrm{Vol}(D)/\det(\Delta). \tag{12}$$

In the periodic case, the same is true after averaging over all periodic trajectories.

Proof. The statement follows from the proposition 2.13, because the dynamics of the model is equivalent to the flow with the constant rate $\mathbf{v}$ in $\mathbf{R}^V$ modulo $\mathcal{L}$ and $(\mathbf{n}_i, \mathbf{v}) = v_i$ where $\mathbf{n}_i$ is the unit normal to $\mathcal{R}_i$.

Corollary 5.2. *For a quasiperiodic trajectory of an AA model (in the periodic case, for a randomly chosen periodic trajectory) the mean number of avalanches started at a site i is equal to*

$$p_i(\mathcal{R}_i) = v_i \mathrm{Vol}(\mathcal{R}_i)/\det(\Delta). \tag{13}$$

Corollary 5.3. *For a quasiperiodic trajectory of an AA model (in the periodic case, for a randomly chosen periodic trajectory) the mean (per avalanche) number of topples at a* *site j during avalanches started at a site i is equal to*

$$m_{ij} = \frac{\det(\Delta)(\Delta^{-1})_{ij}}{\mathrm{Vol}(\mathcal{R}_i)}, \tag{14}$$

independent of the loading rate $\mathbf{v}$.

Proof. From (12), the topple rate (per unit time) at a site j after an avalanche started at a site i is equal to

$$\frac{v_i}{\det(\Delta)} \int_{\mathcal{R}_i} n_j(\mathbf{h}) d\mathbf{h}_i \tag{15}$$

where $n_j(\mathbf{h})$ is the number of topples at a site j during an avalanche started at $\mathbf{h}$. Hence the mean number r_j of topples at a site j per unit time is equal to

$$r_j = \sum_i \frac{v_i}{\det(\Delta)} \int_{\mathcal{R}_i} n_j(\mathbf{h}) d\mathbf{h}_i \tag{16}$$

Due to the proposition 2.3, the topple rate vector $\mathbf{r} = \{r_j\}$ satisfies $\Delta'\mathbf{r} = \mathbf{v}$. Hence

$$\int_{\mathcal{R}_i} n_j(\mathbf{h}) d\mathbf{h}_i = \det(\Delta)(\Delta^{-1})_{ij}. \tag{17}$$

The value m_{ij} can be now computed as

$$\frac{1}{p_i(\mathcal{R}_i)} \frac{v_i}{\det(\Delta)} \int_{\mathcal{R}_i} n_j(\mathbf{h}) d\mathbf{h}_i = \frac{\det(\Delta)(\Delta^{-1})_{ij}}{\mathrm{Vol}(\mathcal{R}_i)}. \tag{18}$$

Theorem 5.4. *For a codissipative matrix Δ, i.e.*

$$\sum_{i \in V} \Delta_{ij} > 0, \text{ for all } j, \tag{19}$$

we have

$$\mathrm{Vol}(\mathcal{R}_i) = \det(\Delta(i)) = \det(\Delta)(\Delta^{-1})_{ii}, \tag{20}$$

the mean number of avalanches started at a site i per unit time is

$$p_i(\mathcal{R}_i) = v_i \det(\Delta(i)) / \det(\Delta) = v_i (\Delta^{-1})_{ii}, \tag{21}$$

and the mean (per avalanche) number of topples at a site j during avalanches started at a site i is

$$m_{ij} = (\Delta^{-1})_{ij}/(\Delta^{-1})_{ii}. \tag{22}$$

Here $\Delta(i)$ is Δ with i-th row and i-th column deleted.

Proof. If (19) holds, the values of $\mathbf{h}' = \{h_j,\ j \in V,\ j \neq i\}$ in $\mathcal{R}_i$ are defined, due to (9), by the same inequalities as the set of all recurrent configurations for an AA model on $V \setminus \{i\}$ with a matrix $\Delta(i)$. The claim of the theorem 5.4 follows now from the proposition 5.1 and corollaries 5.2 and 5.3.

Remark 5.5. If the matrix Δ is weakly codissipative but not codissipative, the volume of $\mathcal{R}_j$ is less than $\det(\Delta(j))$. Due to (9), for every subset $X \subset V \setminus \{j\}$ such that $\Delta_{jj} + \sum_{i \in X} \Delta_{ij} = 0$, configurations with $h_j = \Delta_{jj}$ and $h_\nu < -\Delta_{j\nu} - \sum_{i \in F} \Delta_{i\nu}$, for $\nu \in F$, do not belong to $\mathcal{R}_j$. It can be shown, however, that the formulas (20)-(22) are still valid for the following modification of the model.

We allow every site to topple at most once in an avalanche. At the end of an avalanche started at a site i, the value h_i can be still at the threshold level Δ_{ii}. In this case we immediately start a new avalanche at a site i, and so on until finally we arrive at a stable configuration.

For a weakly codissipative matrix, (20)-(22) are true if we count every avalanche with the multiplicity of the number of topples at its starting site.

6. Avalanches in AA and ASP models. In this section, we compare the distributions of avalanches in a discrete, stochastic ASP model and a continuous, deterministic AA model with the same redistribution matrix and loading rate. We show that these distributions are identical when the behavior of the AA model is quasiperiodic. In case of the periodic behavior, the same is true after averaging over all periodic trajectories.

Definition 6.1. Let Δ be an integer avalanche-finite redistribution matrix with indices in a set V, and let a non-negative loading vector $\mathbf{v}$ satisfy $\sum_i v_i = 1$. An *Abelian sandpile* (ASP) model (Dhar, [D1]) is defined as follows.

The time is discrete, and the heights $h_i,\ i \in V$, are integer. For a stable configuration ($h_i < \Delta_{ii}$, for all i) a time step consists of adding 1 to h_i at a site i chosen with the probability v_i. If this site remains stable, we proceed with the next time step, otherwise it topples according to (2) starting an avalanche which terminates at a stable configuration, and the process continues.

A configuration is called *recurrent* if it appears with a non-zero probability in the steady-state regime of the ASP model.

Remark 6.2. (a) Due to its physical origin, the matrix Δ is usually supposed to be weakly dissipative. This guarantees that Δ is avalanche-finite. Otherwise this condition plays no special role.

(b) Most of the previous papers on ASP models consider uniform loading (all v_i are equal). However, the arguments are usually valid for any properly loaded (definition 2.1) ASP model.

Definition 6.3. A subset $U \subset \mathbf{Z}^V$ is a $\mathbf{Z}$-*fundamental domain* for a lattice $\mathcal{L}$ in $\mathbf{Z}^V$ if, for every $\mathbf{h} \in \mathbf{Z}^V$, there exists precisely one configuration in U equivalent to $\mathbf{h}$ modulo $\mathcal{L}$.

Theorem 6.4. (Dhar, [D1].) *For a properly loaded ASP model with a redistribution matrix Δ, the set $\mathcal{Z}$ of recurrent configurations is a $\mathbf{Z}$-fundamental domain for the lattice $\mathcal{L}$ generated by the row vectors of Δ. The number $\#(\mathcal{Z})$ of configurations in $\mathcal{Z}$ is equal to* $\det(\Delta)$. The set $\mathcal{Z}$ is invariant under the dynamics of the model, and every configuration in $\mathcal{Z}$ is attended with equal probability.

Proposition 6.5. *The set $\mathcal{Z}$ coincides with the set of integer points in the set $\mathcal{R}$ of recurrent configurations of the AA model with the matrix Δ.*

Proof. The same arguments as in the proof of the lemma 2.10 show that an integer configuration is recurrent for the ASP model if and only if it belongs to $\mathcal{A}(\mathcal{S}_u \cap \mathbf{Z}^N)$ when

$u_i \geq \Delta_{ii}$, for all i. The same property identifies the integer points in $\mathcal{R}$.

Proposition 6.6. *For $i \in V$, the set $\mathcal{Z}_i$ of marginally stable recurrent configurations with $h_i = \Delta_{ii}$ of the ASP model starting avalanches at the site i coincides with the set of integer points in the set $\mathcal{R}_i$ of the marginally stable recurrent configurations for the AA model defined in (11). The number $\#(\mathcal{Z}_i)$ of configurations $\mathcal{Z}_i$ is equal to $\mathrm{Vol}(\mathcal{R}_i) = \int_{\mathcal{R}_i} d\mathbf{h}_i$.*

Proof. By definition, $\mathcal{Z}_i$ is the set of configurations $\mathbf{h}$ with $h_i = \Delta_{ii}$ such that $\mathbf{h} - \mathbf{1}_i \in \mathcal{Z}$. Due to (5) this coincides with the set of integer points in $\mathcal{R}_i = \overline{\mathcal{R}} \cap \{h_i = \Delta_{ii}\}$ for an integer matrix Δ. The same formula (5) guarantees that the number of integer points in $\mathcal{R}_i$ is equal to $\mathrm{Vol}(\mathcal{R}_i)$.

Proposition 6.7. *For a randomly chosen configuration in $\mathcal{Z}$, the probability p_i of initiation of an avalanche at a site i after one time step is equal to $\#(\mathcal{Z}_i)v_i/\det(\Delta)$. In particular, p_i is equal to the mean number of avalanches $p_i(\mathcal{R}_i)$ initiated at i defined in (13) for the AA model.*

Proof. The probability p_i is the product of the probability of the configuration to belong to $\mathcal{Z}_i$, equal to $\#(\mathcal{Z}_i)/\det(\Delta)$ due to the theorem 6.4, and the probability to choose the site i at a time step, equal to v_i. The second statement follows from the proposition 6.6.

Theorem 6.8. *For a non-negative integer vector $\mathbf{k}$ wit $k_i > 0$, let $\mathcal{R}_{i,\mathbf{k}} \subset \mathcal{R}_i$ be the set of configurations where avalanches for the AA model with the script $\mathbf{k}$ start, and let $\mathcal{Z}_{i,\mathbf{k}}$ be the corresponding set for the ASP model. Then $\mathcal{Z}_{i,\mathbf{k}} = \mathcal{R}_{i,\mathbf{k}} \cap \mathbf{Z}^N$ and $\#(\mathcal{Z}_{i,\mathbf{k}}) = \mathrm{Vol}(\mathcal{R}_{i,\mathbf{k}})$.*

The mean number per time step of avalanches for the ASP model with the script $\mathbf{k}$ started at the site i is equal to $\#(\mathcal{Z}_{i,\mathbf{k}})v_i/\det(\Delta)$, which coincides with the mean number per unit time of avalanches of the same type for any quasiperiodic trajectory (in the periodic case, for a randomly chosen periodic trajectory) of the AA model.

Proof. Let us show first that

$$\mathcal{R}_{i,\mathbf{k}} = (\mathcal{R} + \Delta'\mathbf{k}) \cap \mathcal{R}_i. \tag{23}$$

An avalanche started at any configuration $\mathbf{h} \in \mathcal{R}_i$ terminates at a (unique) recurrent configuration equivalent to $\mathbf{h}$. Hence an avalanche started at $\mathbf{h} \in (\mathcal{R}+\Delta'\mathbf{k})\cap\mathcal{R}_i$ terminates at $\mathbf{h} - \Delta'\mathbf{k} \in \mathcal{R}$, which is recurrent and equivalent to $\mathbf{h}$. This means that $\mathbf{h} \in \mathcal{R}_{i,\mathbf{k}}$. The opposite implication is trivial. The same arguments show that $\mathcal{Z}_{i,\mathbf{k}} = (\mathcal{Z}+\Delta'\mathbf{k})\cap\mathcal{Z}_i$. Due to the propositions 6.5 and 6.6, this coincides with $\mathcal{R}_{i,\mathbf{k}} \cap \mathbf{Z}^N$. Due to (5) the number of integer points in (23) coincides with its volume.

The mean number per time step of avalanches for the ASP model with the script $\mathbf{k}$ started at the site i is equal to the product of the probability that a configuration belongs to $\mathcal{Z}_{i,\mathbf{k}}$ which, due to the theorem 6.4, is equal to $\#(\mathcal{Z}_{i,\mathbf{k}})/\det(\Delta)$, and the probability to choose the site i at a time step, equal to v_i. This, according to (12), is equal to

$$p_i(\mathcal{R}_{i,\mathbf{k}}) = v_i \mathrm{Vol}(\mathcal{R}_{i,\mathbf{k}})/\det(\Delta). \tag{24}$$

Corollary 6.9. *The distribution of avalanche sizes for the ASP model with a redistribution matrix Δ and a loading rate vector $\mathbf{v}$ coincides with the distribution of avalanche sizes for any quasiperiodic trajectory of the AA model with the same redistribution matrix and loading rate vector or, in the periodic case, with the average of this distribution over all periodic trajectories of the AA model.*

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REFERENCES

[B] G. Bergman, Adv. Math., **29**, 178 (1978).

[BL] A. Bjorner and L. Lovasz, J. Alg. Combinatorics **1**, 305 (1992).

[BLS] A. Bjorner, L. Lovasz and P.W. Shor, Europ. J. Combinatorics **12**, 283 (1991).

[BTW] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. **A 38**, 364 (1988).

[BuK] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Amer. **57**, 341 (1967).

[Ca] J.M. Carlson, Phys. Rev. **A 44**, 6226 (1991).

[CaL] J. Carlson and J.S. Langer, Phys. Rev. Lett. **62**, 2632 (1989); Phys. Rev. **A 40**, 6470 (1989).

[CC] H.F. Chau and K.S. Cheng, Phys. Rev. **A 44**, 6233 (1991); Phys. Lett. **A 157**, 103 (1991); Phys. Rev. **A 46**, R2981 (1992); J. Math. Phys. **34**, 5109 (1993).

[CFKKP] A.B Cchabra, M.J. Feigenbaum, L.P. Kadanoff, A.J. Kolan, and I. Procaccia. Sandpiles, Avalanches and the statistical mechanics of non-equilibrium stationary states. Preprint, 1992.

[CO] K. Christensen and Z. Olami, Phys. Rev. **A 46**, 1829 (1992).

[Cr] M. Creutz, Nucl. Phys. B, Proc. Suppl. **20**, 758 (1991); Comput. Phys. **5**, 198 (1991).

[D1] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).

[D2] D. Dhar, Physica A **186**, 82 (1992).

[DM] D. Dhar and S.N. Majumdar, J. Phys. A: Math. Gen. **23**, 4333 (1990).

[DMan] D. Dhar and S.S. Manna, Phys. Rev. **E 49**, 2684 (1994).

[D-G] A. Diaz-Guilera, Phys. Rev. **A 45**, 8551 (1992).

[E] K. Eriksson, Strongly Convergent Games and Coxeter Groups. Dissertation, KTH, Stockholm 1993.

[FF] H.J.S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991).

[G1] A. Gabrielov, Physica A, **195**, 253 (1993)

[G2] A. Gabrielov, Avalanches, Sandpiles and Tutte Decomposition. In: L. Corwin, et al., Editors, The Gelfand Mathematical Seminars 1990-1992, p.19-26, Birkhauser 1993.

[GNK] A. Gabrielov, W.I. Newman and L. Knopoff, Lattice models of failure: sensitivity to the local dynamics, Preprint (1992).

[Go] E. Goles, Ann. Inst. Henri Poincare - Phys. Theor. **56**, 75 (1992).

[GrM] P. Grassberger and S.S. Manna, J. Phys. France **51**, 1077 (1990).

[KNWZ] L.P. Kadanoff, S.R. Nagel, L. Wu, and S. Zou, Phys. Rev. **A 39**, 6524 (1989)

[LLT] S.-C. Lee, N.Y. Liang, and W.-J. Tzeng, Phys. Rev. Lett. **67**, 1479 (1991).

[LKM] J. Lomnitz-Adler, L. Knopoff and G. Martinez-Mekler, Phys. Rev. **A 45**, 2211 (1992).

[LT] S.-C. Lee and W.-J. Tzeng, Phys. Rev. **A 45**, 1253 (1992).

[MD] S.N. Majumdar and D. Dhar, J. Phys. A: Math. Gen. **24**, L357 (1991). S.N. Majumdar and D. Dhar, Physica A **185**, 129 (1991).

[MT] M. Matsuzaki and H. Takayasu, J. Geophys. Res. **96**, 19925 (1991).

[N] H. Nakanishi, Phys. Rev. **A 41**, 7086 (1990); Phys. Rev. **A 43**, 6613 (1991).

[OFC] Z. Olami, H.J.S. Feder and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).

[PTZ] L. Pietronero, P. Tartaglia and Y.-C. Zhang, Physica A **173**, 22 (1991).

[S] E.R. Speer, J. Stat. Phys., **71**, 61 (1993)

[T] W.T. Tutte, Graph Theory, Addison-Wesley, 1984.

[WTM] K. Wiesenfeld, J. Theiler, B. McNamara, Phys. Rev. Lett. **65**, 949 (1990).

[Z] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).