FAILURE OF HIERARCHICAL DISTRIBUTIONS OF FIBER BUNDLES. II

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January 23, 1990

Abstract

Using the insight derived from the computational investigation by Newman and Gabrielov (1989) of the failure threshold of fiber bundles organized in hierarchical fashion, we prove by analytic methods independent of the specific failure properties of an individual fiber and independent of the specific hierarchical organization employed that the threshold for failure of a hierarchical fiber bundle obeys a universal scaling law with respect to the size of the bundle.

The investigation of the failure properties of bundles of fibers was first carried out extensively by Daniels (1945). Smalley et al. (1985) suggested that a renormalization approach was applicable to this problem when the bundles were grouped in a hierarchical fashion. Newman and Gabrielov (1989), hereafter referred to as paper I, employed an exact renormalization scheme to computationally investigate the failure properties of large fiber bundles. Given the insight derived from this computational investigation, we have developed a rigorous mathematical proof of these asymptotic properties which we present below.

Consider an assembly of fibers with a given cumulative strength distribution $P(\sigma)$ where σ is the applied stress, i.e. a fiber will break under a stress σ with probability $P(\sigma$. The distribution $P(\sigma)$ is a nondecreasing function with values between 0 and 1 defined for $0 < \sigma < \infty$, where $P(\sigma) \to 1$ when $\sigma \to \infty$. The problem here is to define the strength of the assembly or bundle of fibers given a specific rule of stress redistribution among the fibers. Let us denote individual fibers of the bundle as fibers of order 0. Suppose that these fibers are paired in sequential order and consider *each* pair of fibers of order 0 as though it were itself a fiber, which we will denote as a fiber or, alternatively, a fiber bundle of order 1. Suppose that these fibers of order 1 are also paired sequentially and that these pairs then form fibers of order 2, and so on, with fibers of order n consisting of 2^n individual fibers. The two fibers of order n which are paired are called neighbors of order n and form a fiber of order n+1. It is natural to equate $P_0(\sigma)$ with $P(\sigma)$, and we denote by $P_n(\sigma)$ the strength distribution for fiber bundles of order n.

We will assume that the fibers in a given bundle share equally the load supported by that bundle. Thus, a given pair of fibers breaks under an applied stress σ only if both constituent fibers fail under that stress σ, or one fiber fails under its load and redistributes that load to the surviving fiber which then fails under a load between σ and 2σ. Extending this concept to larger structures, a fiber bundle of order n breaks when its constituent fiber bundles of order $n-1$ break according to the above scenario. By induction, we can readily show that this means that all individual fibers contained within the fiber bundle of order n are broken. We now want to calculate the strength distribution $P_n(\sigma)$.

The probability that both fibers fail simultaneously under a stress σ is $P_{n-1}(\sigma)^2$. The probability that one fiber fails under stress σ with the surviving fiber failing under a stress between σ and 2σ is

$$2P_{n-1}(\sigma)[(P_{n-1}(2\sigma) - P_{n-1}(\sigma)]\ .$$

The factor of two appears here since failure can occur in two ways, according to which of the two fibers fails first. Thus, the probability of failure of a fiber bundle of order n is equal to the sum of the previous two expressions, namely

$$P_n(\sigma) = P_{n-1}(\sigma)[(2P_{n-1}(2\sigma) - P_{n-1}(\sigma)]\ . \tag{1}$$

The problem can be formulated similarly for the m-fold case when a fiber of order n consists of m fibers of order $n-1$.

Smalley et al. (1985) considered fiber bundles as a metaphor for the strength of a fault in the Earth's crust, which was considered to be a hierarchically organized system of asperities. This problem was reduced to the investigation of the properties of the transformation (1) for "one-dimensional fault" and of the corresponding transformation in the 4–fold bundled case for a two-dimensional fault. Strictly speaking, they investigated the transformation that emerges when $P(2\sigma)$ is replaced by a functional of $P(\sigma)$ that corresponded only to the special case $P(\sigma) = 1 - \exp(-a\sigma^2)$. This is not correct, as $P(\sigma)$ and the correspondence between $P_n(\sigma)$ and $P_n(2\sigma)$ depend on n.

In paper I, we investigated transformation (1) by computational means and observed that, for different initial distributions $P(\sigma)$, the corresponding distributions $P_n(\sigma)$ for large values of n approach a step-like function that is close to 0 for $\sigma < \sigma_n$ and close to 1 for $\sigma > \sigma_n$, and whose critical value σ_n decreases as $\dfrac{1}{\ln(n)}$. Qualitatively similar behavior was observed in the 4-fold case. We provide below a rigorous mathematical proof of these asymptotic properties for a class of transformations acting on distribution functions that includes transformation (1) and corresponding transformations in the m–fold case for all m.

Let $\mathbf{L}$ denote the space of the distribution functions, i.e. non-decreasing functions $P(\sigma)$, $0 < \sigma < \infty$, with values between 0 and 1. For any transformation $\mathbf{F} : \mathbf{L} \to \mathbf{L}$ and for $P(\sigma) \in \mathbf{L}$ define $P_n(\sigma) = \mathbf{F}^n(P)(\sigma)$ for $n \geq 0$. For our purpose it is useful to define a set of conditions on $\mathbf{F}$ which are generally met by hierarchically structured fiber bundles.

(A) $F(P)(\sigma) = F[(P(\sigma), P(a\sigma), \ldots]$, where $a > 1$, $F(x, y, \ldots)$ is a continuously twice differentiable function and $\ldots$ means a possible dependence of F on the values of P at $a'\sigma$, $a''\sigma$ etc., $1 < a < a' < a'' \ldots$.
(B) For $0 \leq x \leq y \leq \ldots \leq 1$, we have $0 \leq F(x, y, \ldots) \leq 1$ and all the components of the vector grad$(F)(x, y, \ldots)$ are nonnegative.
(C) $F(x, y, \ldots) \leq Qxy$ for $0 \leq x \leq y \leq \ldots \leq 1$ for some constant $Q > 1$.
(D) Let $f(x) = F(x, 1, \ldots, 1)$. Then $f(0) = 0$, $f(1) = 1$, $f(x) > x$ for $0 < x < 1$, $f'(0) > 1$, $f'(1) < 1$.
(E) $F(x, y, \ldots) \geq qxy^v$ for $0 \leq x \leq y \leq \ldots \leq 1$ for some constants $q > 0$, $v \geq 1$.
(F) $F(x, \ldots, x) < x$, for $0 < x < 1$.

Remark 1. Condition (B) is equivalent to the requirement that $\mathbf{F}(\mathbf{L}) \subset \mathbf{L}$. For simplicity, let $F = F(x, y)$. Suppose, to the contrary, that $grad(F)(x, y)$ has a negative component for some x, y with $0 \leq x \leq y \leq 1$. As $grad(F)$ is continuous, it is sufficient to consider $0 < x < y < 1$. Then, for a small ϵ, we have $0 < x < x + \epsilon < y < y + \epsilon < 1$ and $F(x, y) > F(x+\epsilon, y+\epsilon)$. For any small δ there exists a function $P \in \mathbf{L}$ such that $P(1) = x$, $P(1 + \delta) = x + \epsilon$, $P(a) = y$, $P[a(1 + \delta)] = y + \epsilon$. Then $\mathbf{F}(P)(1) > \mathbf{F}(P)(1 + \delta)$ which is contrary to the properties of a distribution function, and $\mathbf{F}(P) \notin \mathbf{L}$. Conversely, if all the components of $grad(F)$ are nonnegative, then it is easy to show that $F(x, y) \leq F(x', y')$ for $x < x'$, and $y < y'$. Thus condition (B) implies $\mathbf{F}(\mathbf{L}) \subset L$.

Remark 2. Conditions (A)–(F) are evidently valid for the transformation (1) with $F(x, y) = x(2y - x)$, $a = 2$, $Q = 2$, $q = 1$, and $n = 1$. Let us show that these conditions are valid for corresponding transformations in the m–fold case for any m.

It is easy to verify that the probability of failure of a fiber bundle of order $n + 1$ in the m-fold case under given stress σ is a polynomial F of the probabilities of failure of its constituent fiber bundles of order n under initial stress σ, as well as under stresses $\frac{m\sigma}{m-1} \frac{m\sigma}{m-2}, \ldots, m\sigma$ emerging when some of the fiber bundles of order n are broken. See Phoenix and Smith (1983) and Kuo and Phoenix (1987) and the references therein. This means that condition (A) is valid for the polynomial F and $a = \frac{m}{m-1}$, $a' = \frac{m}{m-2}$, etc.

Condition (B) is automatically valid, as the transformation by definition acts on the space $\mathbf{L}$ of distribution functions.

Condition (C) is valid since the failure of a fiber bundle of order $n+1$ under a stress σ always requires either the failure of at least two fibers of order n under that stress σ or the failure of one fiber of order n under the stress σ and at least one other fiber under stress $a\sigma$. Thus, $P_{n+1}(\sigma) = P_n(\sigma)^2)[\dots] + P_n(\sigma)P_n(a\sigma)[\dots]$. Accordingly, $F = x^2[\dots] + xy[\dots]$. It is easy to check that any such polynomial satisfies (C).

Condition (D) follows since $f(x)$ is defined to have the functional form $F(x, 1, \dots, 1)$, i.e. the probability of failure of a fiber of order n under the initial stress σ is x and the probability of failure of a fiber of order n under a higher stress $\frac{m\sigma}{m-1}, \frac{m\sigma}{m-2}$, etc. is given as 1. This is called the "weakest link approximation" since it presumes that if the weakest fiber fails then all surviving fibers *necessarily* fail. Therefore, $f(x)$ has the form $f(x) = 1 - (-x)^m$.

To verify condition (E) we note that the probability $P_{n+1}(\sigma)$ of failure of a fiber bundle of order $n+1$ under stress σ is not less than $P_n^m(\sigma) + mP_n(\sigma)\left\{P_n\left(\frac{m\sigma}{m-1}\right) - P_n(\sigma)\right\}^{m-1}$ where the first term is the probability of failure of one fiber of order n under the stress σ, and the second term is the probability of failure of all m fibers or order n under the stress σ and of $m-1$ other fibers under a stress between σ and $\frac{m\sigma}{m-1}$, i.e. $F(x, y, \dots) \geq x^m + mx(y-x)^{m-1}$. Therefore, $\frac{F(x,y,\dots)}{(xy^{m-1})} \geq \theta^{m+1} + m(1-\theta)^{m-1}$ where $\theta = \frac{x}{y}$. It is easy to show that $\theta^{m-1} + m(1-\theta)^{m-1} \geq \left(1 + m^{\frac{1}{m-2}}\right)^{2-m}$ for $0 < \theta \leq 1$, $m > 2$. Hence, condition (E) is valid with $q = \left(1 + m^{\frac{1}{m-2}}\right)^{2-m}$, $v = m-1$.

Condition (F) is valid as $F(x, \dots, x)$ is the probability of failure of a fiber bundle if the probability x of failure of its constitutent fibers does not depend on σ. So, in the m–fold case $F(x, \dots, x) = x^m$.

Now we are able to state our main theorems.

Theorem 1. *Let $\mathbf{F} : \mathbf{L} \to \mathbf{L}$ satisfy the conditions (A) –(D).*

Let $P \in \mathbf{L}$, and suppose that $P(\sigma) < \frac{1}{Q}$ when $\sigma \to 0$ for Q given above in condition (C). For a given value of σ, let n_σ be the minimal number n such that $P_n(\sigma) > \frac{1}{Q}$.

Then

$$\ln(n_\sigma) > \frac{c}{\sigma^\mu} \qquad \text{when} \quad \sigma \to 0. \tag{2}$$

Here $\mu = \frac{1}{\log_2(a)}$, a being a constant in condition (A), and a constant $c > 0$ depends upon the initial distribution P.

Theorem 2. *Let* $\mathbf{F} : \mathbf{L} \to \mathbf{L}$ *satisfy the conditions (A)–(E).*

Then, for any $P \in \mathbf{L}$ that satisfies $\ln\{-\ln[P(\sigma)]\} \ll \sigma^{\mu}$ when $\sigma \to 0$, $P(\sigma) < \frac{1}{Q}$ when $\sigma \to 0$, $1 - P(\sigma) < \sigma^{-\rho}$ when $\sigma \to \infty$, we have with n_σ defined as before

$$\ln(n_\sigma) = c\sigma^{\mu} + O(1) \quad \text{when} \quad \sigma \to 0. \tag{3}$$

Here $\mu = \frac{1}{\log_2(a)}$, a constant $\rho > 0$ depends only on $\mathbf{F}$, and a constant $c > 0$ depends on the initial distribution P. For the case $v = 1$ we have a better estimate

$$\ln(n_\sigma) = c\sigma^{\mu} + \ln(2) + O(\sigma^{\mu/2}) \text{ when } \sigma \to 0. \tag{4}$$

Remark 3. The asymptotic limitations on $P(\sigma)$ stated above are necessary for these theorems to be valid. It is essential that $P(\sigma)$ not vanish for finite σ, a condition employed by virtually all previous investigations on fiber bundles. The first of these conditions states that $P(\sigma)$ must not approach zero too rapidly as $\sigma \to 0$. However, this condition is met by all physically reasonable distributions, including power laws and even exponentials, such as $exp(-1/\sigma)$, although the last distribution does not satisfy the condition of Theorem 2 when $\sigma \to \infty$. Physically, the second condition requires that there be at least a "reasonable" likelihood of survival for a fiber subjected to a small amount of stress. It is easy to check that condition (F) implies $P_n(\sigma) < \frac{1}{Q}$ when $\sigma \to 0$ for some n, if $P(\sigma) < 1$ when $\sigma \to 0$, instead of $P(\sigma) < \frac{1}{Q}$ in the conditions of Theorems 1 and 2. The third condition requires that the distribution approaches unity at least as fast as the power law σ^{ρ} as $\sigma \to \infty$, where ρ depends on $\mathbf{F}$ and is given in the proof of lemma 2. For example, the case $P(\sigma) = \left(1 - \frac{c}{\sigma}\right)_+$, c a constant, which happens to be a fixed point for the transformation (1), is specifically excluded. From thermodynamic considerations, $P(\sigma)$ approaches unity exponentially fast as $\sigma \to \infty$, so we can expect this third condition to always be met in real problems.

In the following considerations we shall always suppose that $\mathbf{F}$ satisfies the conditions (A)–(F). For the proof of Theorem 1 it is not essential, for if $\mathbf{F}$ satisfies the conditions (A)–(D) then it is possible to replace it by a function $\mathbf{F}_1 \geq \mathbf{F}$ that satisfies the condition (E) as well. If the estimate (2) is valid for $\mathbf{F}_1$ then it is also valid for $\mathbf{F}$.

For the proof of the theorems 1 and 2 it will be convenient to introduce a logarithmic variable $k = -\log_a\left(\frac{\sigma}{\sigma_0}\right)$, so that $\sigma = \sigma_0 a^{-k}$. Then $\mathbf{L}$ becomes the space of all non-increasing functions $P(k)$, $-\infty < k < \infty$, $0 \leq P(k) \leq 1$, and

$$\mathbf{F}(P)(k) = F[P(k), P(k-1), \ldots]$$

where $\ldots$ means a possible dependence of F on the values $P(k')$, $k' < k - 1$. We shall write n_k instead of n_σ for $\sigma = \sigma_0 a^{-k}$ and we want to show that $\ln(n_k) \propto 2^k \propto \frac{1}{\sigma^{\mu}}$ when $k \to \infty$. Let $\Delta_k = n_k - n_k - 1$.

Lemma 1. *Δ_k increases faster than $r2^k$ when $k \to \infty$, where r is some positive constant.*

Proof. It is sufficient to consider the case when $P(k) < \theta < \dfrac{1}{Q}$ for $k \geq 0$. According to (C), $P_{n+1}(k) < QP_n(k)P_n(k-1)$. Therefore $QP_1(k) < (\theta Q)^2$ for $k \geq 1$, $QP_2(k) < (\theta Q)^4$ for $k > 2, \dots, QP_k(k) < (\theta Q)^{2^k}$. By the definition of n_k, we have $P_n(k-1) < \dfrac{1}{Q}$ for $n < n_{k-1}$. Condition (C) implies, then, that $P_k(k) > P_{k+1}(k) > P_{n_{k-1}}(k)$. Therefore

$$\frac{1}{Q} < P_{n_k}(k) < Q^{\Delta_k} P_{n_{k-1}}(k) < Q^{\Delta_k} P_k(k) < Q^{\Delta_k - 1}(\theta Q)^{2^k}. \tag{5}$$

This implies $\Delta_k > r2^k$ where $r = -\log_Q(\theta) - 1 > 0$.

Lemma 2. *There exist positive numbers ϵ, $\gamma < 1$, and j depending only on $\mathbf{F}$, such that for any distribution P with $1 - P(K - \kappa) < \epsilon\gamma^\kappa$ for $\kappa \geq 0$ the following holds:*

$$1 - P_n)(K - \kappa) < \epsilon\gamma^{n/j+\kappa} \textit{ for } n \geq 0,\ \kappa \geq 0. \tag{6}$$

Proof. Since F is a differentiable function, condition (D) implies, for any $\alpha > f'(1)$ and some $\beta > 0$,

$$1 - F(\ l(x, y, \dots) < \alpha(1 - x) + \beta(1 - y) \text{ for } \epsilon \geq 1 - x \geq 1 - y \geq \dots, \tag{7}$$

if ϵ is small enough. As $f'(1) < 1$, we can choose $\alpha < 1$ in (7). Let γ be a positive number, $\gamma < \min\left(1, \dfrac{1-\alpha}{\beta}\right)$. Since $\alpha + \beta\gamma < 1$, we have $(\alpha + \beta\gamma)^j < \gamma$ for some $j \geq 1$. Let us prove (6) using induction on n. For $n = 0$ it follows from the conditions of the lemma. Suppose it is valid for some n. Then

$$1 - P_n(K - \kappa) < \epsilon\gamma^{n/j+\kappa}, 1 - Pn(K - \kappa - 1) < \epsilon\gamma^{n/j+\kappa+1} \quad \text{for } \kappa \geq 0.$$

It follows then from (7) that

$$1 - P_{n+1}(K - \kappa) < \epsilon\gamma^{n/j+\kappa}(\alpha + \beta) < \epsilon\gamma^{(n+1)/j+\kappa} \quad \text{for } \ \kappa \geq 0,$$

q.e.d.

Remark 4. A constant ρ in the formulation of theorem 2 can be taken as $-\log_a \gamma$. For the transformation (1) we have $f'(1) = 0$, $\alpha = \epsilon$, $\beta = 2$, and for ϵ small enough any $\gamma < \dfrac{1}{2}$ can be taken in lemma 2. This means that $\rho > 1$ and $P(\sigma)$ must approach unity faster than $\dfrac{1}{\sigma}$ as $\sigma \to \infty$.

Lemma 3. *If a distribution P satisfies conditions of Lemma 2 for $K = k-1$ and $P(k) > \frac{1}{Q}$ then P_n satisfies conditions of Lemma 2 for $K = k$ for all $n \geq M$. Here a constant M depends only on* $\mathbf{F}$.

Proof. It is sufficient to show that $1 - P_M(k) < \epsilon$ for some M depending only on $\mathbf{F}$. It follows from the conditions (A) and (C) that F is divisible by x, $F(x, y, \dots) = xG(x, y, \dots)$ where G is continuously differentiable. So,

$$F((x, y, \dots) > f(x) - \zeta x(1 - y)$$

with $\zeta > 0$ depending upon F, where $f(x) = F(x, 1, \dots)$. Condition (D) implies that for some constants $m > 0$, $A > 1$ depending on F,

$$f(x) - \zeta x \in \gamma^{m/j} > Ax \text{ for } 0 < x < 1 - \epsilon.$$

The statement of the lemma follows from these two estimates, condition (E), and Lemma 2 for $K = k - 1$.

Proof of Theorem 1. Condition (B) implies that for any two distribution functions P and R such that $P(k) \leq R(k)$ for all k we have $P_n(k) \leq R_n(k)$ for all k and all $n \geq 0$. Therefore it is enough to prove the inequality (2) for distribution functions satisfying conditions of lemma 2 with $K = 0$. It follows from lemma 1 that $\Delta_k > M$, if k is large enough. We can suppose that $\Delta_k > M$ for all $k \geq 0$. Lemma 3 applied to $P = P_{n_k}$ implies then that for $k \geq 0$ the distribution function $P = P_{n_k+M}$ satisfies the conditions of lemma 2 with $K = k$. In particular, we have

$$1 - P_{n_{k-1}+n}(k - 1) < \epsilon\gamma^{(n-M)/j} \text{ for } n \geq M,$$

i.e. $P_n(k - 1) \approx 1$ and $F[P_n(k), P_n(k - 1), \dots] \approx f[P_n(k)]$ for $n \gg n_{k-1}$. Condition (D) implies that the transformation $x \to f(x)$ has an unstable fixed point with a basin of repellance $0 \leq x < 1$ and $f(x) \approx Sx$ for $x \to 0$ where $S = f'(0)$. As $P_{n_k-n}(k) < \frac{1}{Q}$ for $n > 0$, this implies

$$bS^{-n} < P_{n_k-n}(k) < BS{-}n \text{ for } 0 \leq n \leq \Delta_k \tag{8}$$

where constants b and B depend only on $\mathbf{F}$. The essential idea of the proof of (8) emerges, since is is easy to show that $P_{n_k-n}(k) < B_1 S_1^{-n}$ for $0 \leq n \leq \Delta_k$ with some constants $B_1 > 0$, $S > S_1 > 1$ depending on $\mathbf{F}$. It follows from the conditions (A) and (C) that F is divisible by x, $F(x, y, \dots) = xG(x, y, \dots)$ where $\mathbf{G}$ is continuously differentiable. So, $\mathbf{F}(x, y, \dots) = f(x) + xO(1 - y) = Sx((1 + O(x) + O(1 - y))$. Substituting here $x_n = P_{n_k-n}(k) = O(S_1^{-n})$, $y_n = P_{n_k-n}(k-1) = 1 - O(\gamma^{(\Delta_k n)/j})$, we have for $0 \leq n \leq \Delta_k$,

$$x_{n-1} = F(x_n, y_n, \dots) = Sx_n[1 + O(S_1^{-n}) + O(\gamma^{(\Delta_k - n)/j})].$$

Let $z_n = \log(x_n)$, $Z = \log(S)$. Then,

$$z_{n-1} = Z + z_n + O(S_1^{-n}) + O(\gamma^{(\Delta_k - n)/j}).$$

As O here contains a convergent series,

$$z_n = z_0 - nZ + O(1).$$

As $x_0 = P_{n_k}(k) \approx 1$, we have $bS^{-n} < x_n = P_{n_k - n}(k) < BS^{-n}$. Then, condition (C) implies $P_{n+1}(k+1) < QP_n(k+1)P_n(k)$. Therefore

$$\begin{aligned} P_{n_k}(k+1) &< QBP_{n_k-1}(k+1)S^{-1} < (QB)^2 P_{n_k-2}(k+1)S^{-3} \\ &< (QB)^{\Delta_k} P_{n_k-1}(k+1)S^{-\Delta_k(\Delta_k-1)/2} < (QB)^{\Delta_k} S^{-\Delta_k(\Delta_k-1)/2}. \end{aligned}$$

Let J be a number such that $S^J > QB$. Then

$$P_{n_k}(k+1) < S^{-\Delta_k(\Delta_k - J - 1)/2}.$$

Inequality (8) with $k+1$ instead of k implies that $P - n_k(k+1) > bS^{-\Delta_{k+1}}$. Therefore

$$bS^{-\Delta_{k+1}} < S^{-\Delta_k(\Delta_k - J - 1)/2}$$

and

$$\Delta_{k+1} > \frac{\Delta_k^2}{2} - O(\Delta_k).$$

Hence

(9) $$\ln(\Delta_{k+1} > 2\ln(\Delta_k) - \ln(2) - O(\Delta_k^{-1}).$$

As the sequence Δ_k increases according to lemma 1, inequality (9) implies that the sequence $\ln(\Delta_k)/2^k$ is bounded from below by a positive constant c, i.e. $\ln(\Delta_k) > c2^k$, q.e.d.

Proof of Theorem 2. The condition $1 - P(\sigma) < \sigma^{-\rho}$ of theorem 2 means that $1 - P(k) < \gamma^k$ for some $\gamma < 1$ when $k \to \infty$. Let us take γ to be the same as in Lemma 2. Then, $P(k)$ satisfies the conditions of Lemma 2; hence, inequality (8) is applicable, namely

(10) $$bS^{-n} < P_{n_k - n}(k) < BS^{-n} \quad \text{for } 0 \le n \le \Delta_k$$

where the constants b and B depend only on $\mathbf{F}$, and $S = f'(0)$. Condition (E) implies $P_{n+1}(k+1) > qP_n(k+1)P_n(k)^\nu$. Therefore

$$\begin{aligned} P_{n_k}(k+1) &> qb^\nu P_{n_k-1}(k+1)S^{-\nu} > qb^\nu)^2 P_{n_k-2}(k+1)S^{-3\nu} \\ &> (qb^\nu)^{\Delta_k} P_{n_{k-1}}(k+1)S^{-\nu\Delta_k(\Delta_k-1)/2}. \end{aligned}$$

Let J be a number such that $S^{-\nu J} < qb^{\nu}$. Then

$$P_{n_k}(k+1) > P_{n_k-1}(k+1)S^{-\nu\Delta_k(\Delta_k+2J-1)/2}. \tag{11}$$

Condition (C) implies that $P_n(k)$ decrease with increasing n when $n < n_{k-1}$. This, together with condition (E) implies

$$P_{n_{k-1}}(k+1) > q^{n_{k-1}}P(k+1)[P_{n_{k-1}}(k)]^{\nu n_{k-1}}.$$

Applying (10) for $n = n_{k-1}$, we have

$$P_{n_{k-1}}(k+1) > q^{n_{k-1}}P(k+1)(bS^{-\Delta_k})^{\nu n_{k-1}}.$$

Replacing qb^{ν} by $S^{-\nu J}$ in the latter, we have

$$P_{n_{k-1}}(k+1) > P(k+1)S^{-\nu(\Delta_k+J)n_{k-1}}. \tag{12}$$

Theorem 1 implies $n_{k-1} = O(\Delta_{k-1})$. This, together with formulas (11) and (12), gives

$$P_{n_k}(k+1) > P(k+1)S^{-\nu\Delta_k(\Delta_k+O(\Delta_{k-1}))/2}.$$

Applying inequality (10) with $k+1$ instead of k, we have

$$S^{-\Delta_{k+1}} > P(k+1)S^{-\nu\Delta_k(\Delta k+O(\Delta_{k-1})/2)},$$

i.e.

$$\Delta_{k+1} < \frac{\nu\Delta_k(\Delta_k + O(\Delta_{k-1}))}{2} - \log S[P(k+1)]. \tag{13}$$

It follows from theorem 1 that $\ln(\Delta_k) > c2^k$. According to the conditions of theorem 2, $\ln\{-\ln[P(k)]\} \ll 2^k$ when $k \to \infty$, hence $-\log_S[P(k+1)] = o(\Delta_k)$. Morover, it follows from (9) that $\Delta_k < O(\Delta_k^{1/2})$. This, together with (13), gives

$$\Delta_{k+1} < \nu\Delta_k^2/2 + O(\Delta_k^{3/2}),$$

$$\ln(\Delta_{k+1}) < 2\ln(\Delta_k) + \ln(\nu/2) + O(\Delta_k^{-1/2}). \tag{14}$$

It follows from the inequalities (9) and (14) that

$$\begin{aligned}\ln(\Delta_k)/2k - \ln(2)/2^{k+1} + O(\Delta_k^{-1}/2^{k+1}) &< \ln(\Delta_{k+1})/2^{k+1} \\ &< \ln(\Delta_k)/2^k + \ln(\nu/2)/2^{k+1} + O(\Delta_k^{-1/2})/2^{k+1}).\end{aligned}$$

Therefore the sequence $\ln(\Delta_k)/2^k$ converges to a constant c, and

$$c - \ln(\nu/2)/2^k + O(\Delta_k^{-1/2}/2^k) < \ln(\Delta_k)/2^k < c + \ln(2)/2^k + O(\Delta_k^{-1}/2^{k+1}).$$

Since, according to lemma 1, the sequence Δ_k increases, we have $c > 0$. Therefore $\ln(\Delta_k) = c2^k + O(1)$, and if $\nu = 1$, then $\ln(\Delta_k) = c2^k + \ln(2) + O(2^{-k/2})$, q.e.d.

In paper I, we explored by computational means the failure properties of hierarchically organized fiber bundles with equal load sharing. The outcome of our computational investigation of this exactly renormalizeable problem revealed an apparently universal asymptotic scaling law relating the stress threshold for failure to the size of the system. In particular, we showed that the bound identified by the widely held weakest link theory did not provide a realistic indication of the threshold for failure. Indeed, we showed that the threshold decreased so slowly as to be practically indistinguishable from a critical point. In this paper, we have employed the insight developed in paper I to construct a rigorous derivation of the asymptotic properties of a hierarchy of fiber bundles. As a practical outcome of the universality of the failure properties of this mode of assembling fiber bundles, we believe that there may be some advantage to constructing fiber bundles in engineering applications in this way.

Acknowledgements

This work was performed under the auspices of Environmental Protection Agreement 02.09-13 between the United States of America and the Union of Soviet Socialist Republics. The authors wish to thank the host institutions, including the U.S. Geological Survey, the University of California, and the USSR Academy of Science, respectively, for their hospitality during this scientific exchange. We wish to thank the National Science and Engineering Research Council of Canada for its support of this work during a visit to the Department of Physics at the University of Alberta in Edmonton, Canada. One of us (W.I.N.) wishes to thank the John Simon Guggenheim Memorial Foundation for its support of this work during an extended visit to the Department of Astronomy at Cornell University. The authors are also grateful to D.L Turcotte, S.L. Phoenix, A. Ruina, J. Stephenson, V. Pisarenko, and L. Knopoff for fruitful discussions. We also wish to thank an anonymous referee for very careful and helpful comments.

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