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On the Number of Connected Components of the Relative Closure of a Semi-Pfaffian Family

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Abstract. The notion of relative closure $(X,Y)_0$ of a semi-Pfaffian couple (X,Y) was introduced by Gabrielov to give a description of the o-minimal structure generated by Pfaffian functions. In this paper, an effective bound is given for the number of connected components of $(X,Y)_0$ in terms of the Pfaffian complexity of X and Y.

Introduction

Pfaffian functions are real analytic functions that are solutions to certain triangular systems of polynomial partial differential equations (see Definition 1). They were introduced by Khovanskiĭ [**12**], who showed that these transcendental functions exhibit, in the real domain, global finiteness properties similar to the properties of polynomials. It follows in turn that the geometrical and topological characteristics of sets defined using Pfaffian functions are also well controlled. Effective upper bounds for those geometric properties can be found in [**3, 5, 7, 8, 18, 22**].

O-minimality is a natural framework for the study of Pfaffian functions. (The reader can refer to van den Dries [**2**] for definitions.) Wilkie proved in [**21**] that the structure generated by Pfaffian functions is o-minimal (see also [**9, 19**] for generalizations of this result.)

In [**6**], Gabrielov introduced the notion of *relative closure* of a semi-Pfaffian couple, as an alternative to Wilkie's construction. In this way, he obtained an o-minimal structure in which definable sets (called *limit sets*) have a simple presentation. As a result, this structure supports a notion of complexity which naturally extends the usual Pfaffian complexity, and allows one to estimate the complexity of Boolean operations in that structure.

In this paper, we use this notion of complexity to give an explicit bound on the number of connected components of a limit set. In the first part, we recall the usual definitions related to Pfaffian functions, then we cover the essential material we will use from [**6**]. In the second part, we give explicit bounds for the smooth case (Theorem 15) and for the general case (Theorem 17). Section 3 is devoted to two applications to the fewnomial case: Theorem 19 and Theorem 20.

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Notations. If X is a subset of $\mathbb{R}^n$, we'll denote by $\overline{X}$ its closure, and $\partial X = \overline{X} \setminus X$ its frontier. Unless otherwise noted, all subsets X are assumed to be relatively compact.

1. Semi-Pfaffian sets and relative closure

We will recall in this section the usual definitions of Pfaffian functions and semi-Pfaffian sets as given by Khovanskiĭ. Then, we will define the relative closure of a semi-Pfaffian couple that appears in [**6**].

1.1. Pfaffian functions. Let $\mathscr{U} \subseteq \mathbb{R}^n$ be an open domain. The following definitions are due to Khovanskiĭ [**12**] (see also [**10, 11**]).

Definition 1. *A sequence $(f_1, \dots, f_\ell)$ of functions that are defined and analytic in $\mathscr{U}$ is called a* Pfaffian chain *if it satisfies on $\mathscr{U}$ a differential system of the form:*

$$df_i = \sum_{j=1}^{n} P_{i,j}(x, f_1(x), \dots, f_i(x)) dx_j, \tag{1}$$

where each $P_{i,j}$ is a polynomial in $x, f_1, \dots, f_i$, and the following holds.

(P1) *The graph Γ_i of f_i is contained in a domain Ω_i defined by polynomial inequalities in $(x, f_1(x), \dots, f_{i-1}(x), t)$, and such that $\partial\Gamma_i \subseteq \partial\Omega_i$.*
(P2) *Γ_i is a* separating submanifold *in Ω_i,* i.e. *$\Omega_i \setminus \Gamma_i$ is a disjoint union of two open sets Ω_i^+ and Ω_i^-. (See* [**12**, p. 38]. *This is also called the* Rolle leaf *condition in the terminology of* [**14, 15**].)

If $\alpha \in \mathbb{N}$ is a bound on the degrees of the polynomials $P_{i,j}$, we say that the degree *of the chain is (at most) α. The integer ℓ is called the* length *of the chain.*

A Pfaffian function *$q(x)$ with the chain $(f_1, \dots, f_\ell)$ is any function that can be written as*

$$q(x) = Q(x, f_1(x), \dots, f_\ell(x)),$$

for some polynomial $Q(x, y_1, \dots, y_\ell)$. If the degree of Q is β, we say that the degree of the function $q(x)$ is (at most) β.

Example 2. *For any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ the function $e_a : x \mapsto \exp(a \cdot x)$ is a Pfaffian function on $\mathbb{R}^n$, since*

$$de_a(x) = \sum_{j=1}^{n} a_j \, e_a(x) dx_j.$$

Pfaffian functions form a rather large class of functions. Note in particular that elementary functions can be seen as Pfaffian functions on appropriate subsets of their domain of definition (see the Chapter 1 of [**12**] or [**7**] for more precise statements).

In sections 2 and 3, we will take $\mathscr{U}$ to be of the form $\mathbb{R}^n$, $\mathbb{R}^{n-1} \times \mathbb{R}_+$, or $(\mathbb{R}_+)^n$. In that case, we will use the following result which is due to Khovanskiĭ [**12**, p. 79]. The reader can also refer to the revised edition (in Russian) [**13**].

THEOREM 3. *Let* $(f_1, \dots, f_\ell)$ *be a Pfaffian chain on* $\mathscr{U}$, *where* $\mathscr{U}$ *is one of* $\mathbb{R}^n$, $\mathbb{R}^{n-1} \times \mathbb{R}_+$, *or* $(\mathbb{R}_+)^n$. *If* $q_1, \dots, q_n$ *are Pfaffian functions of respective degrees* β_i, *the number of solutions of the system*

$$q_1(x) = \cdots = q_n(x) = 0,$$

for which the Jacobian determinant is not zero is bounded by

$$2^{\ell(\ell-1)/2}\beta_1 \cdots \beta_n \left[\beta_1 + \cdots + \beta_n - n + \min(n,\ell)\alpha + 1\right]^\ell. \tag{2}$$

Remark. It is possible to give effective estimates when the domain $\mathscr{U}$ is not of the above form, but one has to modify the estimates to take into account the complexity of the domain.

1.2. Semi-Pfaffian sets. Let $(f_1, \dots, f_\ell)$ be a Pfaffian chain of degree α, defined on a domain $\mathscr{U} \subseteq \mathbb{R}^n$.

DEFINITION 4. *A* basic semi-Pfaffian set *is a set* X *of the form:*

$$X = \{x \in \mathscr{U} \mid \varphi_i(x) = 0,\ \psi_j(x) > 0,\ \textit{for } i = 1,..,I;\ j = 1,..,J\}, \tag{3}$$

where all the functions above are Pfaffian with the chain $(f_1, \dots, f_\ell)$. *If all these functions have degree at most* β, *the set* X *is said to have* format $(I, J, n, \ell, \alpha, \beta)$.

A semi-Pfaffian set *is any finite union of basic semi-Pfaffian sets. A semi-Pfaffian set has format* $(N, I, J, n, \ell, \alpha, \beta)$ *if it is the union of at most* N *basic semi-Pfaffian sets having formats component-wise not exceeding* $(I, J, n, \ell, \alpha, \beta)$.

We say that a semi-Pfaffian set X *is* restricted *if it is relatively compact in* $\mathscr{U}$.

DEFINITION 5. *A basic semi-Pfaffian set is called* (effectively) non-singular *if for all* $x \in X$, *the functions* φ_i *appearing in* (3) *verify*

$$d\varphi_1(x) \wedge \cdots \wedge d\varphi_I(x) \neq 0.$$

1.3. Effective bounds on the Betti numbers. Following ideas of Oleĭnik-Petrovskiĭ [**17**], Milnor [**16**] and Thom [**20**] for algebraic varieties, we can use the bound appearing in Theorem 3 to estimate the sum of the Betti numbers of a semi-Pfaffian set [**12, 22**].

Assume X is a compact semi-Pfaffian set defined only by equations,

$$X = \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_r(x) = 0\}, \tag{4}$$

and let $p = p_1^2 + \cdots + p_r^2$. Then, it can be shown that the sum of the Betti numbers of X is exactly one-half the sum of the Betti numbers of the compact components of the set $\{p = \varepsilon\}$, where ε is a small positive real number.

Counting the number of critical points of a generic projection, we obtain the following bound.

COROLLARY 6. *Assume* $X \subseteq \mathbb{R}^n$ *is of the form* (4), *where* $p_1, \dots, p_r$ *are of degree at most* β *in a Pfaffian chain* $(f_1, \dots, f_\ell)$ *defined in* $\mathscr{U} = \mathbb{R}^n$ *with a length* ℓ *and degree* α. *Then, the sum of the Betti numbers of* X *is bounded by*

$$2^{\ell(\ell-1)/2}\beta(\alpha + 2\beta - 1)^{n-1}[(2n-1)\beta + (n-1)(\alpha - 2) + \min(n,\ell)\alpha]^\ell. \tag{5}$$

1.4. Relative closure. ¿From now on, we consider semi-Pfaffian subsets of $\mathbb{R}^n \times \mathbb{R}_+$ with a fixed Pfaffian chain $(f_1, \dots, f_\ell)$ in a domain $\mathscr{U}$. We write $(x_1, \dots, x_n)$ for the coordinates in $\mathbb{R}^n$ and λ for the last coordinate (which we think of as a parameter.) If X is such a subset, we let $X_\lambda = \{x \mid (x, \lambda) \in X\} \subseteq \mathbb{R}^n$ and consider X as the family of its fibers X_λ.

We let $X_+ = X \cap \{\lambda > 0\}$ and $\check{X} = \overline{X_+} \cap \{\lambda = 0\}$. The following definitions appear in [**6**].

Definition 7. *Let X be a relatively compact semi-Pfaffian subset of $\mathbb{R}^n \times \mathbb{R}_+$. The family X_λ is said to be a* semi-Pfaffian family *if for any $\varepsilon > 0$, the set $X \cap \{\lambda > \varepsilon\}$ is restricted. (See Definition)*

The format *of the family X is the format of the fiber X_λ for a small $\lambda > 0$.*

Remark. Note that the format of X as a semi-Pfaffian set is different from its format as a semi-Pfaffian family.

Definition 8. *Let X and Y be semi-Pfaffian families in $\mathscr{U}$ with a common chain $(f_1, \dots, f_\ell)$. They form a* semi-Pfaffian couple *if the following properties are verified:*

- $(\overline{Y})_+ = Y_+$;
- $(\partial X)_+ \subseteq Y$.

Then, the format *of the couple (X, Y) is the component-wise maximum of the formats of the families X and Y.*

Definition 9. *Let (X, Y) be a semi-Pfaffian couple in $\mathscr{U}$. We define the* relative closure *of (X, Y) at $\lambda = 0$ by*

$$(X, Y)_0 = \check{X} \setminus \check{Y} \subseteq \check{\mathscr{U}}. \tag{6}$$

Definition 10. *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. A* limit set *in Ω is a set of the form $(X_1, Y_1)_0 \cup \dots \cup (X_K, Y_K)_0$, where (X_i, Y_i) are semi-Pfaffian couples respectively defined in domains $\mathscr{U}_i \subseteq \mathbb{R}^n \times \mathbb{R}_+$, such that $\check{\mathscr{U}}_i = \Omega$ for $1 \le i \le K$. If the formats of the couples (X_i, Y_i) are bounded component-wise by $(N, I, J, n, \ell, \alpha, \beta)$, we say that the* format *of the limit set is $(K, N, I, J, n, \ell, \alpha, \beta)$.*

Example 11. *Any (not necessarily restricted) semi-Pfaffian set X is a limit set.*

Proof. It is enough to prove the result for a basic set $X \subseteq \mathscr{U}$ of the form (3). Assume $\mathscr{U} = \{x \in \mathbb{R}^n \mid g_i(x) > 0,\ 1 \le i \le r\}$. Let $\psi = \psi_1 \cdots \psi_J$ and $g = g_1 \cdots g_r$. Define the sets

$$W = \left\{(x, \lambda) \in X \times \Lambda \mid g(x) > \lambda,\ |x| < \lambda^{-1}\right\};$$
$$Y_1 = \left\{(x, \lambda) \in \mathscr{U} \times \Lambda \mid \varphi_1(x) = \dots = \varphi_I(x) = 0,\ \psi(x) = 0,\ g(x) \ge \lambda,\ |x| \le \lambda^{-1}\right\};$$
$$Y_2 = \left\{(x, \lambda) \in \mathscr{U} \times \Lambda \mid \varphi_1(x) = \dots = \varphi_I(x) = 0,\ g(x) = \lambda,\ |x| \le \lambda^{-1}\right\};$$
$$Y_3 = \left\{(x, \lambda) \in \mathscr{U} \times \Lambda \mid \varphi_1(x) = \dots = \varphi_I(x) = 0,\ g(x) \ge \lambda,\ |x| = \lambda^{-1}\right\};$$

where $\Lambda = (0, 1]$. If $Y = Y_1 \cup Y_2 \cup Y_3$, it is clear that (W, Y) satisfies the requirements of Definition 8. Thus (W, Y) is a semi-Pfaffian couple; its relative closure is X. □

For all $n \in \mathbb{N}$ we let $\mathcal{S}_n$ be the collection of limit sets in $\mathbb{R}^n$, and $\mathcal{S} = \cup_{n \in \mathbb{N}} \mathcal{S}_n$. The following theorem sums up the results in [**6**, Theorems 2.9 and 5.1].

THEOREM 12. *The structure $\mathcal{S}$ is o-minimal. Moreover, if X is a definable set defined by a formula involving the limit sets $L_1, \dots, L_N$, X can be presented as a limit set whose format is bounded by an effective function of the formats of $L_1, \dots, L_N$.*

2. Connected components of a limit set

In this section, we establish effective bounds on the number of connected components of the relative closure of a semi-Pfaffian couple (X, Y). In what follows, we will always assume that Y is not empty. Note that if Y is empty, it follows from Definition 8 that $(\partial X)_+$ must be empty too; then, X_λ is compact and the number of connected components of X_0 is at most the number of connected components of X_λ. Effective bounds for this case appear in [**12, 22**].

As announced in the introduction, we assume as in [**6**] that X and Y are relatively compact. To keep the estimates simple, we'll also assume that the Pfaffian chain $(f_1, \dots, f_\ell)$ is defined over the whole of $\mathbb{R}^n \times \mathbb{R}_+$, so that the bound given in Theorem 3 is applicable.

2.1. General considerations.

We show here how to reduce the problem of counting the number of connected components of a limit set to a problem in the semi-Pfaffian setting.

Let Φ be the (squared) distance function on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} \Phi : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x,y) &\mapsto |x-y|^2 \end{aligned} \tag{7}$$

For any $\lambda > 0$, we can define the distance to Y_λ, Ψ_λ on X_λ by:

$$\Psi_\lambda(x) = \min_{y \in Y_\lambda} \Phi(x, y). \tag{8}$$

Define similarly for $x \in \check{X}$:

$$\Psi(x) = \min_{y \in \check{Y}} \Phi(x, y). \tag{9}$$

The following result appears in [**6**, Theorem 2.12]. The proof is presented here for the sake of self-containment.

THEOREM 13. *Let (X, Y) be a semi-Pfaffian couple. Then, there exists $\lambda \ll 1$ such that for every connected component C of $(X, Y)_0$, we can find a connected component D_λ of the set of local maxima of Ψ_λ such that D_λ is arbitrarily close to C.*

PROOF. Let C be a connected component of $(X, Y)_0$. Note that by definition of the relative closure, if x is in C, it cannot be in $\check{Y}$. So we must have $\Phi(x, y) > 0$ for all $y \in \check{Y}$, and since $\check{Y}$ is compact, we must have $\Psi(x) > 0$. Also, any point in ∂C must be in $\check{X}$, but not in $(X, Y)_0$. So we must have $\partial C \subseteq \check{Y}$, hence $\Psi|_{\partial C} \equiv 0$. This means that the restriction of Ψ to C takes its maximum inside of C.

Choose $x_0 \in C$, and let $c = \Psi(x_0) > 0$. For a small λ, there is a point $x_\lambda \in X_\lambda$ close to x_0 such that $c_\lambda = \Psi_\lambda(x_\lambda)$ is close to c, and is greater than the maximum of the values of Ψ_λ over points of X_λ close to ∂C. Hence the set $\{x \in X_\lambda \mid \Psi_\lambda(x) \geq c_\lambda\}$ is nonempty, and the connected component A_λ of this set that contains x_λ is close to C. There exists a local maximum $x_\lambda^* \in A_\lambda$ of Ψ_λ. If D_λ is the connected component in the set of local maxima of Ψ_λ, it is contained in Z_λ and is close to C. $\square$

2.2. A bound for the smooth case. We will now show how the number of connected components of the set of local maxima of Ψ_λ that appear in Theorem 13 can be estimated when the sets X_λ and Y_λ are smooth.

Define for all p,

$$Z^p_\lambda = \{(x, y_0, \dots, y_p) \in W^p_\lambda \mid \Phi(x, y_0) = \dots = \Phi(x, y_p)\}, \tag{10}$$

where

$$W^p_\lambda = \{(x, y_0, \dots, y_p) \in X_\lambda \times (Y_\lambda)^{p+1} \mid y_i \neq y_j, 0 \le i < j \le p\}. \tag{11}$$

LEMMA 14. *Assume (X, Y) is a Pfaffian couple such that X_λ and Y_λ are smooth for all $\lambda > 0$. For a given $\lambda > 0$, let x^* be a local maximum of $\Psi_\lambda(x)$. Then, there exists $0 \le p \le \dim(X_\lambda)$ and a point $z^* = (x^*, y^*_0, \dots, y^*_p) \in Z^p_\lambda$ such that Z^p_λ is smooth at z^*, and z^* is a critical point of $\Phi(x, y_0)$ on Z^p_λ.*

PROOF. Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y^*_0 \in Y_\lambda$ such that $\Phi(x^*, y^*_0) = \min_{y \in Y_\lambda} \Phi(x^*, y) = \Psi_\lambda(x^*)$. In particular, $d_y\Phi(x, y) = 0$ at $(x, y) = (x^*, y^*_0)$. If (x^*, y^*_0) is a critical point of $\Phi(x, y)$ (this is always the case when $\dim(X_\lambda) = 0$) the statement holds for $p = 0$. Otherwise $d_x\Phi(x, y^*_0) \neq 0$ at $x = x^*$. Let ξ be a tangent vector to X at x^* such that $d_x\Phi(x^*, y^*_0)(\xi) > 0$.

Assume that for all $y \in Y_\lambda$ such that $\Phi(x^*, y) = \Psi_\lambda(x^*)$, we have $d_x\Phi(x, y)(\xi) > 0$ when $x = x^*$. Let $\gamma(t)$ be a curve on X_λ such that $\gamma(0) = x^*$ and $\dot\gamma(0) = \xi$. For all $y \in Y_\lambda$, there exists T_y such that for all $0 < t < T_y$, the inequality $\Phi(\gamma(t), y) > \Phi(x^*, y)$ holds. By compactness of Y_λ, this means we can find some t such that that inequality holds for all $y \in Y_\lambda$. Hence, $\Psi_\lambda(\gamma(t)) > \Psi_\lambda(x^*)$, which contradicts the hypothesis that Ψ_λ has a local maximum at x^*.

Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y^*_1 \in Y_\lambda$ such that $d_x\Phi(x, y^*_1)(\xi) \le 0$ at $x = x^*$ and $\Phi(x^*, y^*_1) = \Psi_\lambda(x^*)$. In particular, $y^*_1 \neq y^*_0$, $d_y\Phi(x^*, y) = 0$ at $y = y^*_1$, and $d_x\Phi(x, y^*_1) \neq d_x\Phi(x, y^*_0)$ at $x = x^*$. This implies that $(x^*, y^*_0, y^*_1) \in Z^1_\lambda$, and Z^1_λ is smooth at (x^*, y^*_0, y^*_1). If (x^*, y^*_0, y^*_1) is a critical point of $\Phi(x, y_0)$ on Z^1_λ (this is always the case when $\dim(X_\lambda) = 1$) the statement holds for $p = 1$. Otherwise $d_x\Phi(x, y^*_0)$ and $d_x\Phi(x, y^*_1)$ are linearly independent at $x = x^*$. Since $\dim(X_\lambda) \ge 2$, there exists a tangent vector ξ to X_λ at x^* such that $d_x\Phi(x^*, y^*_0)(\xi) > 0$ and $d_x\Phi(x^*, y^*_0)(\xi) > 0$. Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y^*_2 \in Y_\lambda$ such that $d_x\Phi(x, y^*_2)(\xi) \le 0$ at $x = x^*$ and $\Phi(x^*, y^*_2) = \Psi_\lambda(x^*)$. This implies that $(x^*, y^*_0, y^*_1, y^*_2) \in Z^2_\lambda$, and Z^2_λ is smooth at $(x^*, y^*_0, y^*_1, y^*_2)$. The above arguments can be repeated now for Z^2_λ, Z^3_λ, etc., to prove the statement for all $p \le \dim(X_\lambda)$. □

Assume now that X_λ and Y_λ are *effectively* non-singular, *i.e.* they are of the following form:

$$\begin{aligned} X_\lambda &= \{x \in \mathbb{R}^n \mid p_1(x, \lambda) = \dots = p_{n-d}(x, \lambda) = 0\}; \\ Y_\lambda &= \{y \in \mathbb{R}^n \mid q_1(y, \lambda) = \dots = q_{n-k}(y, \lambda) = 0\}; \end{aligned} \tag{12}$$

where, for all $\lambda > 0$, we assume that $d_xp_1 \wedge \dots \wedge d_xp_{n-d} \neq 0$ on X_λ and that $d_yq_1 \wedge \dots \wedge d_yq_{n-k} \neq 0$ on Y_λ. In particular, we have $\dim(X_\lambda) = d$ and $\dim(Y_\lambda) = k$.

Remark. Note that we assume that no inequalities appear in (12). We can clearly make that assumption for Y_λ, since that set has to be closed for all $\lambda > 0$. For X_λ, we observe the following: if C is a connected component of C^p_λ, the critical set of

$\Phi|_{Z^p_\lambda}$, the function Φ is constant on C. If C contains a local maximum for Ψ_λ, it cannot meet ∂X_λ because $\partial X_\lambda \subseteq Y_\lambda$. Hence, we do not need to take into account the inequalities appearing in the definition of X_λ.

Let us now define for all p,

$$\theta_p : (y_0, \dots, y_p) \in (Y_\lambda)^{p+1} \mapsto \sum_{0 \le i < j \le p} |y_i - y_j|^2. \tag{13}$$

Then, for X_λ and Y_λ as in (12), the sets Z^p_λ are defined for all p by the following conditions.

$$\begin{cases} p_1(x,\lambda) = \cdots = p_{n-d}(x,\lambda) = 0; \\ q_1(y_i,\lambda) = \cdots = q_{n-k}(y_i,\lambda) = 0, \quad 0 \le i \le p; \\ \Phi(x,y_i) - \Phi(x,y_j) = 0, \quad 0 \le i < j \le p; \end{cases} \tag{14}$$

and the inequality

$$\theta_p(y_0, \dots, y_p) > 0. \tag{15}$$

Under these hypotheses, we obtain the following bound.

Theorem 15. *Let (X,Y) be a semi-Pfaffian couple such that for all small $\lambda >$ 0, X_λ and Y_λ are effectively non-singular basic sets of dimension respectively d and k. If the format of (X,Y) is $(1, I, J, n, \ell, \alpha, \beta)$, the number of connected components of $(X,Y)_0$ is bounded by*

$$2 \sum_{p=0}^{d} 2^{q\ell(q\ell-1)/2} \beta_p (\alpha + 2\beta_p)^{nq-1} [nq(2\beta_p + \alpha - 2) + q \min(n,\ell)\alpha]^{q\ell}, \tag{16}$$

where $q = p+2$ and $\beta_p = \max\{1 + (n-k)(\alpha+\beta-1), 1 + (n-d+p)(\alpha+\beta-1)\}$.

Proof. According to Theorem 13, we can chose $\lambda > 0$ such that for any connected component C of $(X,Y)_0$, we can find a connected component D_λ of the set of local maxima of Ψ_λ such that D_λ is close to C. We see that for λ small enough, two connected components C and C' of $(X,Y)_0$ cannot share the same connected component D_λ, since D_λ cannot meet $\check{Y}$ for λ small enough. Indeed, the distance from D_λ to $\check{Y}$ is bounded from below by the distance from D_λ to Y_λ, – which is at least c_λ, – minus the distance between Y_λ and $\check{Y}$. But the latter distance goes to zero, whereas the former goes to a positive constant c when λ goes to zero.

Once that λ is fixed, all we need to do is estimate the number of connected components of the set of local maxima of Ψ_λ. According to Lemma 14, we can reduce to estimating the number of connected components of the critical sets C^p_λ of the restriction $\Phi|_{Z^p_\lambda}$ for $0 \le \lambda \le d$.

For the sake of concision, we will drop λ from the notations in this proof, writing Z^p for Z^p_λ, $p_i(x)$ for $p_i(x,\lambda)$, *etc* ...

A point $z = (x, y_0, \dots, y_p) \in Z^p$ is in C^p if and only if the following conditions are satisfied:

$$\begin{cases} d_y\Phi(x,y_j) = 0, \quad 0 \le j \le p; \\ \operatorname{rank}(d_x\Phi(x,y_0), \dots, d_x\Phi(x,y_p)) \le p. \end{cases} \tag{17}$$

For X_λ and Y_λ as in (12), those conditions translate into:

(18)
$$\begin{cases}\operatorname{rank}\{\nabla_y q_1(y_i),\dots,\nabla_y q_{n-k}(y_i),\nabla_y\Phi(x,y_i)\}\le n-k, \quad 0\le i\le p;\\ \operatorname{rank}\{\nabla_x p_1(x),\dots,\nabla_x p_{n-d}(x),\nabla_x\Phi(x,y_0),\dots,\nabla_x\Phi(x,y_0)\}\le n-d+p.\end{cases}$$

Those conditions translate into all the maximal minors of the corresponding matrices vanishing. These minors are Pfaffian functions in the chain used to define X and Y. Their degrees are respectively $1+(n-k)(\alpha+\beta-1)$ and $1+(n-d+p)(\alpha+\beta-1)$.

The number of connected components of C^p is bounded by the number of connected components of the set D^p defined by the conditions in (14) and (15), and the vanishing of the maximal minors corresponding to the conditions in (18).

Let E^p be the set defined by the equations (14) and (18), so that $D^p=E^p\cap\{\theta_p>0\}$. Then, the number of connected components of D^p is bounded by the number of connected components of E^p plus the number of connected components of $E^p\cap\{\theta_p=\varepsilon\}$ for a choice of $\varepsilon>0$ small enough.

Hence, we're reduced to the problem of estimating the number of connected components of two semi-Pfaffian sets in $\mathbb{R}^{n(p+2)}$ that are defined without inequalities using a Pfaffian chain in $n(p+2)$ variables of degree α and length $(p+2)\ell$. Using the bounds on the Betti numbers from Corollary 6, we obtain (16). □

2.3. Bounds for the singular case. Let's consider now the case where X_λ and Y_λ may be singular. We can use deformation techniques to reduce to the smooth case. First, the following lemma shows we can reduce to the case where X_λ is a basic set.

LEMMA 16. *Let X_1, X_2 and Y be semi-Pfaffian sets such that (X_1,Y) and (X_2,Y) are Pfaffian families. Then, $(X_1\cup X_2,Y)_0=(X_1,Y)_0\cup(X_2,Y)_0$.*

The proof follows from the definition of the relative closure.

THEOREM 17. *Let (X,Y) be a semi-Pfaffian couple. Assume X_λ and Y_λ are unions of basic sets having a format of the form $(I,J,n,\ell,\alpha,\beta)$. If the number of basic sets in X_λ is M and the number of basic sets in Y_λ is N, then the number of connected components of $(X,Y)_0$ is bounded by*

$$2\,MN\sum_{p=0}^{n-1}2^{q\ell(q\ell-1)/2}\gamma_p(\alpha+2\gamma_p)^{nq-1}[nq(2\gamma_p+\alpha-2)+q\min(n,\ell)\alpha]^{q\ell}, \tag{19}$$

where $q=p+2$ and $\gamma_p=1+(p+1)(\alpha+2\beta-1)$.

PROOF. Again, we want to estimate the number of local maxima of the function Ψ_λ defined in (8).

By Lemma 16, we can restrict ourselves to the case where X is basic. Let $Y=Y_1\cup\cdots\cup Y_N$, where all the sets Y_i are basic. For each basic set, we take the sum of squares of the equations defining it: the corresponding positive functions, which we denote by p and $q_1,\dots,q_N$, have degree 2β in the chain. Fix $\varepsilon_i>0$, for $0\le i\le N$, and $\lambda>0$, and let $\mathcal{X}=\{p(x,\lambda)=\varepsilon_0\}$ and for all $1\le i\le N$, let $\mathcal{Y}_i=\{q_i(x,\lambda)=\varepsilon_i\}$.

Since Y_λ is compact, if x^* is a point in X_λ such that Ψ_λ has a local maximum at $x=x^*$, there is a point y^* in some $(Y_i)_\lambda$ such that $\Phi(x,y)=\Psi_\lambda(x)$. Then, we can find a couple $(x', y') \in \mathcal{X}_\lambda \times (\mathcal{Y}_i)_\lambda$ close to (x^*, y^*) such that $\Phi(x', y')$ is a local maximum of the distance (measured by Φ) from $\mathcal{X}_\lambda$ to $(\mathcal{Y}_i)_\lambda$.

Since for small enough $\varepsilon_0, \dots, \varepsilon_N$, the sets $\mathcal{X}_\lambda$ and $(\mathcal{Y}_i)_\lambda$ are effectively non-singular hypersurfaces, the number of local maxima of the distance of $\mathcal{X}_\lambda$ to $(\mathcal{Y}_i)_\lambda$ can be bounded by (16), for appropriate values of the parameters. The estimate (19) follows. □

3. Application to fewnomials

In this section, we will apply our previous results to the case where the Pfaffian functions we consider are fewnomials.

3.1. Fewnomials and low additive complexity. Recall that we can consider the restriction of any polynomial q to an orthant as a Pfaffian function whose complexity depends only on the number of non zero monomials in q. Fix $\mathcal{K} = \{m_1, \dots, m_r\} \in \mathbb{N}^n$ a set of exponents.

DEFINITION 18. *The polynomial q is a $\mathcal{K}$-*fewnomial *if it is of the form:*

$$q(x) = a_0 + a_1 x^{m_1} + \cdots a_r x^{m_r}, \quad \textit{where} \quad a_0, \dots, a_r \in \mathbb{R}.$$

Let $\ell = n + r$, and $(f_1, \dots, f_\ell)$ be the functions defined by:

$$f_i(x) = \begin{cases} x_i^{-1} \text{ if } 1 \le i \le n, \\ x^{m_{i-n}} \text{ if } i > n. \end{cases} \tag{20}$$

It is easy to see that $(f_1, \dots, f_\ell)$ is a Pfaffian chain of length ℓ and degree $\alpha = 2$ in the positive orthant $\mathscr{U} = (\mathbb{R}_+)^n$, since we have:

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} -f_i^2 \text{ if } i = j \le n, \\ f_j f_i \text{ if } i > n. \end{cases}$$

Then, a $\mathcal{K}$-fewnomial as defined above is just a Pfaffian function in the chain $(f_1, \dots, f_\ell)$ of degree $\beta = 1$, involving only the last r functions of the chain. In general, if $q(x)$ is a polynomial, and if $q(x)$ can be expressed as a Pfaffian function of degree at most β in the chain (20), we say that q has a *low additive complexity,* and call β the *pseudo-degree* of q. (For more on additive complexity, see [**1, 12**].)

Example. To make the distinction between fewnomials and polynomials with low additive complexity clearer, if a and b are any integers, the univariate polynomial $(x^a + 1)^b$ has pseudo-degree b in the obvious Pfaffian chain. But under our terminology, we wouldn't call it a fewnomial, and it has indeed many monomials if expanded.

Let now $S \subseteq (\mathbb{R}_+)^n$ be a semi-algebraic set defined by $\mathcal{K}$-fewnomials, and assume S is bounded. We can define from S a semi-Pfaffian family $X \subseteq \mathbb{R}^n \times \mathbb{R}$ by:

$$X = \{(x, \lambda) \in \mathscr{U} \times \mathbb{R}_+ \mid x \in S,\ x_1 > \lambda, \dots, x_n > \lambda\}. \tag{21}$$

We can apply the results from Theorems 15 and 17 to X, to obtain a bound on the number of connected components of $\overline{S} \cap \partial\mathscr{U}$. Note that this is indeed an important case, since an example is given in [**4**] of a fewnomial semialgebraic set for which the closure cannot be described independently of the degrees of the defining polynomials.

3.2. Bounds for fewnomial couples. We now give explicit bounds in the case where the fibers in the semi-Pfaffian couple (X, Y) can both be defined $\mathcal{K}$-fewnomials, *i.e.* by degree 1 Pfaffian function in a chain of the type (20). In particular, if S is a semi-algebriac set such that $\partial S \cap \mathscr{U} = \emptyset$, these bounds will apply to X as in (21) and to $Y = \{x_1 = \lambda\} \cup \cdots \cup \{x_n = \lambda\}$.

Theorem 19. *Let* (X, Y) *be a semi-Pfaffian couple defined by degree 1 functions in the chain* (20)*. Then, the following bounds can be established for the number of connected components of* $(X, Y)_0$.

Case 1. *If for all* $\lambda > 0$, X_λ *and* Y_λ *are effectively non-singular of dimension respectively* d *and* k, *we obtain from* (16) *the bound*

$$\sum_{p=0}^{d} 2^{q^2(n+r)^2/2}(4n+6)^{q(3n+2r)}q^{q(n+r)}. \tag{22}$$

Case 2. *If* X *and* Y *are the union of respectively* M *and* N *basic sets, the number of connected components of* $(X, Y)_0$ *is bounded by:*

$$MN\sum_{p=0}^{n-1} 2^{q^2(n+r)^2/2}(6n+6)^{q(3n+2r)}q^{q(n+r)}. \tag{23}$$

Proof. These bounds are obtained using the results from Theorems 15 and 17, with $\alpha = 2$, $\beta = 1$ and $\ell = n + r$, and then bounding very bluntly the terms β_p and γ_p. □

3.3. Closure relative to the frontier of a fewnomial set. Let X be a semi-Pfaffian family such that for all $\lambda > 0$, the set X_λ is defined by $\mathcal{K}$-fewnomials. By definition of a family, the set ∂X_λ is contained in the domain of the Pfaffian chain, so by the results of [**5**], this set is a semi-Pfaffian set defined by functions in the chain, that is, polynomials of low additive complexity. Moreover, the format of ∂X_λ can be estimated from the format of X_λ. Applying those results together with those of Theorem 17, we can give estimates for the number of connected components of $(X, \partial X)_0$.

Theorem 20. *Let* X *be a semi-Pfaffian family in a Pfaffian chain of type* (20)*. If the format of* X *is of the form* $(N, I, J, n, \ell = n + r, \alpha = 2, \beta = 1)$, *the number of connected components of* $(X, \partial X)$ *is bounded by*

$$N^2(I+J)^{N+rO(n^2)}n^{(n+r)^{n^{O(n^2+nr)}}}. \tag{24}$$

Proof. Following [**5**], the set ∂X_λ can be defined using the same Pfaffian chain as X_λ, using N' basic sets and functions of degree at most β', where, under the hypotheses above, the following bounds hold.

$$\beta' \le n^{(n+r)^{O(n)}}, \quad N' \le N(I+J)^{N+rO(n^2)}N^{(n+r)^{rO(n)}}.$$

The bound on the number of connected components follows readily. □

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