Analytic continuation of eigenvalues of a quartic oscillator

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Abstract

We consider the Schrödinger operator on the real line with even quartic potential $x^4+\alpha x^2$ and study analytic continuation of eigenvalues, as functions of parameter α. We prove several properties of this analytic continuation conjectured by Bender, Wu, Loeffel and Martin. 1. All eigenvalues are given by branches of two multi-valued analytic functions, one for even eigenfunctions and one for odd ones. 2. The only singularities of these multi-valued functions in the complex α-plane are algebraic ramification points, and there are only finitely many singularities over each compact subset of the α-plane.

1. Introduction.

Consider the boundary value problem on the real line:

$$-y'' + (\beta x^4 + x^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0. \tag{1}$$

If $\beta > 0$, then this problem is self-adjoint, it has a discrete spectrum of the form $\lambda_0 < \lambda_1 < \ldots \to +\infty$, and every eigenspace is one-dimensional. The eigenvalues λ_n are real analytic functions of β defined on the positive ray.

In 1969, Bender and Wu [5] studied analytic continuation of λ_n to the complex β-plane. Their main discoveries are the following:

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(i) For every non-negative integers m and n of the same parity, the function λ_m can be obtained by an analytic continuation of the function λ_n along some path in the complex β-plane.

(ii) The only singularities encountered in the analytic continuation of λ_n in the punctured β-plane $\mathbf{C}\backslash\{0\}$ are algebraic ramification points.

(iii) These ramification points accumulate to $\beta = 0$, in such a way that no analytic continuation of any λ_n to 0 is possible.

The last statement gives a good reason why the formal perturbation series of λ_n in powers of β is divergent. Bender and Wu also studied the global structure of the Riemann surfaces of the functions λ_n spread over the β-plane, that is the position of their ramification points, and how the sheets of these Riemann surfaces are connected at these points.

Bender and Wu used a combination of mathematical and heuristic arguments with numerical computation. Since the publication of their paper, several results about analytic continuation of λ_n were proved rigorously. We state some of these results. A broader context in which this problem arises is described in the survey [34].

First of all, we recall a change of the variable which substantially simplifies the problem [32]. Consider the family of differential equations

$$H(\alpha,\beta)y = \lambda y, \quad \text{where} \quad H(\alpha,\beta) = -d^2/dx^2 + (\beta x^4 + \alpha x^2).$$

The change of the independent variable

$$w(x) = y(tx) \tag{2}$$

gives the differential equation $H(t^4\alpha, t^6\beta)w = t^2\lambda w$. Thus, if α and t are real, and $\beta > 0$, we have

$$\lambda_n(t^4\alpha, t^6\beta) = t^2\lambda_n(\alpha,\beta).$$

If $\alpha = 1$ we can take $t = \beta^{-1/6} > 0$ and obtain

$$\lambda_n(1,\beta) = \beta^{1/3}\lambda_n(\beta^{-2/3}, 1). \tag{3}$$

This reduces our problem to the study of the analytic continuation of eigenvalues of the one-parametric family

$$-y'' + (x^4 + \alpha x^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0, \tag{4}$$

depending on the complex parameter α. The study of this family of quartic oscillators is equivalent to the study of the family (1). Indeed, if we know an analytic continuation of $\lambda_n(\alpha, 1)$ along some curve in the α-plane, then equation (3) gives an analytic continuation of $\lambda_n(1, \beta)$ in the β-plane and vice versa. The main advantage of restating the problem in the form (4) is that any analytic continuation of an eigenvalue (eigenfunction) of (4) remains an eigenvalue (eigenfunction) of (4). This is not so for the operator in (1): when we perform an analytic continuation of an eigenfunction the result may no longer be an eigenfunction, because it may fail to satisfy the boundary condition, see [32], [4].

From now on we consider only the family (4), and slightly change our notation: the λ_n will be real analytic functions of $\alpha > 0$ representing the eigenvalues of (4).

Notice that the λ_n have immediate analytic continuations[1] from the positive ray to the whole real line.

Loeffel and Martin [24] proved that the functions λ_n have immediate analytic continuations to the sector $|\arg \alpha| < 2\pi/3$. They conjectured that the radius of convergence of the power series of λ_n at $\alpha = 0$ tends to infinity as $n \to \infty$.

Simon [32, 33] proved that the singularities of the λ_n accumulate to ∞ in the asymptotic direction of the negative ray. More precisely, for every n and for every $\eta \in (0, \pi)$, there exists $B = B_n(\eta)$ such that λ_n has an immediate analytic continuation from the positive ray to the region $\{\alpha : |\arg \alpha| < \eta, |\alpha| > B\}$. On the other hand, he also proved that the λ_n do not have immediate analytic continuations to full punctured neighborhoods of ∞. This proves statement (iii) of Bender and Wu and implies divergence of the perturbation series for λ_n at $\beta = 0$.

Delabaere, Dillinger and Pham in their interesting papers [7, 8, 9] used a version of the WKB method to study operator (4) for large α. They claim to confirm all conclusions of Bender and Wu, however it is not clear to us what is proved rigorously in [7, 8, 9], which statements are heuristic and which are verified numerically. In particular, we could not determine whether these papers contain a complete proof of the statement (i).

[1] We say that a function analytic on a set $X \subset \overline{\mathbf{C}}$ has an immediate analytic continuation to a set $Y \subset \overline{\mathbf{C}}$ if $X \cap Y$ has limit points in X and there exists an analytic function g on Y such that $f(z) = g(z)$ for $z \in X \cap Y$.

Another study of λ_n for large α is [20]. It is not clear whether statements of Theorem 1 can be derived from the results of [20].

In this paper we give complete proofs of statements (i) and (ii) of Bender and Wu and of the conjecture of Loeffel and Martin. Our methods are different from those of all papers mentioned above.

Theorem 1.
a) *All λ_n are branches of two multi-valued analytic functions Λ^i, $i = 0, 1$, of α, one for even n another for odd n.*

b) *The only singularities of Λ^i over the α-plane are algebraic ramification points.*

c) *For every bounded set X in the α-plane, there are only finitely many ramification points of Λ^i over X.*

Statements a) and b) prove (i) and (ii) of Bender and Wu. Statement c) implies the conjecture of Loeffel and Martin stated above.

Statements b) and c) actually hold in greater generality. Let $P(a, z) = z^d + a_{d-1}z^{d-1} + \ldots + a_1 z$ be any monic polynomial of even degree d, with complex coefficients $a = (a_1, \ldots, a_{d-1}) \in \mathbf{C}^{d-1}$. Consider the boundary value problem

$$-y'' + P(a, z)y = \lambda y, \tag{5}$$

$$y(+\infty) = y(-\infty) = 0, \tag{6}$$

where the boundary condition is imposed on the real axis. For non-real a, this problem is not self-adjoint, however it is known [31] that the spectrum of this problem is infinite and discrete, eigenspaces are one-dimensional, and the eigenvalues tend to infinity in the asymptotic direction of the positive ray. Let Λ be the multi-valued function of $a \in \mathbf{C}^{d-1}$ which to every a puts into correspondence the set of eigenvalues of (5), (6). We write the set $\Lambda(a)$ as $\Lambda(a) = \{\mu_0(a), \mu_1(a), \ldots\}$ where

$$|\mu_0(a)| \leq |\mu_1(a)| \leq \ldots \to +\infty.$$

So for real a we have $\mu_n(a) = \lambda_n(a)$ but this does not have to be the case for complex a. (The functions μ_n are not expected to be analytic; they are only piecewise analytic). It is known [31] that there exists an entire function F of d variables with the property that λ is an eigenvalue of the problem (5), (6) if and only if

$$F(a, \lambda) = 0. \tag{7}$$

Equation (7) may be called the characteristic equation of the problem (5), (6). It is the equation implicitly defining our multi-valued function Λ.

Theorem 2. *The only singularities of Λ are algebraic ramification points. For every $R > 0$ there exist a positive integer N such that for $n > N$ the μ_n are single-valued analytic functions on the set $\{a : |a| < R\}$ with disjoint graphs.*

Here are some general properties of implicit functions $\lambda(a)$ defined by equations of the form (7) with arbitrary entire function F. Let g_0 be an analytic germ of such function at some point a_0, and $\gamma_0 : [0,1] \to \mathbf{C}^{d-1}$ a curve in the a-space beginning at a_0. Then for every $\epsilon > 0$ there is a curve γ beginning at a_0, satisfying $|\gamma(t) - \gamma_0(t)| < \epsilon,\ t \in [0,1]$, and such that an analytic continuation of g_0 along γ is possible. In other words, "the set of singularities" of Λ is totally discontinuous. This is called the *Iversen property*, and it was proved by Julia [21], see also [35]. However, the "set of singularities" of Λ in general can have non-isolated points, as can be shown by examples, [15].

Theorem 2 is in fact an easy consequence of the following known result.

Theorem A. *For every $R > 0$ there exists a positive integer N such that for all a in the ball $|a| \le R$ we have the strict inequalities $|\mu_{n+1}(a)| > |\mu_n(a)|$ for all $n \ge N$.*

For a complete proof of this result we refer to Shin [28, Thm 1.7] who used his earlier paper [29] and the results of Sibuya [31]. Theorem A is derived from the asymptotic expansion for the eigenvalues μ_n in powers of n which is uniform with respect to a for $|a| \le R$ [28, Theorem 1.2]. Similar asymptotic formula for the eigenvalue problem (5), (6) is also given in [17, Ch. III, §6] where it is derived with a different method.

To deduce Theorem 2 from Theorem A, we also need the Weierstrass Preparation Theorem [6, 19]:

Theorem B. *Let $Z \subset \mathbf{C}^m$ be the set of solutions of the equation (7), and $(a_0, \lambda_0) \in Z$. Suppose that $F(a_0, \lambda) \not\equiv 0$. Then there is a neighborhood V of (a_0, λ_0) such that in V we have*

$$F(a,\lambda) = \Big((\lambda - \lambda_0)^k + F_{k-1}(a)(\lambda - \lambda_0)^{k-1} + \ldots + F_0(a)\Big)\, G(a,\lambda),$$

where F_j and G are analytic functions in V, and $G(a_0, \lambda_0) \neq 0$, and $F_j(a_0) = 0$ for $0 \le j \le k-1$.

So for each a close to a_0 the equation $F(a,\lambda)=0$ with respect to λ has k roots close to λ_0, and these roots tend to λ_0 as $a \to a_0$.

Proof of Theorem 2. Application of Theorem A shows that for every $R>0$ there exists N such that for $n>N$ the functions μ_n have disjoint graphs over $\{a : |a|<R\}$. Application of Theorem B to the solutions $\mu_n(a)$ of the equation $F(a,\mu_n(a))=0$ with $n>N$ shows that $k=1$ for all points $(a,\mu_n(a_0))$, and then the implicit function theorem implies that the μ_n are analytic for $n>N$. This proves the second part of Theorem 2. To prove the first part, consider a curve $\gamma : [0,1] \to \{a : |a| \le R\} \subset \mathbf{C}^{d-1}$ such that Λ has an analytic continuation g_t, $0 \le t < 1$. Here g_t is an analytic germ of Λ at the point $\gamma(t)$. Suppose that $g_0(\gamma(0)) = \mu_j(\gamma(0))$. By Theorem A there exists $N>j$ such that $|g_t(\gamma(t))| < |\mu_N(\gamma(t))|$ for all $t \in [0,1)$. As $\mu_N(a)$ is bounded for $|a| \le R$, we conclude that $g_t(\gamma(t))$ is a bounded function on $[0,1)$, and there exists a sequence $t_k \to 1$ such that $g_{t_k}(\gamma(t_k))$ has a finite limit λ_1. Application of the Weierstrass Preparation theorem to the point $(\gamma(1),\lambda_1)$ shows that in fact $g_t(\gamma(t)) \to \lambda_1$ as $t \to 1$, and g_t either has an analytic continuation to the point $\gamma(1)$ along γ or $\gamma(1)$ is a ramification point of some order k. This completes the proof.

An alternative proof can be given by using perturbation theory of linear operators instead of the Weierstrass Preparation theorem as it is done in [32]. However we notice that an analog of Theorem 2 does not hold for general linear operators analytically dependent of parameters, as examples in [22, p. 371-372] show. The crucial property of our operators (5), (6) is expressed by Theorem A.

Theorem 2 implies statements b) and c) of Theorem 1.

In the rest of the paper we prove statement a). We briefly describe the idea of the proof. Equation (7) which we now write as

$$F(\alpha,\lambda)=0, \tag{8}$$

defines an analytic set $Z \subset \mathbf{C}^2$ which consists of all pairs (α,λ) for which the problem (4) has a solution. We are going to show that this set Z consists of exactly two irreducible components, which are also its connected components. To do this we introduce a special parametrization of the set Z by a (not connected) Riemann surface G. As this parametrization $\Phi : G \to Z$ comes from the work of Nevanlinna [26], we call it the *Nevanlinna parametrization*.

To study the Riemann surface G we introduce a function $W : G \to \overline{\mathbf{C}}$, which has the property that it is unramified over $\overline{\mathbf{C}} \backslash \{0,1,-1,\infty\}$. More

precisely, this means that

$$W : G \backslash W^{-1}(\{0, 1, -1, \infty\}) \to \overline{\mathbf{C}} \backslash \{0, 1, -1, \infty\}$$

is a covering map. Then we study the monodromy action on the generic fiber of this map W. Using a description of G and W which goes back to Nevanlinna, we label the elements of the fiber by certain combinatorial objects (cell decompositions of the plane) and explicitly describe the monodromy action on this set of cell decompositions. Our explicit description shows that there are exactly two equivalence classes of this action, thus the Riemann surface G consists of exactly two components.

In fact we not only prove that G consists of two components but give in some sense a global topological description of the surface G, and thus of the set Z.

The plan of the paper is the following. In Sections 2 we collect all necessary preliminaries and construct G, Φ and W. In Sections 3 and 4 we discuss the cell decompositions of the plane needed in the study of the monodromy of the map $W : G \to \overline{\mathbf{C}}$. In Section 5 we compute this monodromy and complete the proof of statement a) in Theorem 1. In section 6 we briefly mention several other one-parametric families of linear differential operators with polynomial potentials which can be treated with the same method.

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2. Preliminaries.

Some parts of our construction apply to the general problem (5), (6) so we explain them for this general case. We include more detail than it is strictly necessary for our purposes because the papers [26] and [25] are less known nowadays than they deserve.

First we recall some properties of solutions of the differential equation (5). The proofs of all these properties can be found in Sibuya's book [31]. Every solution of this differential equation is an entire function of order $(d+2)/2$, where d is the degree of P. To avoid trivial exceptional cases, we always assume that $d > 0$. We set $q = d + 2$ and divide the plane into q disjoint open sectors

$$S_j = \{z : |\arg z - 2\pi j/q| < \pi/q\}, \quad j = 0, \ldots, q-1.$$

In what follows we will always understand the subscript j as a residue modulo

q, so that, for example, $S_q = S_0$ etc. We call S_j the *Stokes sectors* of the equation (5).

1. For each solution $y \neq 0$ of the equation (5) and each sector S_j we have either $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in S_j. We say that y is *subdominant* in S_j in the first case and *dominant* in S_j the second case.

2. Of any two linearly independent solutions of (5), at most one can be subdominant in a given Stokes sector.

Let y_1 and y_2 be two linearly independent solutions, and consider their ratio $f = y_2/y_1$. Then f is a meromorphic function of order $q/2$. (The order of a meromorphic function f can be defined as the minimal number ρ such that f is a ratio of two entire functions of order at most ρ.)

3. For each S_j, we have $f(z) \to w_j \in \overline{\mathbf{C}}$ as $z \to \infty$ along any ray in S_j starting at the origin.

4. $w_j \neq w_{j+1}$ for all $j \bmod q$.

5. $w_j \in \{0, \infty\}$ if and only if one of the solutions y_1, y_2 is dominant and another is subdominant in S_j.

A curve $\gamma : [0, 1) \to \mathbf{C}$ is called an *asymptotic curve* of a meromorphic function f if $\gamma(t) \to \infty$ as $t \to 1$, and $f(\gamma(t))$ has a limit, finite or infinite, as $t \to 1$. This limit is called an *asymptotic value* of f. A classical theorem of Hurwitz says that the singularities of the inverse function f^{-1} are exactly the critical values and the asymptotic values of f.

Returning to the function $f = y_2/y_1$, where y_1 and y_2 are linearly independent solutions of equation (5), we notice that f does not have critical points. Indeed, all poles of f are simple because y_1 can have only simple zeros, and $f'(z) \neq 0$ because the Wronskian determinant of y_1, y_2 is constant. Thus f has no critical values, and the only singularities of f^{-1} are the asymptotic values of f.

Next we describe these asymptotic values and associated asymptotic curves. By property 3 above, for each sector S_j, every ray from the origin in S_j is an asymptotic curve. Thus all w_j are asymptotic values. Function f has no other asymptotic values except the w_j.

By Hurwitz theorem we conclude that the only singularities of f^{-1} lie

over the points w_j. More precisely,

$$f : \mathbf{C} \backslash f^{-1}(\{w_0, \ldots, w_{q-1}\}) \to \overline{\mathbf{C}} \backslash \{w_0, \ldots, w_{q-1}\} \tag{9}$$

is an unramified covering.

Let D_j be discs centered at w_j and having disjoint closures. Each component B of the preimage $f^{-1}(D_j)$ is either a topological disc in the plane which is mapped by f onto D_j homeomorphically, or an unbounded domain such that $f : B \to D_j \backslash \{w_j\}$ is a universal covering. Such unbounded domains are called *tracts* over w_j.

6. The tracts are in bijective correspondence with the sectors S_j. More precisely, for each j there exists a unique tract B_j which contains each ray from the origin in S_j, except a bounded subset of this ray, and the total number of tracts is q.

The *Schwarzian derivative* of a function f is

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The main fact about S_f that we need is its relation with equation (5):

7. A ratio $f = y_2/y_1$ of two linearly independent solutions of (5) satisfies the differential equation

$$S_f = -2(P - \lambda), \tag{10}$$

and conversely, every non-zero solution of the differential equation (10) is a ratio of two linearly independent solutions of (5).

Let $Z_d \subset \mathbf{C}^d$ be the set of all pairs (a, λ) such that λ is an eigenvalue of the problem (5), (6). Consider the class G_d of meromorphic functions with the following two properties:

$$-\frac{1}{2}S_f \quad \text{is a monic polynomial of degree } d \tag{11}$$

and

$$f(z) \to 0, \quad z \in \mathbf{R}, \quad z \to \pm\infty. \tag{12}$$

The set G_d is equipped with the usual topology of uniform convergence on compact subsets of $\mathbf{C}$ with respect to the spherical metric in the target.

Now we define a map $\Phi : G_d \to \mathbf{C}^d$ by $\Phi(f) = (a_1, \ldots, a_{d-1}, \lambda)$, where $-\lambda, a_1, \ldots, a_{d-1}$ are the coefficients of the polynomial $-(1/2)S_f$. This map is evidently continuous.

Proposition 1. *The map Φ sends G_d to Z_d surjectively.*

Proof. First we prove that the image of Φ is contained in Z_d. Let f be an element of G_d. By property 7 above, $f = y/y_1$, a ratio of two linearly independent solutions of (5). By (12) and property 5 above, y should be subdominant in S_0 and $S_{q/2}$. So y satisfies the boundary condition (6) and thus $\Phi(f) = (a, \lambda)$ is an element of Z_d.

Now we prove that Φ maps G_d to Z_d surjectively. Let $(a, \lambda) \in Z_d$ and let y be the corresponding eigenfunction. Let y_1 be any solution of the equation (5) which is linearly independent of y. Then $f = y/y_1$ satisfies the differential equation (10), thus (11) holds. Now in view of the boundary condition (6), y is subdominant in S_0 and $S_{q/2}$, so by properties 1 and 2 above y_1 must be dominant in S_0 and $S_{q/2}$. So $f = y/y_1$ satisfies (12). Thus $f \in G_d$, and (10) gives $(a, \lambda) = \Phi(f)$.

Proposition 2. *A meromorphic function g satisfies $g(z) = f(cz)$ for some $f \in G_d$ and $c \in \mathbf{C}^*$ if and only if it has the following three properties:*

(i) *g has no critical points,*

(ii) *g has $q = d + 2$ tracts,*

(iii) *There is a tract B_0 such that if the tracts are ordered counterclockwise as $B_0, B_1, \ldots, B_{q-1}$, and if w_j are the asymptotic values of g in B_j then $w_0 = w_{q/2} = 0$.*

The sufficiency of these conditions is a deep result of R. Nevanlinna [26]. The proof becomes much simpler if one adds the condition

(iv) *g is a meromorphic function of finite order.*

This additional condition will be easy to verify in our setting and we sketch a simpler proof of Proposition 2, using the condition (iv). This proof is based on F. Nevanlinna's work [25].

Proof of Proposition 2 with condition (iv). The necessity of conditions (i)-(iv) has already been established.

Now we prove sufficiency. Condition (i) implies that S_g is an entire function (in general, Schwarzian derivative of a meromorphic function has poles

exactly at its critical points). Condition (iv) combined with the Lemma on the logarithmic derivative [27, 18] gives a growth estimate of S_g which implies that S_g is a polynomial. Now by property 7 above, $g = y/y_1$, where y, y_1 are two linearly independent solutions of the differential equation

$$y'' + \frac{1}{2} S_g y = 0.$$

Property 6 above shows that g has $\deg S_g + 2$ tracts, so we conclude from (ii) that $\deg S_g = d$. Now we can find $c \in \mathbf{C}^*$ so that for $f(z) = g(z/c)$ the polynomial $(-1/2)S_f$ is monic. Such c is defined up to multiplication by a q-th root of unity. Using (iii), we choose this root of unity in such a way that (12) is satisfied.

Now we define a map $W : G_d \to \overline{\mathbf{C}}^q$, $W(f) = (w_0, \ldots, w_{q-1})$, whose image is evidently contained in the subspace H of codimension 2 given by the equations $w_0 = w_{q/2} = 0$. This map is known to be a local homeomorphism into H [2], and its image can be described precisely using a result of R. Nevanlinna [26]. We do not use these results in our paper. We only remark that the local homeomorphism W permits to define a structure of a complex analytic manifold of dimension d on G_d, so that W becomes holomorphic. The map Φ we introduced earlier is also holomorphic with respect to this analytic structure.

Proposition 3. *Let f_0 be an element of G_d, and $\gamma : [0,1] \to H$,*

$$\gamma(t) = (w_0(t), \ldots, w_{q-1}(t)), \quad w_0 \equiv w_{q/2} \equiv 0,$$

be a path with the properties $\gamma(0) = W(f_0)$, and

$$w_j(0) \neq w_k(0) \quad \Longleftrightarrow \quad w_j(t) \neq w_k(t)$$

for all $j \neq k$ and all $t \in [0,1]$. Then there is a lift of the path γ to G_d, that is a continuous family $f_t \in G_d$, $t \in [0,1]$ such that $W(f_t) = \gamma(t)$.

Proof. There is a continuous family of diffeomorphisms $\psi_t : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, $\psi_0 = \mathrm{id}$, such that $\psi_t(w_j(0)) = w_j(t)$, $0 \leq j \leq q-1$. These diffeomorphisms are quasiconformal [1]. Then the Fundamental existence theorem for quasiconformal maps [1, Chap. V] implies the existence of a continuous family of quasiconformal maps ϕ_t, $\phi_0 = \mathrm{id}$, such that $g_t = \psi_t \circ f_0 \circ \phi_t$ are meromorphic functions. These meromorphic functions evidently have no critical points, because g_0 does not. They have the same number of tracts as g_0 and their tracts

satisfy the condition (iii) of Proposition 2. Thus all conditions of Proposition 2 are satisfied. We can also check the additional condition (iv): it follows from the general property of quasiconformal mappings $|\phi_t(z)| \le |z|^C, |z| > r_0$, where C is a constant. Thus by Proposition 2, $g_t = f_t(c_t z)$, $f_t \in G_d$, where the constants c_t are determined from the condition that the polynomials $-(1/2)S_{f_t}$ are monic. Evidently the correspondence $t \mapsto c_t$ is continuous, and we have $W(f_t) = \gamma(t)$ by construction.

Centrally symmetric case.

Suppose now that the polynomial P in (5) is even. We write it as $P(a, z) = z^d + a_{d-2}z^{d-2} + \ldots + a_2 z^2$ and consider the set Z_d^e of all pairs $(a, \lambda) \in \mathbf{C}^{d/2}$ such that λ is an eigenvalue of the problem (5), (6). Then each eigenfunction y of the problem (5), (6) is either even or odd. Indeed, $y(-z)$ is also an eigenfunction with the same eigenvalue, so $y(z) = cy(-z)$ because the eigenspace is one-dimensional. Putting $z = 0$ we obtain that either $c = 1$ (so the eigenfunction is even) or $y(0) = 0$. In the latter case, differentiate to obtain $y'(z) = -cy'(-z)$ and put $z = 0$ to conclude that $c = -1$, so the eigenfunction is odd.

Equation (5) with even P always has even and odd solutions: to obtain an even solution we solve the Cauchy problem with the initial conditions $y_1(0) = 1, y_1'(0) = 0$; to obtain an odd solution we use the initial conditions $y_1(0) = 0, y_1'(0) = 1$.

Let y be an eigenfunction, and y_1 a solution of (5) of the *opposite parity* to y. Then y and y_1 are linearly independent, and the ratio $f = y/y_1$ is odd. Let G_d^o be the set of all odd functions in G_d. Then Φ maps G_d^o to Z_d^e because the Schwarzian derivative of an odd function is even. Thus we have a centrally symmetric version of Proposition 1: the map

$$\Phi : G_d^o \to Z_d^e \quad \text{is well defined and surjective.}$$

Similarly, Proposition 2 has a centrally symmetric analog: *for an odd meromorphic function g to be of the form $f(cz)$, where $f \in G_d^o$, it is necessary and sufficient that conditions* (i)-(iii) *(or* (i)-(iv)*) be satisfied.* Finally, Proposition 3 has a centrally symmetric analog:

Proposition 3′. *Let f_0 be an element of G_d^o, and $\gamma : [0, 1] \to H$,*

$$\gamma(t) = (w_0(t), \ldots, w_q(t)), \quad w_0 \equiv w_{q/2} \equiv 0,$$

be a path with the properties $\gamma(0) = W(f_0)$, $w_j(t) = -w_{j+q/2}(t)$ *and*

$$w_j(0) \neq w_k(0) \quad \Longleftrightarrow \quad w_j(t) \neq w_k(t)$$

for all $j \neq k \bmod q$ *and all* $t \in [0,1]$. *Then there is a lift of the path* γ *to* G_d^o, *that is a continuous family* $f_t \in G_d^o$, $t \in [0,1]$ *such that* $W(f_t) = \gamma(t)$.

The proof is the same as that of the original Proposition 3: one can choose all homeomorphisms ψ_t and ϕ_t to be odd, then $g_t = \psi_t \circ f_0 \circ \phi_t$ will be odd.

Case a) of Theorem 1 which we are proving corresponds to the even potential with $d = 4$. To prove a) we only need to show that G_4^o consists of two components: one containing the functions with $f(0) = 0$ and another containing the functions with $f(0) = \infty$.

Notice that $\mathbf{C}^*$ acts on G_d and on G_d^o by the rule $f \mapsto cf$, and that the map Φ is invariant with respect to this action. Introducing the equivalence relation $f \sim cg$, $c \in \mathbf{C}^*$ on G_d^o we obtain a factor-map $\tilde{\Phi}$ which maps the equivalence classes to Z_d.

Proposition 4. *The map* $\tilde{\Phi} : G_d^o/\sim \to Z_d^e$ *is a homeomorphism.*

Proof. We only have to show that it is injective, that is that any two non-zero odd solutions f_1 and f_2 of the Schwarz differential equation $S_f = -2P$, which tend to zero as $z \to \pm\infty$ on the real line, are proportional. All non-zero solutions of a Schwarz differential equation are related by fractional-linear transformations. So we have $f_1 = T \circ f_2$, where T is a fractional-linear transformation. Changing z to $-z$ we conclude that T is odd. Every odd fractional-linear transformation has the form cz or c/z. The latter case is excluded by the condition that $f_1(z)$ and $f_2(z)$ both tend to zero as $z \to \infty$ on the real line. Thus $f_1 = cf_2$.

In the next section we study the map $W : G_d \to \overline{\mathbf{C}}^q$, and in particular, the monodromy of this map. For this we need a description of the general fiber of this map by certain cell decompositions of the plane. Notice that our map W commutes with multiplication by constants $c \in \mathbf{C}^*$.

3. Some cell decompositions of the plane.

By a *cell decomposition* of a surface X we understand its representation as a locally finite union of disjoint subsets called *cells*. The cells can be of dimension 0 (points or vertices), 1 (edges) or 2 (faces). The edges and faces

are homeomorphic images of an open interval or of an open disc, respectively, and they satisfy the following condition: the boundary (in X) of each cell is a locally finite union of cells of smaller dimension of this decomposition. We do not assume that the homeomorphisms of the open discs defining faces have extensions to the closed discs.

Let $w = f(z)$ be a meromorphic function without critical points and with finitely many asymptotic values. Consider a fixed cell decomposition Ψ_0 of the sphere $\overline{\mathbf{C}}_w$ such that all asymptotic values are contained in the faces and each face contains one asymptotic value. Then the preimage $\Psi_f = f^{-1}(\Psi_0)$ is a cell decomposition of the plane $\mathbf{C}_z$ with connected 1-skeleton (The 1-skeleton of a cell decomposition is the union of edges and vertices.) That the 1-skeleton is connected is seen from (9), which is a covering, and from the fact that every path in $\overline{\mathbf{C}}\backslash\{w_0, \ldots, w_{q-1}\}$ can be deformed to a path in the 1-skeleton of Ψ_0. The closures of the edges of Ψ_f are mapped by f onto the closures of the edges of Ψ_0 homeomorphically. Each face B of Ψ_f is mapped by f onto a face D of Ψ_0 either homeomorphically or as a universal covering over $D\backslash\{w\}$ where w is the asymptotic value in D. In the former case the face B of Ψ_f is bounded, in the latter case it is unbounded.

We label the faces of Ψ_f by the names of their image faces under f. Labeled cell decompositions of the plane $\mathbf{C}_z$ are considered up to equivalences, orientation-preserving homeomorphisms of the plane preserving the labels. If f is an odd function, it is reasonable to choose Ψ_0 to be invariant under the map $w \mapsto -w$. Then Ψ_f will be also invariant under $z \mapsto -z$. For such cell decompositions of $\mathbf{C}_z$, the natural equivalence relation is that they are mapped one onto another by an odd orientation-preserving homeomorphism of $\mathbf{C}_z$ respecting the face labels. We call two such cell decompositions symmetrically equivalent. For a given set of asymptotic values and a given Ψ_0, the labeled cell decomposition Ψ_f almost completely determines f. Namely, we have the following

Proposition 5. *Let f_1 and f_2 be two meromorphic functions without critical points and with the same finite set of asymptotic values. Fix a cell decomposition Ψ_0 of $\overline{\mathbf{C}}_w$ such that the asymptotic values are contained in faces of Ψ_0 and each face contains one asymptotic value. If $\Psi_i = f_i^{-1}(\Psi_0)$ are equivalent cell decompositions of $\mathbf{C}_z$ then $f_1(z) = f_2(cz + b)$, $c \neq 0$.*

If f_i are odd, Ψ_0 is centrally symmetric, and the Ψ_i are symmetrically equivalent then $b = 0$.

Proof. Let ψ be the orientation-preserving homeomorphism of the plane

$\mathbf{C}_z$ that maps Ψ_1 onto Ψ_2 preserving the face labels. We are going to define another homeomorphism ψ' with the same properties, and in addition,

$$f_1 = f_2 \circ \psi'. \tag{13}$$

Let B_1 and $B_2 = \psi(B_1)$ be two faces such that f_i map B_i onto a face B_0 of Ψ_0. If one of the B_1, B_2 is bounded then another is also bounded and the maps $f_i : B_i \to B_0$ are homeomorphisms. So there exists a unique homeomorphism $\psi' : B_1 \to B_2$ such that (13) holds. If both B_1 and B_2 are unbounded, then $f_i : B_i \to B_0 \backslash \{w_0\}$ are universal coverings, and there are infinitely many homeomorphisms $B_1 \to B_2$ that satisfy (13). To choose one, we first notice that every homeomorphism $B_1 \to B_2$ with property (13) has a continuous extension to the boundary ∂B_1 and sends boundary edges of B_1 to boundary edges of B_2. We choose ψ' in B_1 so that it maps the boundary edges in the same way as ψ. This is possible to do as ψ preserves orientation. Now ψ' is defined on all faces of Ψ_1, and it is easy to check that the boundary extensions from different faces to edges match. Thus ψ' is a homeomorphism of the plane satisfying (13), and (13) implies that it is conformal. So $\psi'(z) = az + b$. It is easy to check that in the centrally symmetric case the above construction gives an odd homeomorphism ψ'.

Now we consider a special class of cell decompositions Ψ_0 which is convenient for our purposes.[2]

Let $f : \mathbf{C}_z \to \overline{\mathbf{C}}_w$ be a meromorphic function of order $q/2$ without critical points and with the asymptotic values $w_0, \ldots, w_{q-1}$, ordered according to the cyclic order of their Stokes sectors. We assume that the set $J = \{j : w_j \neq 0\}$ is a fixed subset of $\{0, \ldots, q-1\}$, and that all nonzero asymptotic values of f are finite and distinct. Let $J = \{j_1, \ldots, j_k\}$ with $j_1 < \ldots < j_k$, and let $c_\nu = w_{j_\nu}$. Then the cyclic order of the Stokes sectors with nonzero asymptotic values $c_1, \ldots, c_k$ agrees with that of $\{1, \ldots, k\} \bmod k$. We assume $3 \leq k < q$, so f has at least three distinct non-zero asymptotic values.

We define a cell decomposition Ψ_0 of $\overline{\mathbf{C}}_w$ with a single vertex at ∞ as follows (see Fig. 1).

[2]The usual choice of Ψ_0 as in [13, 18, 26, 27] leads to the cell decompositions of the plane which are called line complexes. We prefer a different choice of Ψ_0, as in [16], which is better compatible with the symmetries of our problem.

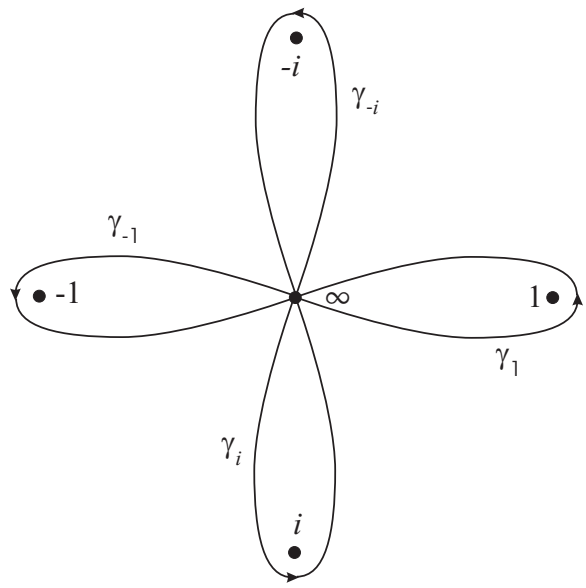

Fig. 1. Cell decomposition Ψ_0 for asymptotic values $0, i, 1, 0, -i, -1$.

In the complex plane $\mathbf{C}_w^\bullet = \overline{\mathbf{C}}_w \backslash \{0\}$, fix a system of directed loops $\gamma_{c_1}, \ldots, \gamma_{c_k}$, starting and ending at ∞, intersecting only at their endpoints, and such that each loop γ_{c_ν} is an oriented boundary of an open domain D_{c_ν} containing c_ν and not containing other asymptotic values of f. Let D_0 be the connected component of 0 in $\overline{\mathbf{C}}_w \setminus (\gamma_{c_1} \cup \ldots \cup \gamma_{c_k})$. The domains D_0, $D_{c_1}, \ldots, D_{c_k}$ are the faces of Ψ_0, and the open loops $\dot{\gamma}_\nu = \gamma_\nu \setminus \{\infty\}$ are its edges.

There is a natural cyclic order $\nu_1 \prec \ldots \prec \nu_k \prec \nu_1$ of the loops γ_{c_ν}, and of the corresponding domains D_{c_ν}. It is defined by the order in which the domains D_{c_ν} cross the oriented boundary of a small disk in $\mathbf{C}_w^\bullet$ centered at ∞. Alternatively, it is opposite to the order in which the loops γ_{c_ν} appear in the boundary of D_0.

So we introduced two cyclic orders on $c_1, \ldots, c_k$, the first one coming from the cyclic order of the Stokes sectors, and the second one from the cyclic order of the loops in the cell decomposition Ψ_0. These two cyclic orders are in general different.

Let $\Psi_f = f^{-1}(\Psi_0)$ be the corresponding cell decomposition of $\mathbf{C}_z$. An example of such cell decomposition is shown at the top of Fig. 2.

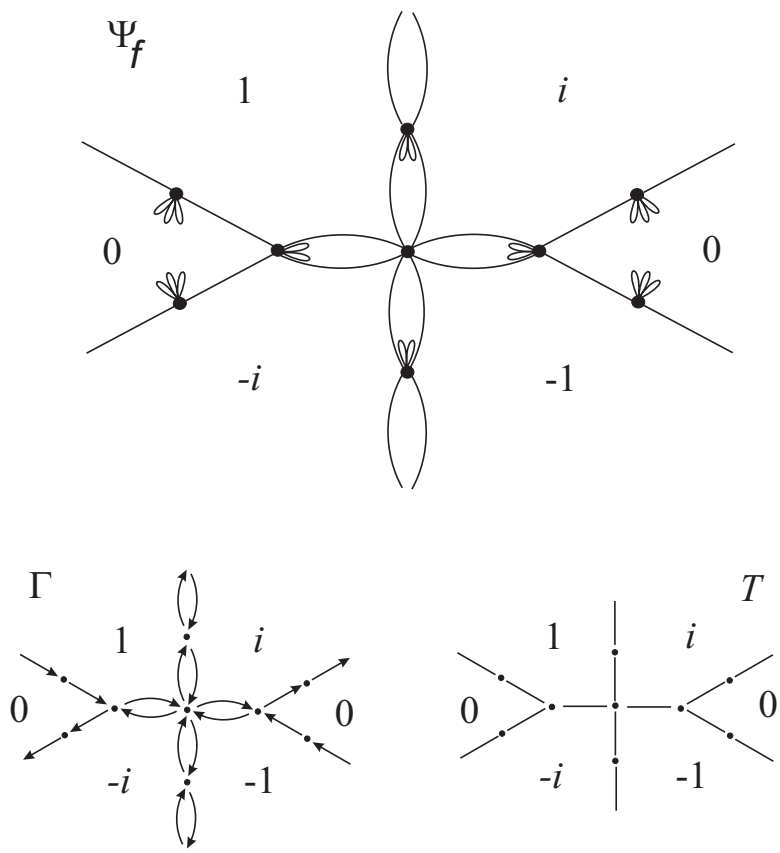

Fig. 2. Cell decomposition Ψ_f for asymptotic values $0, i, 1, 0, -i, -1$, and the corresponding graphs Γ and T of type A_1.

The vertices of Ψ_f are the poles of f. The faces of Ψ_f are labeled with 0, $c_1, \ldots, c_k$ and edges with $c_1, \ldots, c_k$ (only labels of unbounded faces corresponding to the Stokes sectors are shown in Fig. 2). The edges of Ψ_f are directed, being preimages of directed loops γ_{c_ν}. Since f has no critical points, the restriction of f to any face of Ψ_f is either a homeomorphism or a universal covering over the image face of Ψ_0 minus the asymptotic value in this image face. Accordingly, Ψ_f has the following properties:

(1) Unbounded faces of Ψ_f are in one-to-one correspondence with the Stokes sectors of f. This follows from property 6 in Section 2. For each $\nu = 1, \ldots, k$ there is exactly one unbounded face B_{c_ν} labeled with c_ν. The faces $B_{c_1}, \ldots, B_{c_k}$ have the cyclic order
$\{1, \ldots, k\} \bmod k$ at infinity.

(2) Edges of Ψ_f may be either *links* (having two distinct vertices) or *loops* (having both ends at the same vertex). Each edge labeled by c_ν separates two faces, labeled with c_ν and 0, respectively. Its orientation agrees with that of the boundary of its adjacent face labeled with c_ν. If it is a link, it is adjacent to the unbounded face B_{c_ν}.

(3) Each bounded face of Ψ_f labeled with c_ν has as its boundary a loop labeled by c_ν. The unbounded face B_{c_ν} has as its boundary an infinite chain of links labeled with c_ν, and no loops. A bounded face labeled with 0 has as its boundary k edges labeled with c_ν, (oriented opposite to their natural orientation) in the cyclic order opposite to the order $\nu_1, \ldots, \nu_k$ of the loops γ_{c_ν}. An unbounded face labeled with 0 has as its boundary an infinitely repeated sequence of k edges labeled with c_ν, in the cyclic order opposite to $\nu_1, \ldots, \nu_k$.

(4) The cyclic order of the values c_ν labeling the *links* in the boundary of a bounded face labeled by 0 is the same as the cyclic order of the unbounded faces B_{c_ν} adjacent to these links, which agrees with the cyclic order $\{1, \ldots, k\} \bmod k$ of the Stokes sectors.

(5) Each vertex v of Ψ_f has degree $2k$, with the directed edges labeled with $c_{\nu_1}, \ldots, c_{\nu_k}$ consecutively exiting and entering v, where $\nu_1, \ldots, \nu_k$ is the cyclic order of the loops γ_{c_ν}. This means that, as an edge labeled with c_{ν_1} exits v, the next in the cyclic order is (the same or another) edge labeled with c_{ν_1} entering v, then an edge labeled with c_{ν_2} exiting v, and so on, till an edge labeled with c_{ν_k} entering v, followed by the initial edge labeled with c_{ν_1} exiting v.

The one-skeleton of Ψ_f is an infinite directed graph properly embedded in $\mathbf{C}_z$. It is connected.

Removing from the 1-skeleton of Ψ_f all loops, we obtain a directed graph Γ (see Fig. 2). From property **(3)** of Ψ_f, all bounded components of the complement of Γ are labeled by 0, and the unbounded components are in one-to-one correspondence with the Stokes sectors of f. Moreover, the unbounded components corresponding to the Stokes sectors with nonzero asymptotic values are exactly the faces B_{c_ν} of Ψ_f. Replacing two edges in the boundary of each two-gon of Γ by one *undirected* edge inside the two-gon connecting its two vertices, and forgetting orientation of all remaining edges of Γ, we obtain a properly embedded graph T without loops or multiple edges (see Fig. 2), with the components of its complement labeled with 0 and c_ν. That T has no multiple edges follows from property **(2)** of Ψ_f: each link in Ψ_f belongs

to the boundary of some unbounded face, thus there are at most two links in Ψ_f between any pair of vertices. Each edge of T separates two components with different labels. All bounded components are labeled with 0.

Proposition 6. *Suppose that the cyclic order $\nu_1, \ldots, \nu_k$ of the loops γ_{c_ν} in $\mathbf{C}_w^\bullet$ agrees with the cyclic order $\{1, \ldots, k\} \bmod k$ of the Stokes sectors with the asymptotic values $c_1, \ldots, c_k$ in $\mathbf{C}_z$. Then each bounded face of Ψ_f labeled with 0 has exactly two vertices, and its boundary contains exactly two links. Each bounded component of the complement of Γ is a two-gon, and T is an embedded planar tree.*

Proof. First, a bounded face C of Ψ_f labeled with 0 must have at least two vertices. Otherwise, its boundary would consist of loops labeled with $c_\nu \neq 0$, which should be also boundaries of bounded faces labeled with c_ν. This is impossible since the union of this face with the loops and the vertices would be a sphere, and could not be embedded in $\mathbf{C}_z$. Hence there are at least two links in the boundary of C.

From property **(3)** of Ψ_f, the cyclic order of the links labeled with c_ν in the boundary of C should be opposite to the cyclic order of the loops γ_{c_ν}. From property **(4)**, it should agree with the cyclic order $\{1, \ldots, k\} \bmod k$ of the faces B_{c_ν}. If the loops γ_{c_ν} and the faces B_{c_ν} have the same cyclic order, this is impossible when the boundary of C contains more than two links.

Since Γ is obtained by removing loops from the one-skeleton of Ψ_f, its complement has no bounded components other than two-gons, hence the complement of T has no bounded components, so T is a forest (a union of disjoint trees).

To prove that it is connected, we notice that the 1-skeleton of Ψ_f is connected, and the removal of loops and multiple edges from Ψ_f does not affect this connectedness.

Proposition 7. *The embedded planar directed graph Γ and the cell decomposition Ψ_f are determined by the embedded planar graph T uniquely up to an orientation-preserving homeomorphism of $\mathbf{C}_z$ preserving their common vertices.*

Proof. The components of the complement of T labeled with 0 coincide with the components of the complement of Γ labeled with 0. A unique unbounded component C_ν of the complement of T labeled by c_ν contains the unbounded face B_{c_ν} of Ψ_f. Each of its boundary edges separates C_ν either from a component of the complement of T labeled with 0 or from C_μ with

$\mu \neq \nu$. In the first case, the edge belongs to Γ. We make it directed as part of the boundary of C_ν and label it with c_ν. Otherwise, we connect the two vertices of the edge by a new edge labeled with c_ν inside C_ν, directed according to the orientation of the boundary of C_ν, so that all these new edges are disjoint (this can be done one edge at a time, in any order). As a result, each edge of T separating two components C_ν and C_μ is included inside a two-gon. Let Γ' be the embedded planar directed graph obtained by removing all such edges of T. We want to show that Γ' can be obtained from Γ by an orientation-preserving homeomorphism of $\mathbf{C}_z$ preserving all vertices and labels. First, two-gons of Γ' have the same vertices and the same labeling of edges as the two-gons of Γ. Hence there exists an orientation-preserving homeomorphism between each two-gon of Γ and the two-gon of Γ' with the same vertices, preserving their common vertices. These homeomorphisms define a homeomorphism between the union of all two-gons of Γ and the union of all two-gons of Γ', preserving their common vertices and the labeling of their edges. It can be extended to unbounded components to obtain a homeomorphism of $\mathbf{C}_z$.

From property **(5)** of Ψ_f, the cyclic order of the components B_{c_ν} adjacent to a vertex v of Γ agrees with the cyclic order $\nu_1, \ldots, \nu_k$ of the loops γ_{c_ν}. The cyclic order of the edges of Γ exiting and entering v is determined by the cyclic order of the unbounded components B_{c_ν} of its complement adjacent to v. Since this order agrees with the cyclic order of the faces D_{c_ν} of Ψ_0 (same as the order of the loops γ_{c_ν}), one can add to Γ non-intersecting loops, having both ends at v, labeled with the missing values c_ν inside the connected components of the complement of Γ labeled by 0 adjacent to v, so that the cyclic order of the edges consecutively exiting and entering v becomes $\nu_1, \ldots, \nu_k$. Labeling the interiors of these loops by the corresponding values c_ν, we obtain a cell decomposition of $\mathbf{C}_z$ that can be obtained from Ψ_f by an orientation-preserving homeomorphism of $\mathbf{C}_z$ preserving vertices and labels (this can be done first for the loops, in any order, then extended to components labeled with 0).

Centrally symmetric case.

Suppose now that the set of asymptotic values w_j (and the corresponding set of non-zero values c_ν) is centrally symmetric. In this case, k is even and $c_{\nu+k/2} = -c_\nu$. We can choose the loops γ_{c_ν} centrally symmetric (e.g., by selecting $k/2$ loops about c_ν^2 and taking square roots of them). If f is odd, the cell decomposition Ψ_f is centrally symmetric, and so are the graphs Γ

and T (assuming the edges of T inside centrally symmetric two-gons of the complement of Γ are chosen centrally symmetric). The origin $0 \in \mathbf{C}_z$ is either a vertex of Ψ_f when $f(0) = \infty$, or the center of a bounded face labeled with 0 when $f(0) = 0$. This bounded face may correspond either to a bounded face of the complement of T (if its boundary has more than two links) or to the middle of an edge of T.

4. Case $q = 6$, $k = 4$: classification of trees.

Consider the centrally symmetric case $q = 6$, $k = 4$, with $w_0 = w_3 = 0$. We assume that the Stokes sector S_0 contains a ray of the positive real axis. We have $c_1 = w_1 = -c_3 = -w_4$ and $c_2 = w_2 = -c_4 = -w_5$ satisfying $\{c_1, c_2\} \cap \{0, \infty\} = \emptyset$ and $c_1 \neq \pm c_2$. Since multiplication of all asymptotic values by a nonzero constant corresponds to multiplication of f by the same constant, we can assume $c_2 = 1$ and $c_1 = c \neq 0$, ∞, ± 1.

Suppose that the cell decomposition Ψ_0 is centrally symmetric and satisfies conditions of Proposition 6, so that T is a centrally symmetric tree. The two components of its complement labeled by 0 and opposite via central symmetry, cannot have a common boundary edge.

Proposition 8. *Consider centrally symmetric embedded trees in* $\mathbf{C}$ *with six ends, two opposite components of the complement labeled by zero, and such that these two components do not have a common edge. Any such tree is equivalent to either one of the trees shown in Fig. 3, or its complex conjugate. The integer parameters satisfy the following restrictions: for* $A_k:\ k \geq 0$, *for* $D_{k,l}:\ k \geq 0, l \geq 1$, *and for* $E_{k,l}:\ k \geq 1, l \geq 0$.

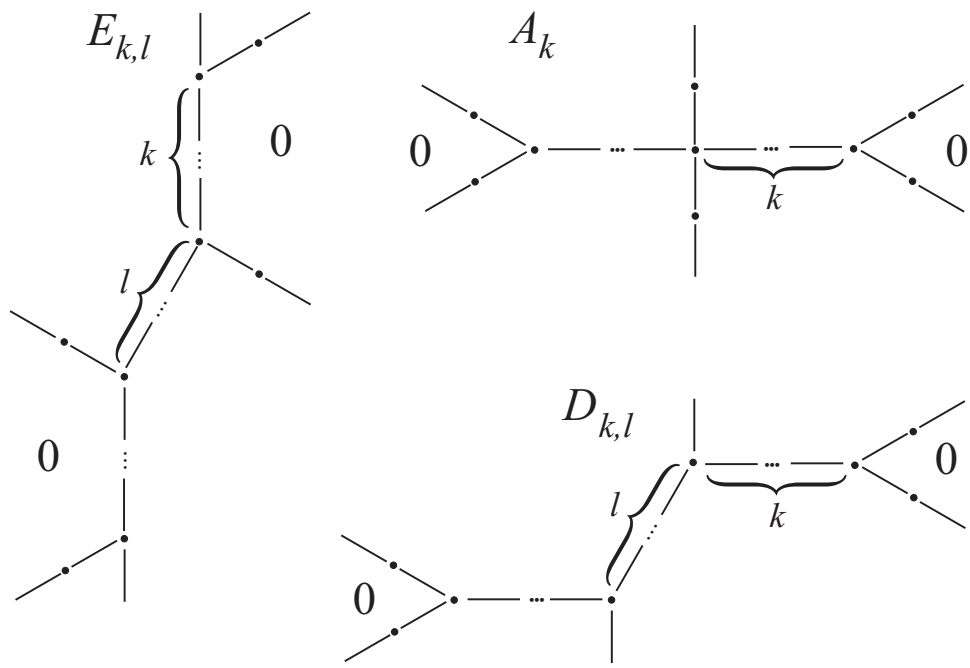

Fig. 3. Classification of trees.

Proof. Let X be a graph in $\mathbf{C}_z$ with the vertices corresponding to components of the complement of T and the edges connecting two of its vertices if the corresponding components have a common edge in T. Then X is combinatorially a hexagon with some of the chords connecting its vertices, so that the chords do not intersect and the vertices 0 and 3 not connected. We can also assume that X is centrally symmetric and that its vertex labeled with 0 is on the positive real axis. There are ten possible cases. Six of them are listed in Fig. 4. They correspond to embedded planar trees A_k, $D_{k,l}$, $E_{k,l}$ shown in Fig. 3. Here indices k and l denote the number of edges in a chain of edges and simple vertices of T corresponding to a chord of X (with l corresponding to the chord passing through the origin). In particular, a tree with even l has a vertex at the origin, while a tree with odd l has the origin as the middle of its edge. The remaining four cases are obtained by complex conjugation.

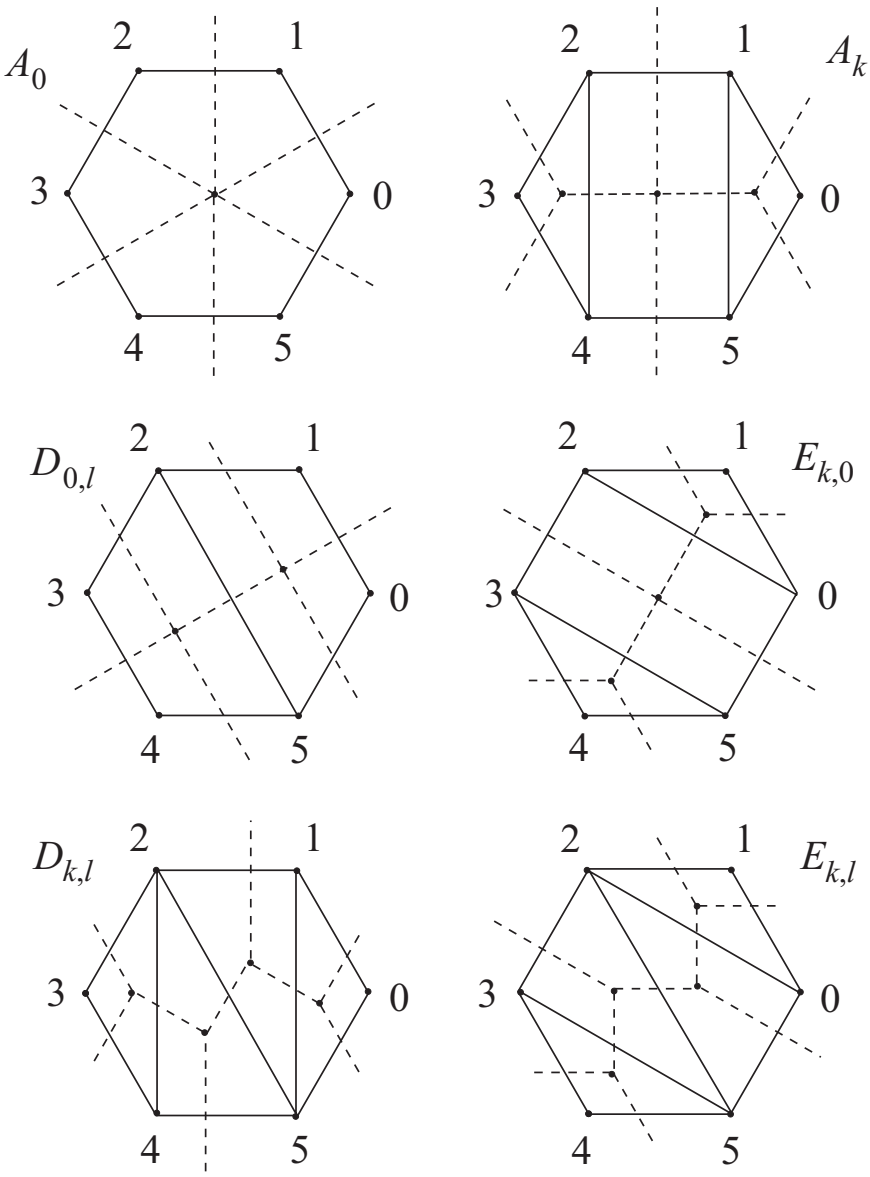

Fig. 4. To the proof of Proposition 8.

The trees A_k are symmetric with respect to complex conjugation. The complex conjugates of the other trees are denoted by $\bar{D}_{k,l}$ and $\bar{E}_{k,l}$.

If the condition of Proposition 6 is not satisfied (i.e., the cyclic order of the loops in Ψ_0 is different from the cyclic order of the Stokes sectors) the

graph T may still be a tree when no three components of its complement labeled by nonzero values are adjacent to any of its vertices. This is the case for the trees $E_{k,l}$ and their complex conjugates $\bar{E}_{k,l}$. To distinguish them from the same trees with the correct cyclic order of face labels we denote them by $E'_{k,l}$ and $\bar{E}'_{k,l}$, respectively. Examples of non-tree graphs T are shown in Fig. 5.

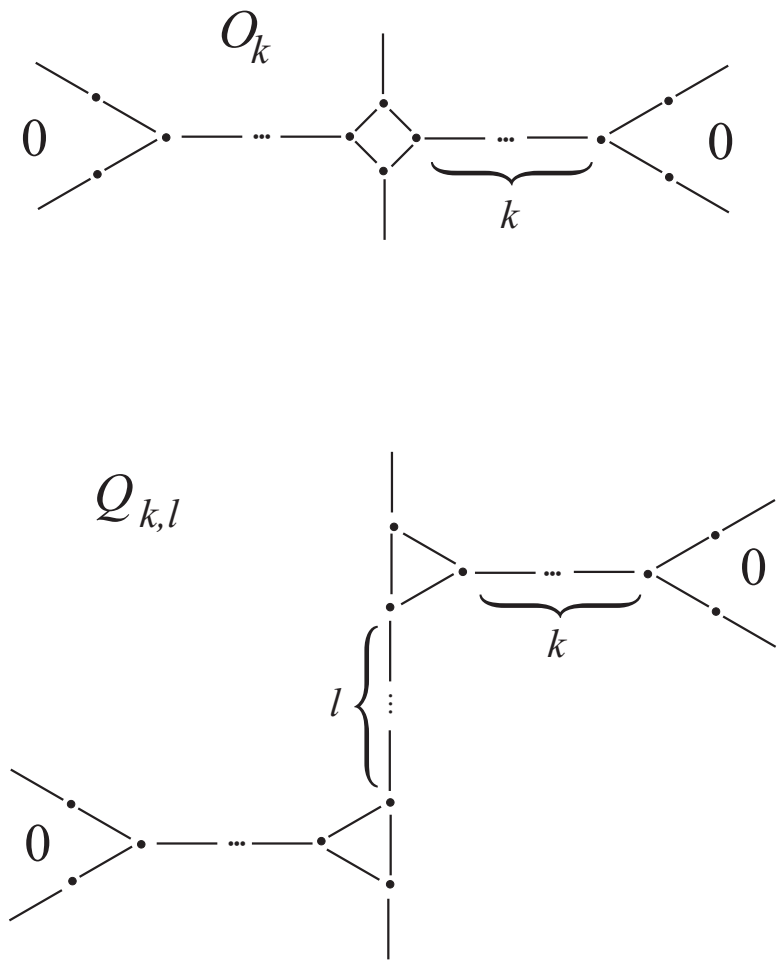

Fig. 5. Non-tree examples of graphs T.

These graphs are O_k and $Q_{k,l}$, with $k, l \geq 0$, and the complex conjugates $\bar{Q}_{k,l}$ of $Q_{k,l}$. We will later show that these graphs T can really occur. Moreover, these are all possible cases of non-tree graphs T corresponding to cell decompositions Ψ_f with the reverse cyclic order of non-zero labels, but we

do not use this fact in the proof; it will rather come as a consequence of our arguments.

5. Monodromy. Proof of Theorem 1 a).

Let Ξ be the space of all four-point sequences $\xi = \{c_1, \ldots, c_4\}$ in $\overline{\mathbf{C}}_w$ such that $c_3 = -c_1$, $c_4 = -c_2$, $c_\mu \neq 0, \infty, \pm c_\nu$ for $\mu \neq \nu$. Let Ξ_0 be the quotient space of Ξ with respect to multiplication by a nonzero constant. A point of Ξ_0 can be represented by a sequence $\{c, 1, -c, -1\}$ such that $c \neq 0, \infty, \pm 1$. Let $b = c^2 \neq 0, \infty, 1$. The fundamental group of $\overline{\mathbf{C}} \setminus \{0, \infty, 1\}$ with a base point -1 is a free group with two generators s_0 and s_∞ corresponding to the loops starting at -1, going along the negative real axis towards 0 (resp. ∞), going counterclockwise about 0 (resp. ∞) and returning to -1 along the negative real axis. Each of these generators defines two paths in the space Ξ_0 (denoted also by s_0 and s_∞) starting at $c = i$ (resp. $c = -i$) and ending at $c = -i$ (resp. $c = i$) (see Fig. 6).

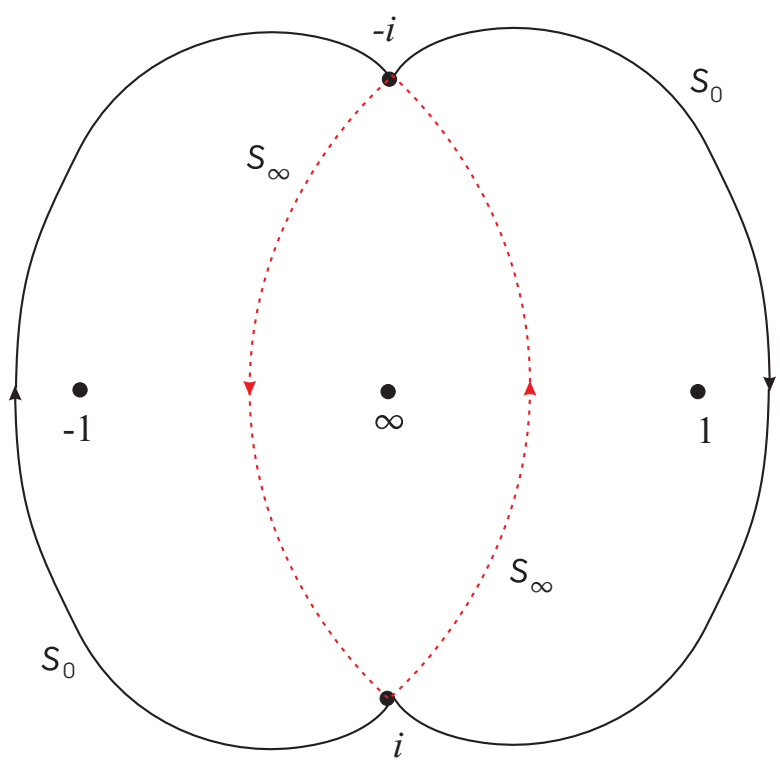

Fig. 6. Paths s_0 and s_∞.

The fundamental group of Ξ_0 with the base point i is generated by $q_0 = s_0^2$, $q_\infty = s_\infty^2$, $q_{-1} = (s_0 s_\infty)^{-1}$, and $q_1 = (s_\infty s_0)^{-1}$, with the relation $q_0 q_1 q_\infty q_{-1} = 1$.

Suppose that f has nonzero asymptotic values $c_1 = i$, $c_2 = 1$, $c_3 = -i$, $c_4 = -1$. Define four loops γ_i, γ_1, γ_{-i}, γ_{-1} in $\mathbf{C}_w^\bullet$ by following either real or imaginary axis from ∞ to one of the points i, 1, $-i$, -1, moving about that point counterclockwise, and returning to ∞ along the same axis (Fig. 1). These four loops generate the free group $\pi_1(\mathbf{C}_w^\bullet \setminus \{\pm i, \pm 1\})$. We can assume that the cell decomposition Ψ_0 of $\mathbf{C}_w^\bullet$ defined by these four loops is both centrally symmetric and invariant with respect to complex conjugation. The cyclic order of the loops γ_i, γ_1, γ_{-i}, γ_{-1} at the point ∞ agrees with the cyclic order of the corresponding Stokes sectors in $\mathbf{C}_z$. According to propositions 6 and 7, the cell decomposition Ψ_f of $\mathbf{C}_z$ can be defined by a tree T.

Each path $\{c(t), 1, -c(t), -1\}$ in Ξ_0 starting at $c(0) = i$ defines a continuous deformation
$\gamma_i(t)$, $\gamma_1(t)$, $\gamma_{-i}(t)$, $\gamma_{-1}(t)$ of the original four loops, each of the deformed loops starting and ending at ∞ and avoiding 0, ∞, ± 1, $\pm c(t)$. This deformation is unique up to isotopy. We can choose the deformation so that the corresponding cell decomposition $\Psi_0(t)$ of $\mathbf{C}_w^\bullet$ remains centrally symmetric. For the paths corresponding to s_0 and s_∞, we have $c(1) = -i$. Hence the loops $\hat{\gamma}_{-i} = \gamma_i(1)$, $\hat{\gamma}_1 = \gamma_1(1)$, $\hat{\gamma}_i = \gamma_{-i}(1)$, $\hat{\gamma}_{-1} = \gamma_{-1}(1)$ belong to the same space $\mathbf{C}_w^\bullet \setminus \{\pm i, \pm 1\}$ as the original loops γ_i, γ_1, γ_{-i}, γ_{-1}. See Figs. 7, 8.

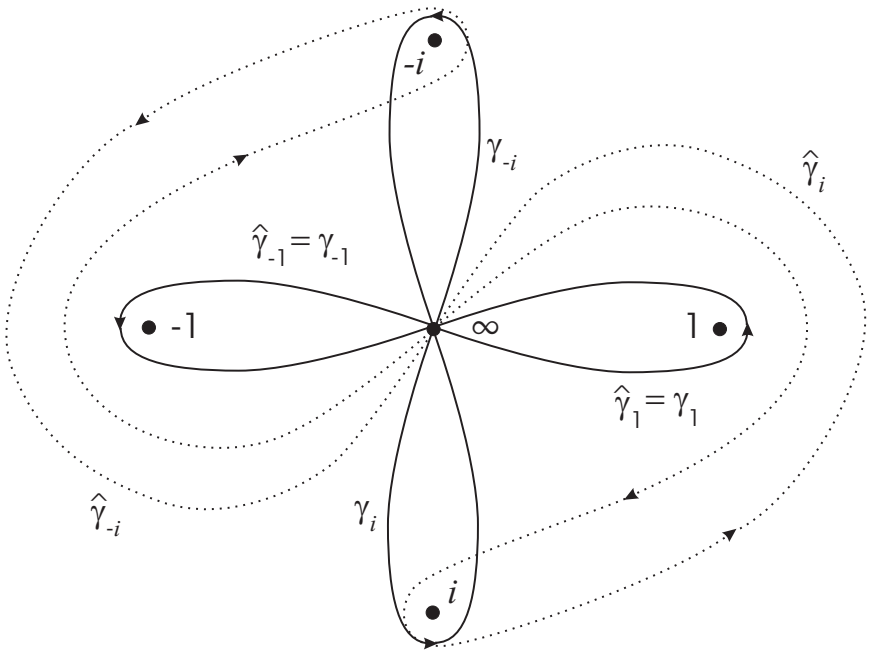

Fig. 7. Action of s_0.

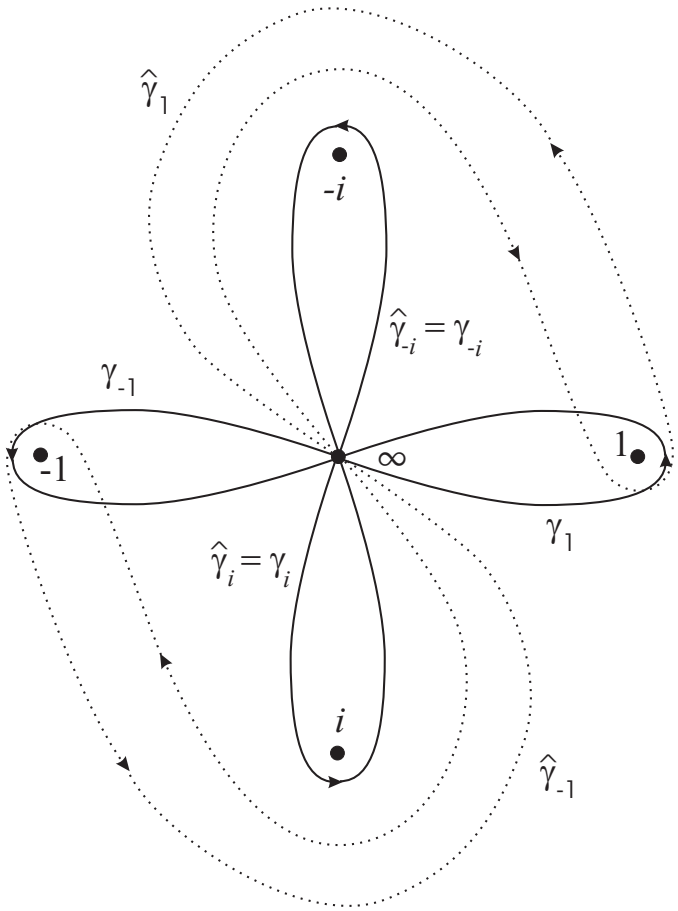

Fig. 8. Action of s_∞.

The paths in Ξ_0 corresponding to s_0 and s_∞ can be considered as elements of the braid group $\mathcal{B}_4$ on four strands in $\mathbf{C}_w^\bullet$ (leaving ± 1 fixed). From the classical formulas for the action of $\mathcal{B}_k$ on the fundamental group of the plane without k points [23], we have (see Fig. 7 for s_0 and Fig. 8 for s_∞).

$$\hat{\gamma}_i = (\gamma_1)^{-1}\gamma_i\gamma_1,\ \hat{\gamma}_1 = \gamma_1,\ \hat{\gamma}_{-i} = (\gamma_{-1})^{-1}\gamma_{-i}\gamma_{-1},\ \hat{\gamma}_{-1} = \gamma_{-1} \text{ for } s_0; \tag{14}$$

$$\hat{\gamma}_i = \gamma_i,\ \hat{\gamma}_1 = (\gamma_{-i})^{-1}\gamma_1\gamma_{-i},\ \hat{\gamma}_{-i} = \gamma_{-i},\ \hat{\gamma}_{-1} = (\gamma_i)^{-1}\gamma_{-1}\gamma_i, \text{ for } s_\infty. \tag{15}$$

Conversely, the original loops can be expressed as products of the new loops:

$$\gamma_i = \hat{\gamma}_1\hat{\gamma}_i(\hat{\gamma}_1)^{-1},\ \gamma_1 = \hat{\gamma}_1,\ \gamma_{-i} = \hat{\gamma}_{-1}\hat{\gamma}_{-i}(\hat{\gamma}_{-1})^{-1},\ \gamma_{-1} = \hat{\gamma}_{-1} \text{ for } s_0; \tag{16}$$

$$\gamma_i = \hat\gamma_i,\ \gamma_1 = \hat\gamma_{-i}\hat\gamma_1(\hat\gamma_{-i})^{-1},\ \gamma_{-i} = \hat\gamma_{-i},\ \gamma_{-1} = \hat\gamma_i\hat\gamma_{-1}(\hat\gamma_i)^{-1},\ \text{for } s_\infty. \qquad (17)$$

Let f_t be the family of functions constructed in Proposition 3′, for the path $\{c(t),\, 1,\, -c(t), -1\}$ in Ξ_0 with $c(0) = i$ and $c(1) = -i$, and let $\Psi_{f_t} = f_t^{-1}(\Psi_0(t))$ be the corresponding cell decompositions. Note that the cell decomposition $\Psi_0(1)$ is defined by the loops $\hat\gamma_{-i} = \gamma_i(1)$, $\hat\gamma_1 = \gamma_1(1)$, $\hat\gamma_i = \gamma_{-i}(1)$, $\hat\gamma_{-1} = \gamma_{-1}(1)$. Then f_1 has the nonzero asymptotic values $-i,\ 1,\ i,\ -1$. The cell decomposition Ψ_{f_1} is obtained by a continuous deformation exchanging i and $-i$ from the cell decomposition Ψ_{f_0}. Accordingly, the directed graph without loops $\hat\Gamma$ corresponding to Ψ_{f_1} is the same as the graph Γ corresponding to Ψ_{f_0}, with the labels i and $-i$ of its edges exchanged.

Let $\Psi'_{f_1} = f_1^{-1}(\Psi_0)$, and let Γ' be the corresponding directed graph without loops. Embedding of the graph Γ' in the plane $\mathbf{C}_z$ is determined, up to an orientation-preserving homeomorphism of $\mathbf{C}_z$, by its combinatorial structure and the cyclic order of its labeled directed edges at each of its vertices of degree greater than 2 [23]. Since this cyclic order agrees with the cyclic order of directed edges of the cell decomposition Ψ_0 at its vertex ∞, to define Γ' it is enough to specify its combinatorial structure, i.e., for a vertex v of Γ' (which can be identified with a vertex of $\hat\Gamma$, since both graphs have $f_1^{-1}(\infty)$ as their vertices) to determine whether an edge of Γ' labeled with one of $i,\ 1,\ -i,\ -1$ exits v, and if yes, which vertex of Γ' does it enter.

Two vertices v and v' of $\hat\Gamma$ are connected by a directed edge labeled by i (resp., $1,\ -i,\ -1$) if and only if the monodromy of f_1^{-1} along the loop $\hat\gamma_i$ (resp., $\hat\gamma_1,\ \hat\gamma_{-i},\ \hat\gamma_{-1}$) maps v to $v' \neq v$. For Γ', the same holds for the monodromy of f_1^{-1} along the loop γ_i (resp., $\gamma_1,\ \gamma_{-i},\ \gamma_{-1}$).

Let us denote the corresponding monodromy transformations by $\sigma_i,\ \sigma_1,\ \sigma_{-i},\ \sigma_{-1}$, and by $\hat\sigma_i,\ \hat\sigma_1,\ \hat\sigma_{-i},\ \hat\sigma_{-1}$, respectively. Then the transformations $\hat\sigma$ can be determined from the transformations σ from the relations (16) and (17) between the two sets of loops. Note that the monodromy is anti-representation of the fundamental group, i.e., the transformation corresponding to the product $\gamma\gamma'$ of two elements of the fundamental group is $\sigma' \circ \sigma$, where σ and σ' are the monodromy transformations corresponding to γ and γ'.

For s_0, the edges of Γ' labeled with 1 and -1 are the same as the edges of $\hat\Gamma$ with the same labels. For each vertex v, the edge of Γ' labeled with i that starts at v ends at the vertex obtained from v by moving along an edge of $\hat\Gamma$ labeled with 1 (or staying at v if there is no such edge), then along an edge of $\hat\Gamma$ labeled with i, then backward along an edge of $\hat\Gamma$ labeled with 1.

(There is no edge when the last vertex coincides with v.)

The edge of Γ' labeled with $-i$ that starts at v ends at the vertex obtained from v by moving along an edge of $\hat{\Gamma}$ labeled with -1, then along an edge of $\hat{\Gamma}$ labeled with $-i$, then backward along an edge of $\hat{\Gamma}$ labeled with -1.

For s_∞, the edges of Γ' labeled with i and $-i$ are the same as the edges of $\hat{\Gamma}$ with the same labels. For each vertex v, the edge of Γ' labeled with 1 that starts at v ends at the vertex obtained from v by moving along an edge of $\hat{\Gamma}$ labeled with $-i$, then along an edge of $\hat{\Gamma}$ labeled with 1, then backward along an edge of $\hat{\Gamma}$ labeled with $-i$. The edge of Γ' labeled with -1 that starts at v ends at the vertex obtained from v by moving along an edge of $\hat{\Gamma}$ labeled with i, then along an edge of $\hat{\Gamma}$ labeled with -1, then backward along an edge of $\hat{\Gamma}$ labeled with i.

Fig. 9 shows the action of s_0 for the tree T and graph Γ of type A_1 from Fig. 2. The graph Γ' corresponds to an undirected graph T' of type $Q_{1,0}$.

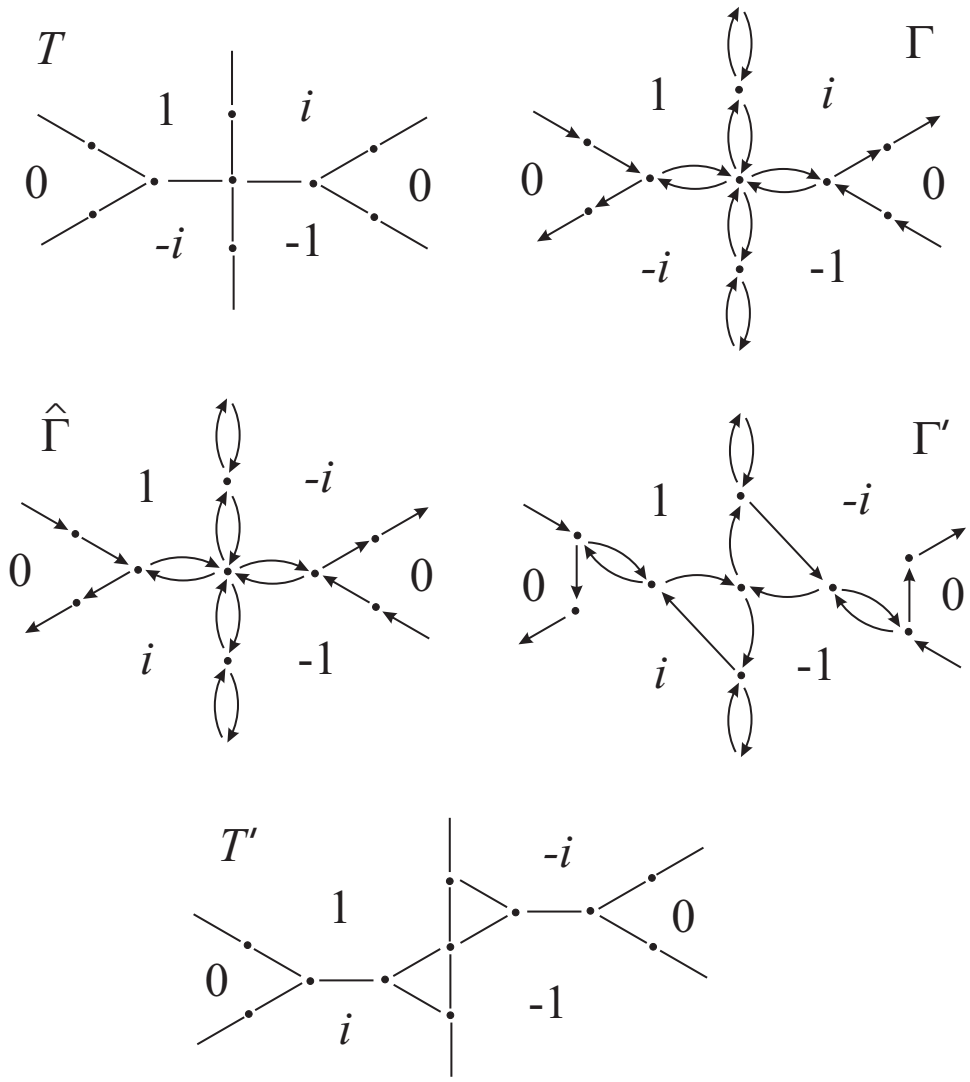

Fig. 9. Action of s_0 transforms A_1 to $Q_{1,0}$.

Similarly, for the paths $(s_0)^{-1}$ and $(s_\infty)^{-1}$, the transformations $\hat{\sigma}$ can be determined from the transformations σ from the relations (14) and (15). However, we do not need this, since they can also be obtained from the symmetry with respect to complex conjugation, sending a function $f(z)$ to $\bar{f}(\bar{z})$ and the asymptotic values $0, c, 1, 0, -c, -1$ (after normalization) to $0, 1/\bar{c}, 1, 0, -1/\bar{c}, -1$. Since the cell decomposition Ψ_0 in Fig. 1 is symmetric with respect to complex conjugation (acting as $w \mapsto i\bar{w}$ after normalization),

the cell decomposition Ψ_f and the corresponding graph Γ is exchanged with its complex conjugate, c with $1/\bar{c}$, α with $\bar{\alpha}$, and the action of s_0 with the action of s_∞^{-1}.

The action of s_0 and s_∞ and their inverses on all trees of Proposition 8 is summarized in the following tables (see also Figs. 10, 11, where this action is represented graphically).

$s_0(A_k) = Q_{k,0},\ k \geq 0;$	$s_0(Q_{k,0}) = A_{k+1},\ k \geq 0;$
$s_0(D_{k,l}) = Q_{k,l},\ k, l \geq 0;$	$s_0(Q_{k,l}) = D_{k+1,l},\ k, l \geq 0;$
$s_0(E_{k,l}) = E'_{k,l},\ k \geq 1,\ l \geq 0;$	$s_0(E'_{k,l}) = E_{k-1,l},\ k \geq 1,\ l \geq 0;$
$s_0(\bar{D}_{k,l}) = \bar{Q}_{k,l-2},\ k \geq 0,\ l \geq 2;$	$s_0(\bar{Q}_{k,l}) = \bar{D}_{k,l+2},\ k, l \geq 0;$
$s_0(\bar{E}_{k,l}) = \bar{E}'_{k,l},\ k \geq 1,\ l \geq 0;$	$s_0(\bar{E}'_{k,l}) = \bar{E}_{k,l},\ k \geq 1,\ l \geq 0;$
$s_0(\bar{D}_{k,1}) = O_k,\ k \geq 0$	$s_0(O_k) = \bar{D}_{k,1},\ k \geq 0;$
$s_0(E'_{1,l}) = D_{0,l},\ l \geq 0;$	$s_0(E'_{1,0}) = A_0.$

Table 1. The action of s_0.

$s_\infty(A_k) = \bar{Q}_{k-1,0},\ k \geq 1;$	$s_\infty(\bar{Q}_{k,0}) = A_k,\ k \geq 0;$
$s_\infty(D_{k,l}) = Q_{k,l-2},\ k \geq 0,\ l \geq 2;$	$s_\infty(Q_{k,l}) = D_{k,l+2},\ k, l \geq 0;$
$s_\infty(E_{k,l}) = E'_{k,l},\ k \geq 1,\ l \geq 0;$	$s_\infty(E'_{k,l}) = E_{k,l},\ k \geq 1,\ l \geq 0;$
$s_\infty(\bar{D}_{k,l}) = \bar{Q}_{k-1,l},\ k \geq 1,\ l \geq 1;$	$s_\infty(\bar{Q}_{k,l}) = \bar{D}_{k,l},\ k \geq 1,\ l \geq 1;$
$s_\infty(\bar{E}_{k,l}) = \bar{E}'_{k+1,l},\ k \geq 1,\ l \geq 0;$	$s_\infty(\bar{E}'_{k,l}) = \bar{E}_{k,l},\ k \geq 1,\ l \geq 0;$
$s_\infty(D_{k,1}) = O_k,\ k \geq 0;$	$s_\infty(O_k) = D_{k,1},\ k \geq 0;$
$s_\infty(\bar{D}_{0,l}) = \bar{E}'_{1,l},\ l \geq 0;$	$s_\infty(A_0) = \bar{E}'_{1,0}.$

Table 2. The action of s_∞.

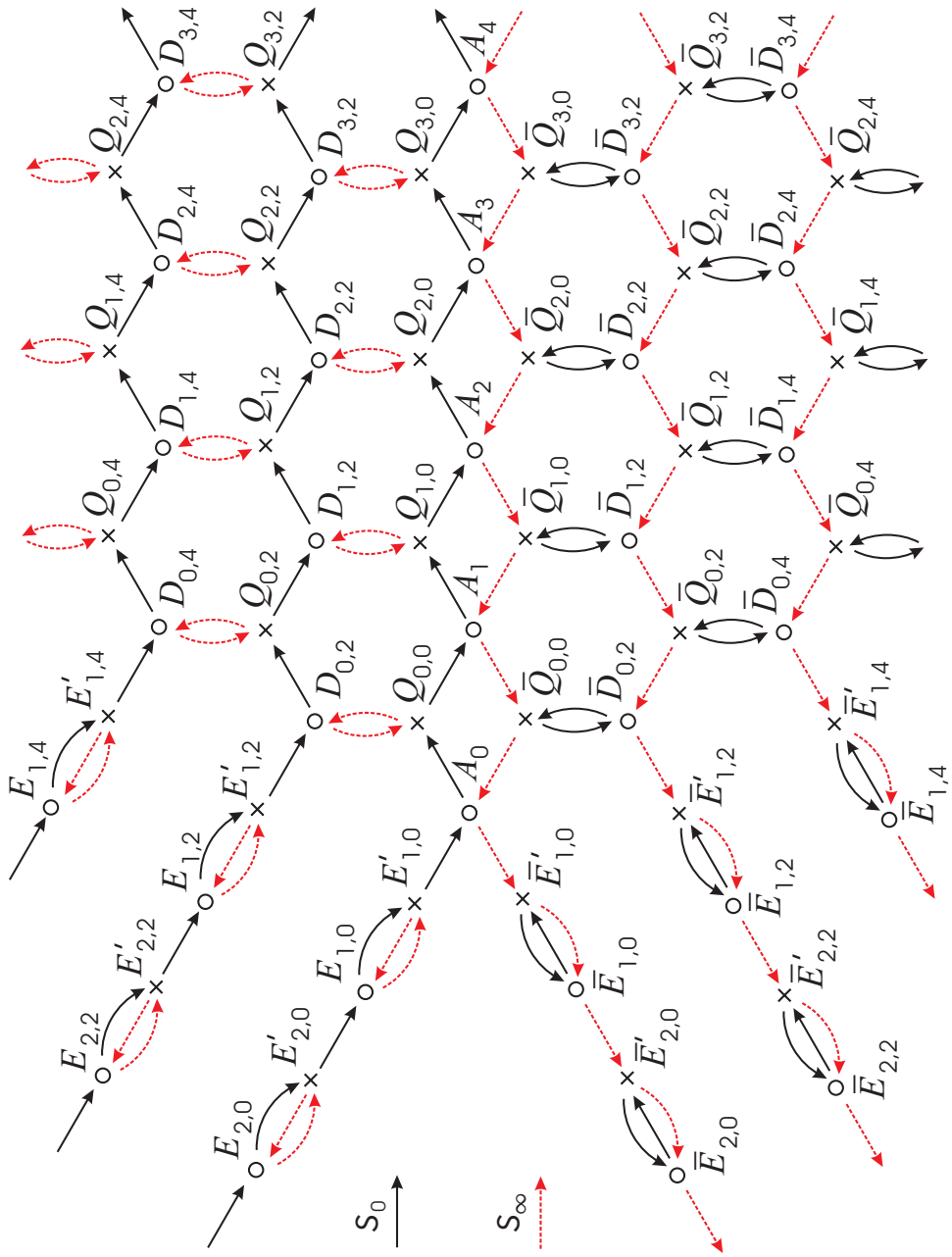

Fig. 10. Monodromy action on the even part of G.

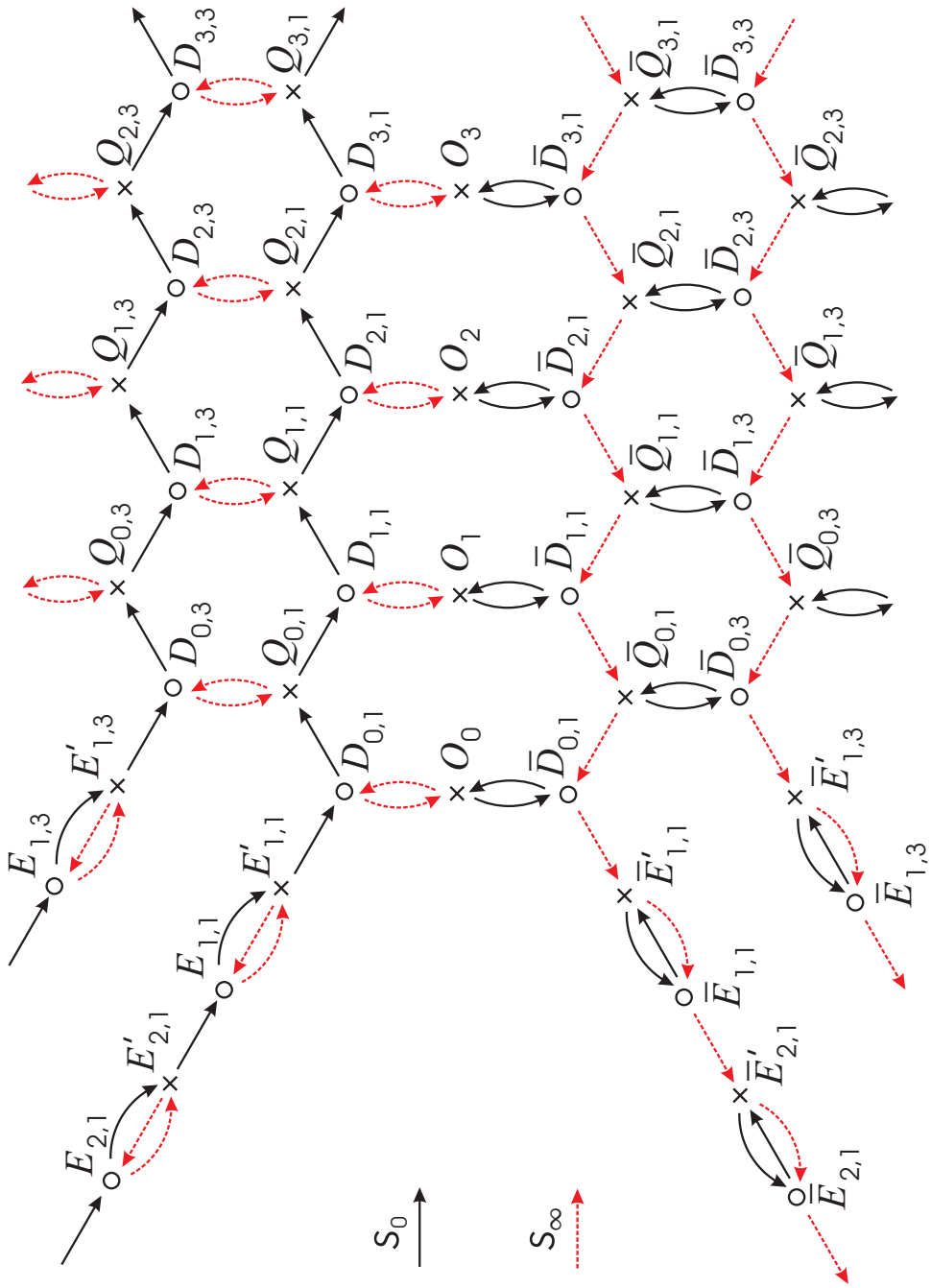

Fig. 11. Monodromy action on the odd part of G.

Completion of the proof of Theorem 1 a).

To prove that the Riemann surface G has exactly two connected components, corresponding to functions with a pole at the origin and the functions with a zero at the origin, respectively, it is enough to show that, for each point $c_0 \in \overline{\mathbf{C}}$ over which the mapping $W : G \to \overline{\mathbf{C}}$ is not ramified (i.e., $c_0 \neq 0, \infty, \pm 1$) the monodromy of W^{-1} acts transitively on the fiber $W^{-1}(c_0)$ of W. Let us choose a point $c_0 = i$. With the loops γ_i, γ_1, γ_{-i}, γ_{-1} defined above, the fiber of $W^{-1}(i)$ consists of the functions f such that the cell decomposition Ψ_f of $\mathbf{C}_z$ corresponds to one of the trees A_k, $D_{k,l}$, $E_{k,l}$, $\bar{D}_{k,l}$, $\bar{E}_{k,l}$. From the tables for the action of s_0 and s_∞, for the trees with a vertex at the origin we have:
(i) A_k can be obtained from A_0 applying $(s_0)^{2k}$;
(ii) $D_{k,2l}$ can be obtained from A_k applying $(s_0 s_\infty)^l$;
(iii) $E_{k,2l}$ can be obtained from $D_{0,2l}$ applying $(s_0)^{-2k}$;
(iv) $E_{k,0}$ can be obtained from A_0 applying $(s_0)^{-2k}$;
(v) The points corresponding to $\bar{D}_{k,2l}$ and $\bar{E}_{k,2l}$ can be obtained from A_0 combining the paths in (i-iv) with the complex conjugation.

For the trees with no vertex at the origin we have:
(i) $D_{k,1}$ can be obtained from $D_{0,1}$ applying $(s_0)^{2k}$;
(ii) $D_{k,2l+1}$ can be obtained from $D_{k,1}$ applying $(s_0 s_\infty)^l$;
(iii) $E_{k,2l+1}$ can be obtained from $D_{0,2l+1}$ applying $(s_0)^{-2k}$;
(iv) $\bar{D}_{k,1}$ can be obtained from $D_{k,1}$ applying $s_\infty s_0$;
(v) The points corresponding to $\bar{D}_{k,2l+1}$ and $\bar{E}_{k,2l+1}$ can be obtained from $D_{0,1}$ combining the paths in (i-iv) with the complex conjugation.

Eigenfunctions of self-adjoint operators.

Suppose that α in (4) is real, i.e., the problem is self-adjoint and the eigenvalues are real. Let $\lambda_0 < \lambda_1 < \dots$ be the eigenvalues of (4) and $y_1(z)$, $y_2(z), \dots$ the corresponding eigenfunctions. Then $y_n(z)$ has n real zeros (see [31]). Let $\lambda = \lambda_n$ be one of these eigenvalues. Since α and λ are real, the function $f(z)$ defined in Section 2 is a real odd meromorphic function. Hence its nonzero asymptotic values satisfy $c_2 = -\bar{c}_1$, $c_3 = -c_1$, $c_4 = \bar{c}_1$. These asymptotic values can be neither real nor pure imaginary (see [16]). Let $a = c_1$, and let Ψ_a be a cell decomposition of $\overline{\mathbf{C}}_w$ defined similarly to the cell decomposition Ψ_0 in Fig. 1, except the four loops of Ψ_a contain the asymptotic values $\pm a$, $\pm \bar{a}$ of f (see Fig. 12).

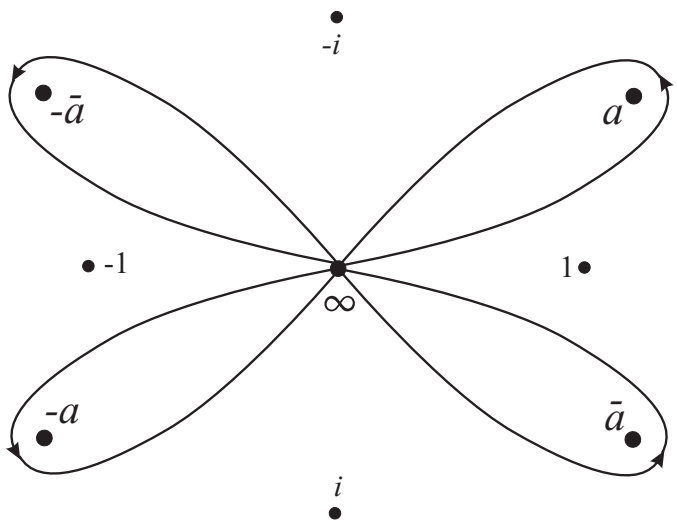

Fig. 12. Cell decomposition for a real function.

We assume that Ψ_a is centrally symmetric and invariant under complex conjugation. Then the cell decomposition $\Psi_f = f^{-1}(\Psi_a)$ of $\mathbf{C}_z$ is also centrally symmetric and invariant under complex conjugation.

Proposition 9. *The cell decomposition Ψ_f is of the type A_k for $n = 2k$, and of the type O_k for $n = 2k + 1$.*

Proof. Since Ψ_f is invariant under complex conjugation in $\mathbf{C}_z$, the corresponding graphs Γ and T are symmetric under complex conjugation.

If a is in the second or fourth quadrant (as in Fig. 12) the cyclic order of the nonzero asymptotic values of f in $\mathbf{C}_w^\bullet$ is consistent with the cyclic order of the corresponding Stokes sectors $S_1,\ S_2,\ S_4,\ S_5$. Thus conditions of Proposition 6 are satisfied, and the graph T should be one of the trees listed in Proposition 8. Since the only such trees symmetric under complex conjugation are A_k, we have $T = A_k$ for some k. A real function f with the cell decomposition Ψ_f of the type A_k has $2k$ real zeros (one zero in each two-gon of Ψ_f corresponding to an edge of T on the real line). Hence $n = 2k$ in this case.

If a is either in the first or third quadrant, the cyclic order of the asymptotic values of f is opposite to the cyclic order of the corresponding Stokes sectors. In this case, operations s_0' and s_∞' similar to s_0 and s_∞ (Figs. 7 and 8) can be applied to reverse the cyclic order. More precisely, for $a = \pm e^{i\pi/4}$,

the function $g(z) = e^{i\pi/4} f(z)$ has the asymptotic values ± 1, $\pm i$ and the cell decomposition $\Psi_g = g^{-1}(\Psi_0)$, where Ψ_0 is the cell decomposition in Fig. 1, coincides with Ψ_f. The action of s_0' and s_∞' on Ψ_f is defined as the action of s_0 and s_∞ on Ψ_g. For any a in the first or third quadrant, one can deform $\pm a$ to $\pm e^{i\pi/4}$ without crossing the real and imaginary axes, apply the two operations, and deform $\pm a$ back to initial values. Thus the action of s_0' and s_∞' on Ψ_f is the same as the action of s_0 and s_∞ described in Tables 1 and 2. In particular, the result of this action is a cell decomposition satisfying the condition of Proposition 6, hence having the type of one of the trees listed in Proposition 8. Accordingly, the cell decomposition Ψ_f itself should correspond to a graph T obtained from one of these trees by the operations s_0^{-1} and s_∞^{-1}. From Tables 1 and 2, all such graphs are of the types O_k, $Q_{k,l}$, $E'_{k,l}$, $\bar{Q}_{k,l}$, $\bar{E}'_{k,l}$. The only graphs in this list symmetric under complex conjugation are O_k. A real function f with the cell decomposition Ψ_f of the type O_k has $2k+1$ real zeros (one zero at the origin and one inside each two-gon of Ψ_f corresponding to an edge of T on the real line). Hence $n = 2k+1$ in this case.

6. Other potentials

The following three one-parametric families of potentials can be treated with the same method. The details will appear elsewhere.

1. PT-symmetric cubic [11]. A differential equation $-y'' + P(z)y = \lambda y$, with a general cubic polynomial P, by an affine change of the independent variable and a shift of λ can be brought to the form

$$-y'' + (iz^3 + i\alpha z)y = \lambda y. \tag{18}$$

We impose the boundary condition

$$y(z) \to 0, \quad \text{as} \quad y \in \mathbf{R}, \quad y \to \pm\infty. \tag{19}$$

This problem is not self-adjoint but for real α it has the so-called PT-symmetry property. It is known [12, 30] that for $\alpha \geq 0$ the spectrum is real.

Theorem 3. *Let $Z_3 \in \mathbf{C}^2$ be the set of all pairs (α, λ) such that the problem (18), (19) has a non-trivial solution. Then Z_3 is an irreducible non-singular curve.*

2. Quasi-exactly solvable sextic family [36, 37]. Consider the equation

$$-y'' + (z^6 + 2\alpha z^4 + \{\alpha^2 - (4m + 2p + 3)\}z^2)y = \lambda y, \tag{20}$$

with the same boundary condition (19). This problem is self-adjoint for real α. It was shown by Turbiner and Ushveridze that for real α this problem has exactly $m + 1$ linearly independent "elementary" eigenfunctions of the form Qe^T with polynomials Q and T. The degree of Q is $2m+p$, so the eigenfunction has $2m + p$ zeros in the complex plane. If $p = 0$ then these elementary eigenfunctions correspond to the first $m+1$ even-numbered eigenvalues, and if $p = 1$ to the first $m + 1$ odd-numbered eigenvalues.

Theorem 4. *Let m be a non-negative integer and $p \in \{0, 1\}$. Let $Z_{6,m,p} \in \mathbf{C}^2$ be the set of all pairs (α, λ) such that λ is an eigenvalue of the problem (20), (19) corresponding to an elementary eigenfunction. Then $Z_{6,m,p}$ is a non-singular irreducible curve.*

3. Quasi-exactly solvable PT-symmetric quartic family [3]. Consider the equation

$$-y'' + (-z^4 - 2\alpha z^2 - 2imz)y = \lambda y. \tag{21}$$

Here the boundary condition is

$$y(re^{i\theta}) \to 0, \quad \text{as} \quad r \to \infty, \quad \theta \in \{-\pi/6, -\pi + \pi/6\}. \tag{22}$$

Similarly to the previous example, this problem has m elementary eigenfunctions.

Theorem 5. *Let m be a non-negative integer, and $Z_{4,m} \subset \mathbf{C}^2$ be the set of pairs (α, λ) such that λ is an eigenvalue of the problem (21), (22), with an elementary eigenfunction. Then $Z_{4,m}$ is a smooth irreducible curve.*

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