LIPSCHITZ GEOMETRY OF REAL SEMIALGEBRAIC SURFACES

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Abstract. We present here basic results in Lipschitz Geometry of semialgebraic surface germs. Although bi-Lipschitz classification problem of surface germs with respect to the inner metric was solved long ago, classification with respect to the outer metric remains an open problem. We review recent results related to the outer and ambient bi-Lipschitz classification of surface germs. In particular, we explain why the outer Lipschitz classification is much harder than the inner classification, and why the ambient Lipschitz Geometry of surface germs is very different from their outer Lipschitz Geometry. In particular, we show that the ambient Lipschitz Geometry of surface germs includes all of the Knot Theory.

1. Introduction

Lipschitz classification of semialgebraic surfaces has become in recent years one of the central questions of the Metric Geometry of Singularities. It was stimulated by the finiteness theorems of Mostowski, Parusinski and Valette. (see [15, 16, 20]). They proved that there are finitely many Lipschitz equivalence classes in any semialgebraic family of semialgebraic sets. Lipschitz classification is intermediate between Smooth (too fine) and Topological (too coarse) classifications. For example, smooth classification of most singularities is not finite. It may be even infinite dimensional for non-isolated singularities.

Here we review recent developments in Lipschitz Geometry of semialgebraic surfaces (two-dimensional real semialgebraic sets). Since we are mainly interested in singularities of semialgebraic surfaces, our main object is a semialgebraic surface germ $(X, 0)$ at the origin of $\mathbb{R}^n$. Note that most results presented in this paper remain true for subanalytic sets, and for the sets definable in a polynomially bounded o-minimal structure.

A connected semialgebraic set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics: the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y. Note that $dist(x, y) \leq idist(x, y)$. A semialgebraic set is called Lipschitz Normally Embedded if these two metrics are equivalent (see Definition 3.1).

For the surface germs, there are three natural equivalence relations:

1) Inner Lipschitz equivalence: $(X, 0) \sim_i (Y, 0)$ if there is a homeomorphism $h : (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the inner metric.

2) Outer Lipschitz equivalence: $(X, 0) \sim_o (Y, 0)$ if there is a homeomorphism $h : (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the outer metric.

3) Ambient Lipschitz equivalence: $(X, 0) \sim_a (Y, 0)$ if there is an orientation preserving bi-Lipschitz homeomorphism $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H(X) = Y$.

Inner Lipschitz Geometry of surface germs is relatively simple. The building block of the inner Lipschitz classification of surface germs is a β-**Hölder triangle** (see Definition 2.1). A surface germ $(X, 0)$ with an isolated singularity is inner Lipschitz equivalent to a β-**horn** (see Definition 2.2). If the singularity is not isolated, classification is made by the theory of Hölder Complexes (see [1]). A **Hölder Complex** is a triangulation (decomposition into Hölder triangles) of a surface germ. Canonical Hölder Complex (see Definition 2.9) is a complete invariant of the inner Lipschitz equivalence of surface germs.

For example, the germs of all irreducible complex curves are inner Lipschitz equivalent to $(\mathbb{C}, 0)$, while the outer Lipschitz classification of the germs of complex plane curves is described by their sets of essential Puiseux pairs (see [17], [12]). Even for the union of two normally embedded Hölder triangles, the outer Lipschitz Geometry is not simple (see [3]).

A special case of a surface germ is the union of a Hölder triangle T and a graph of a Lipschitz semialgebraic function f defined on T. The outer Lipschitz equivalence of two such surface germs is equivalent to the Lipschitz contact equivalence of the two functions. This relates outer Lipschitz geometry of surface germs with the Lipschitz geometry of functions. In [9] a complete invariant of the contact equivalence class of a Lipschitz function f defined on a Hölder triangle T, called a "pizza," is defined. Informally, a pizza is a decomposition of T into "slices," Hölder sub-triangles $\{T_i\}$ of T, such that the order of f on each arc $\gamma \subset T_i$ depends linearly on the order of contact of γ with a boundary arc of T_i.

For the general surface germs, Lipschitz classification with respect to the outer metric is still an open problem. The set of semialgebraic arcs in $(X, 0)$ parameterized by the distance to 0 is called the Valette link $V(X)$ of the germ $(X, 0)$ (see Definition 3.7). The order of contact of the arcs (see Definition 3.10) defines a non-archimedean metric on $V(X)$.

The study of Lipschitz Normally Embedded, or simply Normally Embedded, sets was initiated by Kurdyka and Orro [14]. Kurdyka proved that any semialgebraic set admits a finite partition into normally embedded subsets. Using this partition, Kurdyka and Orro proved that any semialgebraic sets admits a semialgebraic "pancake metric" equivalent to the inner metric. Normal Embedding theorem of Birbrair and Mostowski states that, for any semialgebraic set X, there is a semialgebraic and bi-Lipschitz with respect to the inner metric embedding $\Psi : X \to \mathbb{R}^m$, where $m \geq 2\dim(X) + 1$ (see [10]). Lipschitz Normal Embedding of complex analytic sets is addressed in the paper by Anne Pichon in the present volume.

A pancake decomposition is called minimal if it is not a refinement of another pancake decomposition. A natural question related to Lipschitz Normal Embedding of surface germs is uniqueness of a minimal pancake decomposition. The answer is negative even for a Hölder triangle. Gabrielov and Sousa in [13] gave examples of Hölder triangles having several combinatorially non-equivalent minimal pancake decompositions.

Relations between ambient and outer equivalence of surface germs were studied in [2, 5, 6]. In the paper [2] the authors presented several outer Lipschitz and ambient topologically equivalent families of surface germs $(X_i, 0)$, which were pairwise ambient Lipschitz non-equivalent. In [5, 6], several "Universality Theorems" were formulated. Informally, these theorems state that, even when the link of a surface germ is topologically a trivial knot, ambient Lipschitz classification of such surface germs "contains all of the Knot Theory."

2. Inner Lipschitz Equivalence

Definition 2.1. For $1 \leq \beta \in \mathbb{Q}$, the standard β-Hölder triangle T_β is the germ at the origin of $\mathbb{R}^2$ of the surface $\{x \geq 0,\ 0 \leq y \leq x^\beta\}$ (see Figure 1a). A *β-Hölder triangle* is a surface germ inner Lipschitz equivalent to T_β.

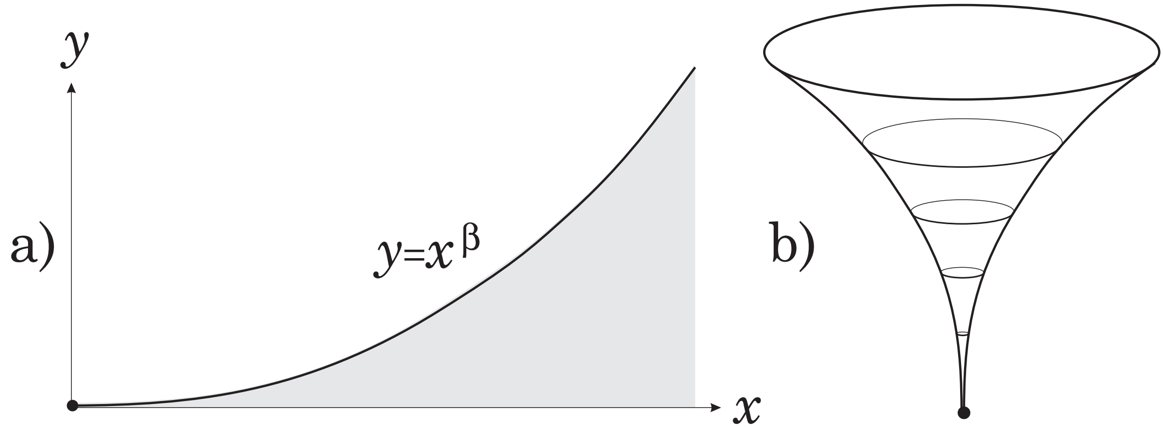

FIGURE 1. A β-Hölder triangle and a β-horn.

Definition 2.2. For $1 \leq \beta \in \mathbb{Q}$, the standard β-horn C_β is the germ at the origin of $\mathbb{R}^3$ of the surface $\{z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$ (see Figure 1b). A *β-horn* is a surface germ inner Lipschitz equivalent to C_β.

Theorem 2.3. *Given the germ $(X, 0)$ of a semialgebraic surface with isolated singularity and connected link, there is a unique rational number $\beta \geq 1$ such that $(X, 0)$ is inner Lipschitz equivalent to the standard β-horn C_β.*

Birbrair's theory of Hölder Complexes (see [1]) is a generalization of Theorem 2.3 for the surface germs with non-isolated singularities.

Definition 2.4. A *Formal Hölder Complex* is a pair (G, β), where G is a graph and $\beta : E_G \to \mathbb{Q}_{\geq 1}$ is a function, where E_G the set of edges of G.

Definition 2.5. A *Geometric Hölder Complex* corresponding to a Formal Hölder Complex (G, β) is a surface germ $(X, 0)$ such that
1. For small $\varepsilon > 0$, the intersection of X with the ε-ball B_ε is homeomorphic to the cone over G, and the intersection of X with the ε-sphere S_ε is homeomorphic to G.
2. For any edge $g \in E_G$, the subgerm of $(X, 0)$ corresponding to g is a $\beta(g)$-Hölder triangle.

Theorem 2.6. *For any surface germ $(X, 0) \subset \mathbb{R}^n$, there exists a Formal Hölder Complex (G, β) such that $(X, 0)$ is a Geometric Hölder Complex corresponding to (G, β).*

Remark 2.7. For a given surface germ $(X, 0)$, the Formal Hölder complex (G, β) in Theorem 2.6 is not unique. The simplification procedure described below reduces it to the unique Canonical Hölder Complex corresponding to $(X, 0)$.

Definition 2.8. We say that a vertex v_0 of the graph G is *non-critical* if it is adjacent to exactly two edges g_1 and g_2 of G, and these edges connect v_0 with two different vertices of G. A vertex v_0 of G is called a *loop vertex* if it is adjacent to exactly two different edges g_1 and g_2 of G, and these edges connect v_0 with the same vertex v_1 of G. The other vertices of G (neither non-critical nor loop vertices) are called *critical*.

Definition 2.9. An Abstract Hölder Complex (G, β) is called *Canonical*, if
1. All vertices of G are either critical or loop vertices;
2. For any loop vertex v of G adjacent to the edges g_1 and g_2, one has $\beta(g_1) = \beta(g_2)$.

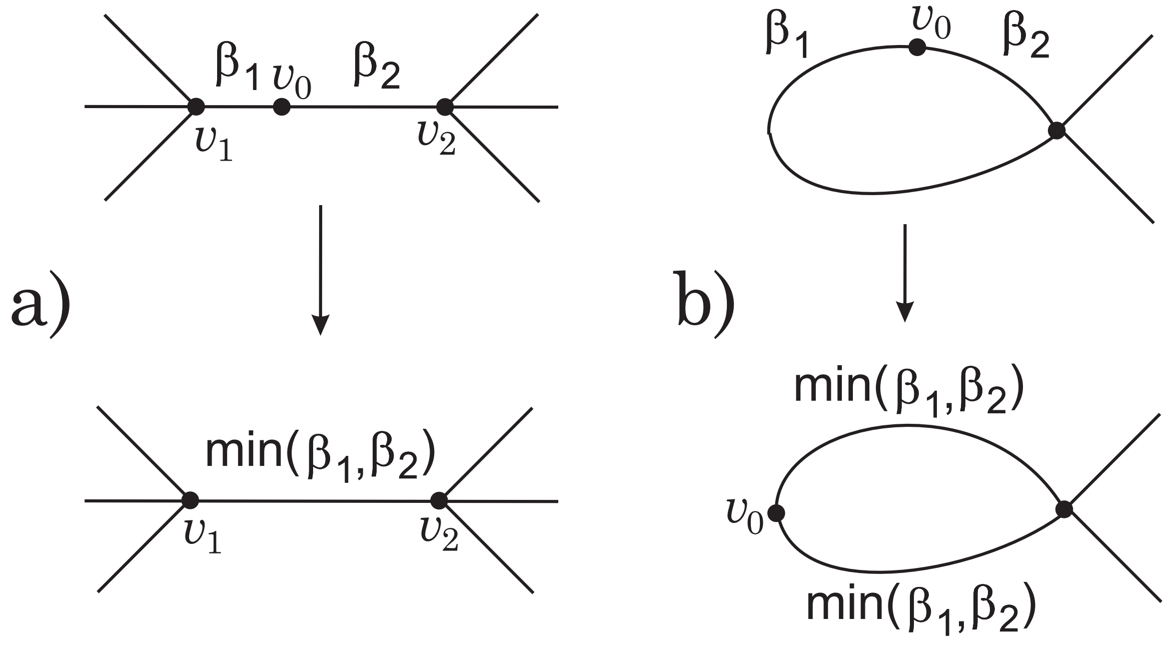

FIGURE 2. Simplification of Hölder complexes

Now we define a **simplification procedure**, reducing an Abstract Hölder Complex (G, β) to a Canonical one.

We start with eliminating non-critical vertices. Let v_0 be a non-critical vertex of G, connected with the vertices v_1 and v_2 of G by the edges g_1 and g_2. Then we remove the vertex v_0 from $V(G)$, and replace the edges g_1 and g_2 of G with the single edge g_0 connecting v_1 with v_2. Let G' be the graph obtained from G by this operation. We define an abstract Hölder complex (G', β'), setting $\beta'(g_0) = \min\{\beta(g_1), \beta(g_2)\}$ and $\beta'(g) = \beta(g)$ on all edges g of G' other than g_0, which are also the edges of G. (see Figure 2a)

We repeat this operation until there are no non-critical vertices. After that, we take care of the loop vertices of G.

Let (G, β) be an Abstract Hölder Complex without non-critical vertices. If a loop vertex v_0 of G is connected by the edges g_1 and g_2 with the same vertex v_1, such that $\beta(g_1) \neq \beta(g_2)$, we define an Abstract Hölder Complex (G, β'), replacing $\beta_1 = \beta(g_1)$ and $\beta_2 = \beta(g_2)$ with $\beta'(g_1) = \beta'(g_2) = \min(\beta_1, \beta_2)$ (see Figure 2b). We repeat this operation for all loop vertices of G.

The main results of the paper [1] are the following:

Theorem 2.10. (**Inner Lipschitz Classification Theorem.**) *The surface-germs $(X, 0)$ and $(X', 0)$ are Lipschitz equivalent with respect to the inner metric if, and only if, the corresponding Canonical Hölder Complexes are combinatorially equivalent.*

Theorem 2.11. (**Realization Theorem.**) *Let (G, β) be an Abstract Hölder Complex. Then there exists a surface germ $(X, 0)$ which is a Geometric Hölder Complex corresponding to (G, β).*

Remark 2.12. The theory of Hölder Complexes implies that a germ $(X, 0)$ of an irreducible complex curve, considered as a real surface germ, is inner Lipschitz equivalent to the germ $(\mathbb{C}, 0)$. Otherwise $(X, 0)$ is inner Lipschitz equivalent to the union of finitely many germs $(\mathbb{C}, 0)$ pinched at the origin, corresponding to irreducible components of $(X, 0)$.

3. Normal Embedding Theorem, Lipschitz Normally Embedded Sets

Definition 3.1. A semialgebraic set X is called *Lipschitz Normally Embedded* (LNE) if the inner and outer metrics on X are equivalent: $dist(x,y) \le idist(x,y) \le C\, dist(x,y)$ for some constant $C > 0$ and all $x, y \in X$.

For example, the germ of an algebraic curve $\{x^3 = y^2\}$ is not LNE, while the standard β-horn C_β is LNE. A germ of an irreducible complex curve is LNE if, and only if, it is smooth.

There are many examples of not normally embedded surface germs. On the other hand, we have the following result:

Theorem 3.2. (*See* [7].) *Let $X \subset \mathbb{R}^m$ be a connected semialgebraic set. Then there exist a normally embedded semialgebraic set $\tilde{X} \subset \mathbb{R}^q$ and an inner bi-Lipschitz homeomorphism $p : X \to \tilde{X}$. This map is called a normal embedding of X.*

Definition 3.3. A subset $\tilde{X} \subset \mathbb{R}^m$ is called Lipschitz Normally Embedded if there exist a bi-Lipschitz homeomorphism $\Psi : \tilde{X}_{inner} \to \tilde{X}_{outer}$.

Here $\tilde{X}_{inner}$ means the space $\tilde{X}$ equipped with the inner metric, and $\tilde{X}_{outer}$ means $\tilde{X}$ equipped with the outer metric. The difference with Definition 3.1 is that in Definition 3.3 we do not a priori suppose that Ψ is the identity map.

Proposition 3.4. *The two definitions of Lipschitz Normally Embedding are equivalent.*

Pancake Decomposition of Kurdyka implies that there exists a decomposition of any semialgebraic set $(X, 0)$ into LNE semialgebraic subsets.

Theorem 3.5. (*See* [14].) *There is a decomposition of any semialgebraic set X into subsets X_i such that :*

1. X_i are semialgebraic LNE sets.

2. $\dim(X_i \cap X_j) < \min(\dim X_i, \dim X_j)$ for $i \neq j$.

Remark 3.6. Using pancake decomposition, Kurdyka and Orro defined the so-called *pancake metric* (see [14], [7]). It is a semialgebraic metric equivalent to the inner metric.

Definition 3.7. (See [19].) An *arc* in $\mathbb{R}^n$ is (a germ at the origin of) a mapping $\gamma : [0, \epsilon) \to \mathbb{R}^n$ such that $\gamma(0) = 0$. Unless otherwise specified, arcs are parameterized by the distance to the origin, i.e., $|\gamma(t)| = t$. We usually identify an arc γ with its image in $\mathbb{R}^n$. The *Valette link* of a surface germ $(X, 0)$ is the set $V(X)$ of all arcs $\gamma \subset X$.

Theorem 3.8. (See [19].) *Let $(X, 0)$ and $(Y, 0)$ be germs of semialgebraic sets in $\mathbb{R}^n$. If these germs are semialgebraically (inner, outer or ambient) Lipschitz equivalent, then there exists a bi-Lipschitz map $h : X \to Y$ (or $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(X) = Y$ in the case of ambient equivalence) such that $h(X \cap S_\varepsilon) = Y \cap S_\varepsilon$ for small $\varepsilon > 0$.*

Definition 3.9. Let $f \not\equiv 0$ be (a germ at the origin of) a Lipschitz function defined on an arc γ. The *order* of f on γ is $q = ord_\gamma f \in \mathbb{Q}$ such that $f(\gamma(t)) = ct^q + o(t^q)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, we set $ord_\gamma f = \infty$.

Definition 3.10. The *tangency order* of arcs γ and γ' is $tord(\gamma, \gamma') = ord_\gamma|\gamma(t) - \gamma'(t)|$. The tangency order of an arc γ and a set of arcs $Z \subset V(X)$ is $tord(\gamma, Z) = \sup_{\lambda \in Z} tord(\gamma, \lambda)$. The tangency order of two subsets Z and Z' of $V(X)$ is $tord(Z, Z') =$

$\sup_{\gamma\in Z} tord(\gamma, Z')$. Similarly, $itord(\gamma,\gamma')$, $itord(\gamma, Z)$ and $itord(Z, Z')$ are the tangency orders with respect to the inner metric.

Remark 3.11. (See [4].) If $(X, 0)$ is a germ of a semialgebraic curve, i.e., $X = \cup\gamma_i$ is a finite family of semialgebraic arcs, then the outer Lipschitz Geometry of $(X, 0)$ is totaly determined by the tangency orders $\{tord(\gamma_i, \gamma_j)\}$.

Proposition 3.12. *A surface germ $(X, 0)$ is LNE if, and only if, for any two arcs γ_1, γ_2 in X one has $tord(\gamma_1, \gamma_2) = itord(\gamma_1, \gamma_2)$.*

Proposition 3.13. *Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a β-horn. The Grassmannian $G(n, 2)$ can be considered as the space of all orthogonal projections $\rho : \mathbb{R}^n \to \mathbb{R}^2$. Then there exist an open semialgebraic subset $\widetilde{G} \subset G(n, 2)$ such that for all $\rho \in \widetilde{G}$ one has $\beta = \min_{\{\gamma_1,\gamma_2\}\subset V(X)} tord(\rho(\gamma_1), \rho(\gamma_2))$.*

The following proposition was proved first by Alexandre Fernandes [12]. A special case of this is the Arc Criterion of Normal Embedding [11].

Proposition 3.14. *Let $(X, 0)$ and $(Y, 0)$ be surface germs. A semialgebraic homeomorphism $\Phi : (X, 0) \to (Y, 0)$ preserving the distance to the origin is outer bi-Lipschitz if, and only if, for any two arcs $\gamma_1, \gamma_2 \in V(X)$ one has*

$$tord(\gamma_1, \gamma_2) = tord(\Phi(\gamma_1), \Phi(\gamma_2)). \tag{1}$$

A special case of Pancake Decomposition for surface germs can be stated as follows:

Theorem 3.15. *Let $(X, 0)$ be a surface germ. Then there exists a decomposition of $(X, 0)$ into the germs $(X_i, 0)$ such that*

1. Each $(X_i, 0)$ is a LNE β_i-Hölder triangle.

2. For $i \neq j$, the intersection $(X_i, 0)\cap(X_j, 0)$ is either the origin or a common boundary arc of $(X_i, 0)$ and $(X_j, 0)$.

Definition 3.16. A pancake decomposition of a surface germ is *minimal* if the union of any two adjacent Hölder triangles X_i and X_j is not normally embedded. Two pancake decompositions are *combinatorially equivalent* if they are combinatorially equivalent as Hölder Complexes.

The answer to a natural question “Are any two minimal pancake decompositions of the same surface germ combinatorially equivalent?” is negative (see Section 5).

4. Pizza Decomposition of the Germ of a Semialgebraic Function

This section is related to the outer Lipschitz Geometry of a special kind of a surface germ: The union of a LNE Hölder triangle and the graph of a semialgebraic Lipschitz function defined on it.

Definition 4.1. For a semialgebraic Lipschitz function f defined on a β-Hölder triangle T, let

$$Q_f(T) = \bigcup_{\gamma\in V(T)} ord_\gamma f. \tag{2}$$

It was shown in [9] that $Q_f(T)$ is either a point or a closed interval in $\mathbb{Q} \cup \{\infty\}$.

Definition 4.2. A Hölder triangle T is *elementary* with respect to a Lipschitz function f if, for any $q \in Q_f(T)$ and any two arcs γ and γ' in T such that $ord_\gamma f = ord_{\gamma'} f = q$, the order of f is q on any arc in the Hölder triangle $T(\gamma, \gamma') \subseteq T$ bounded by the arcs γ and γ'.

Definition 4.3. Let T be a Hölder triangle and f a Lipschitz function defined on T. For each arc $\gamma \subset T$, the *width* $\mu_T(\gamma, f)$ of the arc γ with respect to f is the infimum of exponents of Hölder triangles $T' \subset T$ containing γ such that $Q_f(T')$ is a point. For $q \in Q_f(T)$ let $\mu_{T,f}(q)$ be the set of exponents $\mu_T(\gamma, f)$, where γ is any arc in T such that $ord_\gamma f = q$. It was shown in [9] that, for each $q \in Q_f(T)$, the set $\mu_{T,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{T,f} : Q_f(T) \to \mathbb{Q} \cup \{\infty\}$. If T is elementary with respect to f, then the function $\mu_{T,f}$ is single valued. When f is fixed, we write $\mu_T(\gamma)$ and μ_T instead of $\mu_T(\gamma, f)$ and $\mu_{T,f}$.

Definition 4.4. Let T be a Hölder triangle and f a semialgebraic Lipschitz function defined on T. We say that T is a *pizza slice* associated with f if it is elementary with respect to f and, unless $Q_f(T)$ is a point, $\mu_{T,f}(q) = aq + b$ is an affine function on $Q_f(T)$. If T is a pizza slice such that $Q_f(T)$ is not a point, then the *supporting arc* $\tilde{\gamma}$ of T with respect to f is the boundary arc of T such that $\mu_T(\tilde{\gamma}, f) = \max_{q \in Q_f(T)} \mu_{T,f}(q)$. In that case, $\mu_T(\gamma, f) = tord(\gamma, \tilde{\gamma})$ for any arc $\gamma \subset T$ such that $tord(\gamma, \tilde{\gamma}) \leq \mu_T(\tilde{\gamma}, f)$.

Definition 4.5. (See [9, Definition 2.13].) Let f be a non-negative semialgebraic Lipschitz function defined on a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ oriented from γ_1 to γ_2. A *pizza* on T associated with f is a decomposition $\{T_\ell\}_{\ell=1}^p$ of T into β_ℓ-Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ ordered according to the orientation of T, such that $\lambda_0 = \gamma_1$ and $\lambda_p = \gamma_2$ are the boundary arcs of T, $T_\ell \cap T_{\ell+1} = \lambda_\ell$ for $0 < \ell < p$, and each triangle T_ℓ is a pizza slice associated with f.

A pizza $\{T_\ell\}$ on T is *minimal* if $T_{\ell-1} \cup T_\ell$ is not a pizza slice for any $\ell > 1$.

Definition 4.6. (See [9, Definition 2.12].) An *abstract pizza* is a finite ordered sequence $\{q_\ell\}_{\ell=0}^p$, where $q_\ell \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$, and a finite collection $\{\beta_\ell, Q_\ell, \mu_\ell\}_{\ell=1}^p$, where $\beta_\ell \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$, $Q_\ell = [q_{\ell-1}, q_\ell] \subset \mathbb{Q}_{\geq 1} \cup \{\infty\}$ is either a point or a closed interval, $\mu_\ell : Q_\ell \to \mathbb{Q} \cup \{\infty\}$ is an affine function, non-constant when Q_ℓ is not a point, such that $\mu_\ell(q) \leq q$ for all $q \in Q_\ell$ and $\min_{q \in Q_\ell} \mu_\ell(q) = \beta_\ell$.

Definition 4.7. Two pizzas are *combinatorially equivalent* if the corresponding abstract pizzas are the same.

Theorem 4.8. (See [9, Theorem 4.9].) *Two non-negative semialgebraic Lipschitz functions f and g defined on a Hölder triangle T are contact Lipschitz equivalent if, and only if, minimal pizzas on T associated with f and g are combinatorially equivalent.*

Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be Normally Embedded β-Hölder triangles satisfying the condition :

$$tord(\gamma_1, T') = tord(\gamma_1, \gamma_1') = tord(\gamma_1', T), \quad tord(\gamma_2, T') = tord(\gamma_2, \gamma_2') = tord(\gamma_2', T). \tag{3}$$

For example, the triangles $(T, Graph(f))$ considered in this section satisfy this condition. The following question is natural. Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be Normally Embedded semialgebraic β-Hölder triangles satisfying (3). Is it true that the union $T \cup T'$ is Lipschitz outer equivalent to the union $T \cup Graph(f)$, where f is the distance function: $f(x) = dist(x, T')$? In the paper [3] the authors show that it is not true.

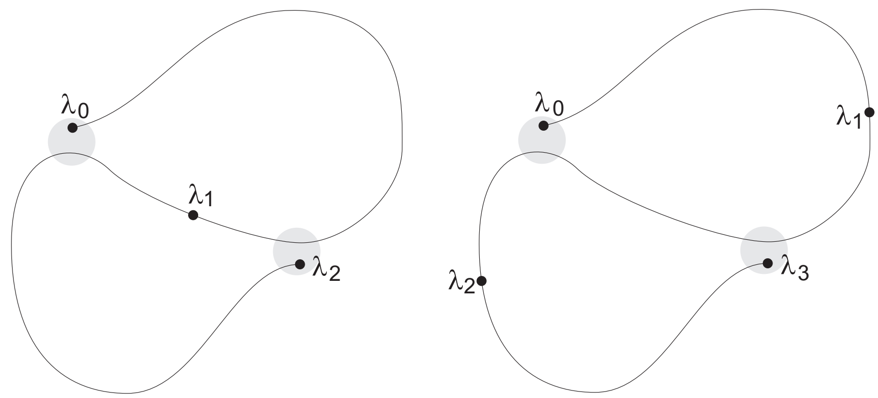

FIGURE 3. Two combinatorially non-equivalent minimal pancake decompositions of a snake. Black dots indicate the boundary arcs of pancakes.

5. Outer Lipschitz Geometry, Snakes

To formulate the results of this section, we need several definitions.

Definition 5.1. Let $(X, 0)$ be a surface germ. An arc γ of X is *Lipschitz non-singular* if there exists a Normally Embedded Hölder triangle $T \subset X$ such that γ is an interior arc of T and $\gamma \not\subset \overline{X \setminus T}$. Otherwise, γ is *Lipschitz singular*. It follows from the pancake decomposition that a surface germ X contains finitely many Lipschitz singular arcs. The union of all Lipschitz singular arcs in X is denoted by $Lsing(X)$. A Hölder triangle $T \subset X$ is *non-singular* if all interior arcs of T are Lipschitz non-singular.

Definition 5.2. If $T = T(\gamma_1, \gamma_2)$ is a non-singular β-Hölder triangle, an arc γ of T is *generic* if $itord(\gamma_1, \gamma) = itord(\gamma, \gamma_2) = \beta$. The set of generic arcs of T is denoted $G(T)$.

Definition 5.3. An arc γ of a Lipschitz non-singular β-Hölder triangle T is *abnormal* if there are two normally embedded Hölder triangles $T' \subset T$ and $T'' \subset T$ such that $T' \cap T'' = \gamma$ and $T \cup T'$ is not normally embedded. Otherwise γ is *normal*. The set $Abn(T)$ of abnormal arcs of T is outer Lipschitz invariant.

Definition 5.4. A non-singular β-Hölder triangle T is called a β*-snake* if $Abn(T) = G(T)$.

The following important property of snakes can be interpreted as "separation of scales" in outer Lipschitz Geometry.

Lemma 5.5. *Let T be a β-snake, and let $\{T_k\}_{k=1}^p$ be a minimal pancake decomposition of T. Then each T_k is a β-Hölder triangle.*

Remark 5.6. Minimal pancake decompositions of a snake may be combinatorially non-equivalent, as shown in Figure 3. We use a planar plot to represent the link of a snake. The points in Figure 3 correspond to arcs of the snake. The points with smaller Euclidean distance inside the shaded disks correspond to arcs with the tangency order higher than their inner tangency order β. Black dots indicate the boundary arcs of pancakes.

Definition 5.7. A β-Hölder triangle T is *weakly normally embedded* if, for any two arcs γ and γ' of T such that $tord(\gamma, \gamma') > itord(\gamma, \gamma')$, we have $itord(\gamma, \gamma') = \beta$.

Proposition 5.8. *Let T be a β-snake. Then T is weakly normally embedded.*

Weak Lipschitz equivalence of snakes is a combinatorial invariant [13, Subsection 6.3].

6. Tangent Cones

Definition 6.1. The *tangent cone* C_0X of a semialgebraic set X at 0 is defined as follows:

$$C_0X = Cone\left(\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(X\cap\{|x|=\epsilon\}\right)\right),$$

where the limit here means the Hausdorff limit.

Remark 6.2. There are several equivalent definitions of the tangent cone of a semialgebraic set. In particular, the tangent cone C_0X can be defined as the set of tangent vectors at the origin to all the arcs in X. The tangent cone of a semialgebraic set is semialgebraic.

The tangent cone is Lipschitz invariant:

Theorem 6.3. (See [18].) *If two germs $(X,0)$ and $(Y,0)$ are outer (resp. ambient) Lipschitz equivalent, then the corresponding tangent cones C_0X and C_0Y are outer (resp. ambient) Lipschitz equivalent.*

The result is used Theory of Metric Knots (see [2],[5]) to prove Universality Theorem below (see also [6],[5]). This result was also used to prove that a complex analytic set, which is a Lipschitz submanifold of $\mathbb{C}^n$,y is a smooth submanifold (See[8]). Moreover, the result was used in the recent study of the Zariski Multiplicity Conjecture (see the paper of Fernandes and Sampaio at the present volume).

7. Ambient Equivalence. Metric Knots.

Definition 7.1. Two germs of semialgebraic sets $(X,0)$ and $(Y,0)$ are *outer Lipschitz equivalent* if there exists a homeomorphism $H:(X,0)\to(Y,0)$ bi-Lipschitz with respect to the outer metric. The germs are *semialgebraic outer Lipschitz equivalent* if the map H can be chosen to be semialgebraic. The germs are *ambient Lipschitz equivalent* if there exists an orientation preserving bi-Lipschitz homeomorphism $\widetilde{H}:(\mathbb{R}^4,0)\to(\mathbb{R}^4,0)$, such that $\widetilde{H}(X)=Y$. The germs are *semialgebraic ambient Lipschitz equivalent* if the map $\widetilde{H}$ can be chosen to be semialgebraic.

Definition 7.2. The *link at the origin* L_X of a germ X is the equivalence class of the sets $X\cap S^3_{0,\varepsilon}$ for small positive ε with respect to the ambient Lipschitz equivalence. The *tangent link* of X is the link at the origin of the tangent cone of X.

Remark 7.3. By the finiteness theorems of Mostowski, Parusinski and Valette (see [15], [16] and [20]) the link at the origin is well defined. We write "the link at the origin" speaking of this notion of the link from Singularity Theory, reserving the word "link" for the notion of the link in Knot Theory. If X has an isolated singularity at the origin then each connected component of L_X is a knot in S^3.

The following result (so called **Universality Theorem**) shows the difference between outer and ambient Lipschitz Geometry of germs of real surfaces:

Theorem 7.4. (**Universality Theorem.**) *Let $K\subset S^3$ be a knot. Then one can associate to K a semialgebraic surface germ $(X_K,0)$ in $\mathbb{R}^4$ so that the following holds:*
1) *The link at the origin of each germ X_K is a trivial knot;*
2) *All germs X_K are outer Lipschitz equivalent;*
3) *Two germs X_{K_1} and X_{K_2} are ambient semialgebraic Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

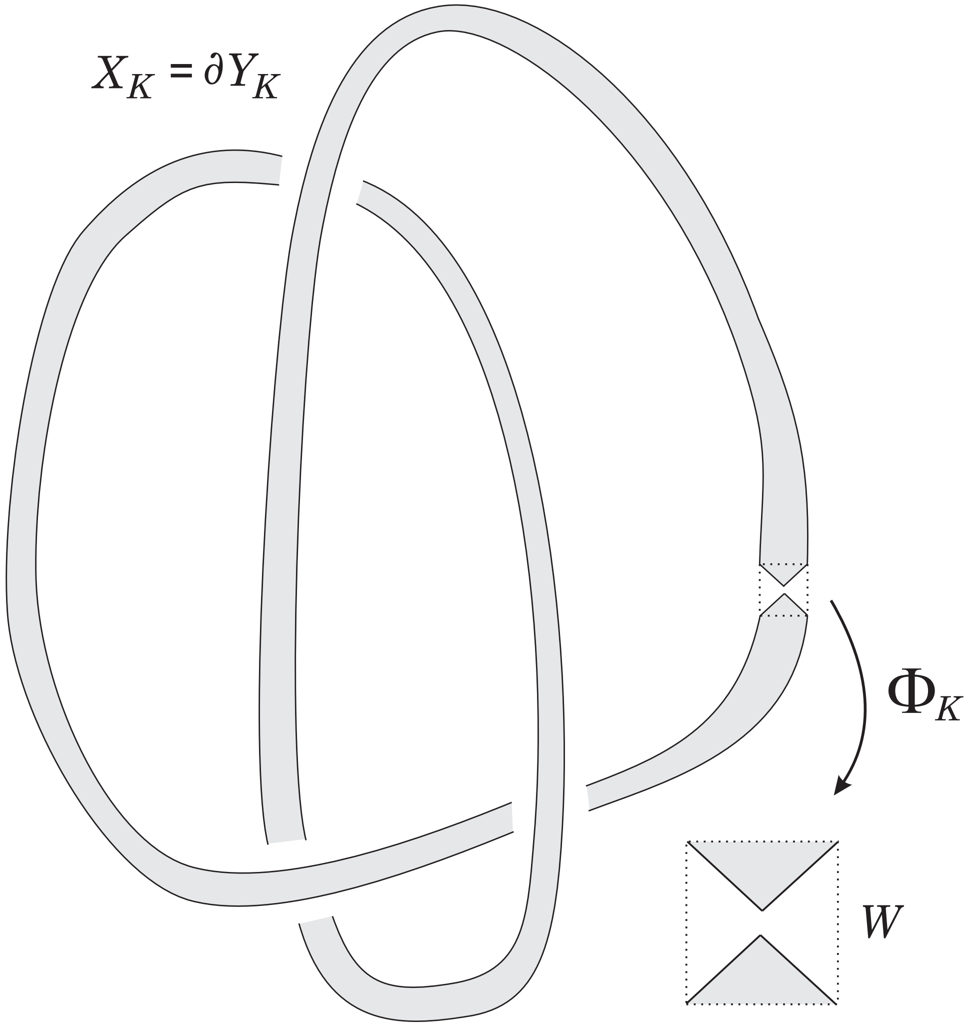

FIGURE 4. The proof of Theorem 7.4.

To show how to proof works, we include the figure 4:

The figure 4, representing the link of the surface X_K. A reader can find a detailed explanation in [6].

The following result shows that for given tangent cone one can find infinitely many Lipschitz outer equivalent, but not Lipschitz ambient equivalent, surface germs.

Theorem 7.5. *For any two knots K and L there exists a semialgebraic surface germ $\tilde{X}_{KL}$ such that:*

1. For any knots K and L, the link at the origin of $\tilde{X}_{KL}$ is isotopic to L.

2. For any knots K and L, the tangent link of $\tilde{X}_{KL}$ is isotopic to K.

3. For fixed α and β, all surface germs $\tilde{X}_{KL}$ are outer bi-Lipschitz equivalent.

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