ON EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A COMPLEX-VALUED POLYNOMIAL POTENTIAL

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ABSTRACT. In this paper, we generalize a recent result of A. Eremenko and A. Gabrielov on irreducibility of the spectral discriminant for the Schrödinger equation with quartic potentials. We consider the eigenvalue problem with a complex-valued polynomial potential of arbitrary degree d and show that the spectral determinant of this problem is connected and irreducible. In other words, every eigenvalue can be reached from any other by analytic continuation.

We also prove connectedness of the parameter spaces of the potentials that admit eigenfunctions satisfying $k > 2$ boundary conditions, except for the case d is even and $k = d/2$. In the latter case, connected components of the parameter space are distinguished by the number of zeros of the eigenfunctions.

CONTENTS

1. INTRODUCTION

In this paper we study analytic continuation of eigenvalues of the Schröodinger operator with a complex-valued polynomial potential. In other words, we are interested in the analytic continuation of eigenvalues $\lambda = \lambda(\alpha)$ of the boundary value problem for the differential equation

$$(1) \qquad\qquad -y'' + P_\alpha(z)y = \lambda y,$$

where

$$P_\alpha(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z \text{ where } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1}), \ d \geq 2.$$

The boundrary conditions are given by either (2) or (3) below. Namely, set $n = d + 2$ and divide the plane into n disjoint open sectors of the form:

$$S_j = \{z \in \mathbf{C} \setminus \{0\} : |\arg z - 2\pi j/n| < \pi/n\}, \quad j = 0, 1, 2, \ldots, n - 1.$$

These sectors are called the *Stokes sectors* of the equation (1). It is well-known that any solution y of (1) is either increasing or decreasing in each open Stokes sector S_j, i.e. $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in S_j, see [Sib75]. In the first case, we say that y is *subdominant*, and in the second case, *dominant* in S_j. We impose the boundary conditions that for two *non-adjacent* sectors S_j and S_k, i.e. for $j \neq k \pm 1$ mod n :

$$(2) \qquad\qquad y \text{ is subdominant in } S_j \text{ and } S_k.$$

For example, $y(\infty) = y(-\infty) = 0$ on the real axis, the boundary conditions usually imposed in physics for even potentials, correspond to y being subdominant in S_0 and $S_{n/2}$. The existence of analytic continuation is a classical fact, see e.g. references in [EG09a].

The main results of this paper are:

Theorem 1. *For any eigenvalue $\lambda_k(\alpha)$ of equation* (1) *and boundary condition* (2), *there is an analytic continuation in the α-plane to any other eigenvalue $\lambda_m(\alpha)$.*

We also prove some stronger results in the case where y is subdominant in more than two sectors:

Theorem 2. *Given $k < n/2$ non-adjacent Stokes sectors $S_{j_1}, \ldots, S_{j_k}$, the set of all $(\alpha, \lambda) \in \mathbf{C}^d$ for which the equation $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$(3) \qquad\qquad y \text{ subdominant in } S_{j_1}, \ldots, S_{j_k}$$

is connected.

Theorem 3. *For n even and $k = n/2$, the set of all $(\alpha, \lambda) \in \mathbf{C}^d$ for which $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$(4) \qquad\qquad y \text{ subdominant in } S_0, S_2, \ldots, S_{n-2}$$

is disconnected. Additionally, the solutions to (1), (3) *have finitely many zeros, and the set of α corresponding to given number of zeros is a connected component of the former set.*

The method we use is based on the "Nevanlinna parameterization" of the spectral locus introduced in [EG09a] (see also [EG09b] and [EG10]).

1.1. **Some previous results.** In the foundational paper [BW69], C. Bender and T. Wu studied analytic continuation of λ in the complex β-plane for the problem

$$-y'' + (\beta z^4 + z^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0.$$

Based on numerical computations, they conjectured for the first time the connectivity of the sets of odd and even eigenvalues. This paper generated considerable further research in both physics and mathematics literature. See e.g. [Sim70] for early mathematically rigorous results in this direction.

In [EG09a], which is the motivation of the present paper, the even quartic potential $P_a(z) = z^4 + az^2$ and the boundary value problem

$$-y'' + (z^4 + az^2)y = \lambda_a y, \quad y(\infty) = y(-\infty) = 0$$

was considered. It is known that the problem has discrete real spectrum for real a, with $\lambda_1 < \lambda_2 < \cdots \to +\infty$. There are two families of eigenvalues, those with even index and those with odd. The main result of [EG09a] is that if λ_j and λ_k are two eigenvalues in the same family, then λ_k can be obtained from λ_j by analytic continuation in the complex α-plane. Similar results have been obtained for other potentials, such as the PT-symmetric cubic, where $P_\alpha(z) = (iz^3 + i\alpha z)$, with $y(z) \to 0$, as $z \to \pm\infty$ on the real line. See for example [EG09b].

Remark 4. *After this project was finished, the authors found out that a result similar to Theorem 2 was proved in a hardly ever quoted Ph.D thesis, [Hab52], page 36. On the other hand, this result is formulated in the setting of Nevanlinna theory, with no connection to properties of (1).*

2. PRELIMINARIES

First, we recall some basic notions from Nevanlinna theory.

Lemma 5 (see [Sib75]). *Each solution $y \neq 0$ of (1) is an entire function, and the ratio $f = y/y_1$ of any two linearly independent solutions of (1) is a meromorphic function, with the following properties:*

(I) *For any j, there is a solution y of (1) subdominant in the Stokes sector S_j. This solution is unique, up to multiplication by a non-zero constant,*

(II) *For any Stokes sector S_j, we have $f(z) \to w \in \bar{\mathbf{C}}$ as $z \to \infty$ along any ray in S_j. This value w is called the asymptotic value of f in S_j.*

(III) *For any j, the asymptotic values of f in S_j and S_{j+1} (index taken modulo n) are different. The function f has at least 3 distinct asymptotic values.*

(IV) *The asymptotic value of f is zero in S_j if and only if y is subdominant in S_j. It is convenient to call such sector subdominant as well. Note that the boundary conditions in (2) imply that the two sectors S_j and S_k are subdominant for f when y is an eigenfunction of (1), (2).*

(V) *f does not have critical points, hence $f : \mathbf{C} \to \bar{\mathbf{C}}$ is unramified outside the asymptotic values.*

(VI) *The Schwartzian derivative S_f of f given by*

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

equals $-2(P_\alpha - \lambda)$. Therefore one can recover P_α and λ from f.

From now on, f always denotes the ratio of two linearly independent solutions of (1), with y being an eigenfunction of the boundary value problem (1), with conditions (2), (3) or (4).

2.1. **Cell decompositions.** Set $n = d + 2$, $d = \deg P$ where P is our polynomial potential and assume that all non-zero asymptotic values of f are distinct and finite. Let w_j be the asymptotic values of f, ordered arbitrarily with the only restriction that $w_j = 0$ if and only if S_j is subdominant. For example, one can denote by w_j the asymptotic value in the Stokes sector S_j. We will later need different orders of the non-zero asymptotic values, see section 2.3.

Consider the cell decomposition Ψ_0 of $\bar{\mathbf{C}}_w$ shown in Fig. 1a. It consists of closed directed loops γ_j starting and ending at ∞, where the index is considered mod n, and γ_j is defined only if $w_j \neq 0$. The loops γ_j only intersect at ∞ and have no self-intersection other than ∞. Each loop γ_j contains a single non-zero asymptotic value w_j of f. For example, the boundary condition $y \to 0$ as $z \to \pm\infty$ for $z \in \mathbf{R}$ for even n implies that $w_0 = w_{n/2} = 0$, so there are no loops γ_0 and $\gamma_{n/2}$. We have a natural cyclic order of the asymptotic values, namely the order in which a small circle around ∞ counterclockwise intersects the associated loops γ_j, see Fig. 1a.

We use the same index for the asymptotic values and the loops, which motivates the following notation:

$j_+ = j + k$ where $k \in \{1, 2\}$ is the smallest integer such that $w_{j+k} \neq 0$.

Thus, γ_{j_+} is the loop around the next to w_j (in the cyclic order mod n) non-zero asymptotic value. Similarly, γ_{j_-} is the loop around the previous non-zero asymptotic value.

2.2. **From cell decompositions to graphs.** We may simplify our work with cell decompositions with the help of the following:

Lemma 6 (See Section 3 [EG09a])**.** *Given Ψ_0 as in Fig. 1a, one has the following properties:*

(a) *The preimage $\Phi_0 = f^{-1}(\Psi_0)$ gives a cell decomposition of the plane $\mathbf{C}_z$. Its vertices are the poles of f, and the edges are preimages of the loops γ_j. These edges are labeled by j, and are called j-edges.*

(b) *The edges of Φ_0 are directed, their orientation is induced from the orientation of the loops γ_j. Removing all loops of Φ_0, we obtain an infinite, directed planar graph Γ, without loops.*

(c) *Vertices of Γ are poles of f, each bounded connected component of $\mathbf{C} \setminus \Gamma$ contains one simple zero of f, and each zero of f belongs to one such bounded connected component.*

(a) Ψ_0

(b) $A_j(\Psi_0)$.
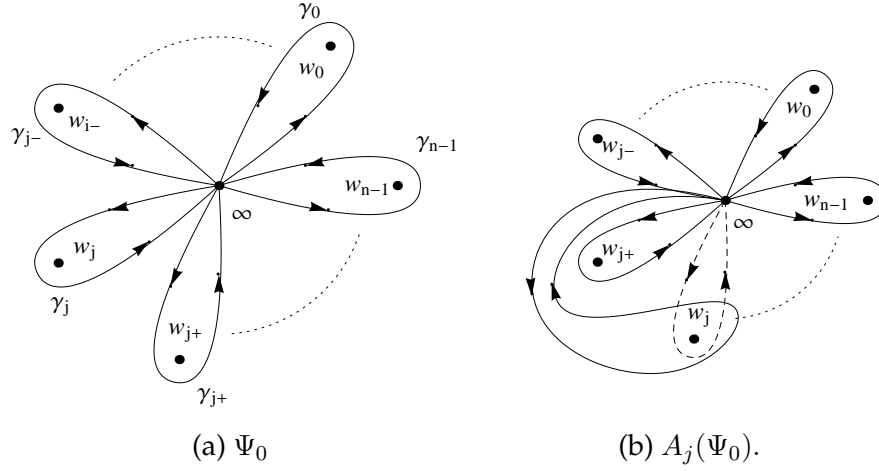
Figure 1: Permuting w_j and w_{j_+} in Ψ_0.

Γ

T_Γ

T_Γ^*
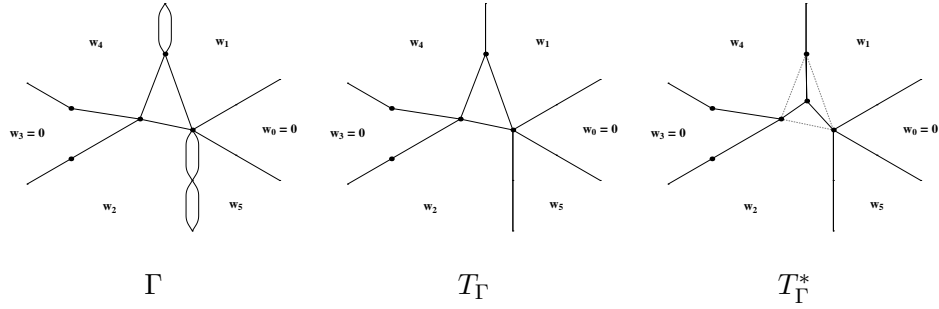
Figure 2: The correspondence between Γ, T_Γ and T_Γ^*.

(d) There are at most two edges of Γ connecting any two of its vertices. Replacing each such pair of edges with a single undirected edge and making all other edges undirected, we obtain an undirected graph T_Γ.

(e) T_Γ has no loops or multiple edges, and the transformation from Φ_0 to T_Γ can be uniquely reversed.

An example of the transformation from Γ to T_Γ is presented in Fig. 2.

A *junction* is a vertex of Γ (and of T_Γ) at which the degree of T_Γ is at least 3. From now on, Γ refers to both the directed graph without loops and the associated cell decomposition Φ_0.

2.3. **The standard order.** For a potential of degree d, the graph Γ has $d+2 = n$ infinite branches and n unbounded faces corresponding to the Stokes sectors. We defined earlier the ordering $w_0, w_1, \ldots, w_{n-1}$ of the asymptotic values of f.

If each w_j is the asymptotic value in the sector S_j, we say that the asymptotic values have *the standard order* and the corresponding cell decomposition Γ is a *standard graph*.

Lemma 7 (See Prop 6. [EG09a]). *If a cell decomposition Γ is a standard graph, the corresponding undirected graph T_Γ is a tree.*

This property is essential in the present paper, and we classify cell decompositions of this type by describing the associated trees.

Below we define the action of the braid group that permute non-zero asymptotic values of Ψ_0. This induces the corresponding action on graphs. Each unbounded face of Γ (and T_Γ) will be labeled by the asymptotic value in the corresponding Stokes sector. For example, labeling an unbounded face corresponding to S_k with w_j or just with the index j, we indicate that w_j is the asymptotic value in S_k.

From the definition of the loops γ_j, a face corresponding to a dominant sector has the same label as any edge bounding that face. The label in a face corresponding to a subdominant sector S_k is always k, since the actions defined below only permute non-zero asymptotic values. We say that an unbounded face of Γ is (sub)dominant if the corresponding Stokes sector is (sub)dominant.

For example, in Fig. 2, the Stokes sectors S_0 and S_3 are subdominant since the corresponding faces have label 0. We do not have the standard order for Γ, since w_2 is the asymptotic value for S_4, and w_4 is the asymptotic value for S_2. The associated graph T_Γ is not a tree.

2.4. Properties of graphs and their face labeling.

Lemma 8 (see [EG09a]). *The following holds:*

 (I) Two bounded faces of Γ cannot have a common edge, since a j-edge is always at the boundary of an unbounded face labeled j.
 (II) The edges of a bounded face of a graph Γ are directed clockwise, and their labels increase in that order. Therefore, a bounded face of T_Γ can only appear if the order of w_j is non-standard.
     (As an example, the bounded face in Fig. 2 has the labels $1, 2, 4$ (clockwise) of its boundary edges.)
 (III) Each label appears at most once in the boundary of any bounded face of Γ.
 (IV) Unbounded faces of Γ adjacent to its junction u always have the labels cyclically increasing counterclockwise around u.
 (V) To each graph T_Γ, we associate a tree by inserting a new vertex inside each of its bounded faces, connecting it to the vertices of the bounded face and removing the boundrary edges of the original face. Thus we may associate a tree T_Γ^* with any cell decomposition, not necessarily with standard order, as in Fig. 2(c). The order of w_j above together with this tree uniquely determines Γ. This is done using the two properties above.
 (VI) The boundary of a dominant face labeled j consists of infinitely many directed j-edges, oriented counterclockwise around the face.
 (VII) If $w_j = 0$ there are no j-edges.
 (VIII) Each vertex of Γ has even degree, since each vertex in $\Phi_0 = f^{-1}(\Psi_0)$ has even degree, and removing loops to obtain Γ preserves this property.

Following the direction of the j-edges, the first vertex that is connected to an edge labeled j_+ is the vertex where the j-edges and the j_+-edges *meet*. The last such vertex is where they *separate*. These vertices, if they exist, must be junctions.

Definition 9. *Let Γ be a standard graph, and let $j \in \Gamma$ be a junction where the j-edges and j_+-edges separate. Such junction is called a j-junction.*

There can be at most one j-junction in Γ, the existence of two or more such junctions would violate property (III) of the face labeling. However, the same junction can be a j-junction for different values of j.

There are three different types of j-junctions, see Fig. 3.

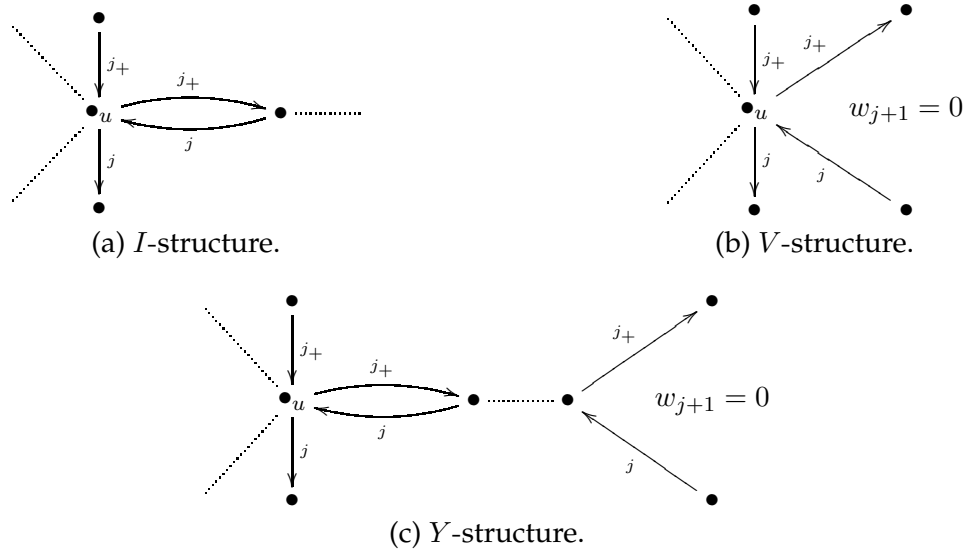
(a) I-structure.

(b) V-structure.

(c) Y-structure.

Figure 3: Different types of j-junctions.

Case (a) only appears when $w_{j+1} \neq 0$. Cases (b) and (c) can only appear when $w_{j+1} = 0$. In (c), the j-edges and j_+-edges meet and separate at different junctions, while in (b), this happens at the same junction.

Definition 10. *Let Γ be a standard graph with a j-junction u. A* structure *at the j-junction is the subgraph Ξ of Γ consisting of the following elements:*

- *The edges labeled j that appear before u following the j-edges.*
- *The edges labeled j_+ that appear after u following the j_+-edges.*
- *All vertices the above edges are connected to.*

If u is as in Fig. 3a, Ξ is called an I-structure at the j-junction. If u is as in Fig. 3b, Ξ is called a V-structure at the j-junction. If u is as in Fig. 3c, Ξ is called a Y-structure at the j-junction.

Since there can be at most one j-junction, there can be at most one structure at the j-junction.

A graph Γ shown in Fig. 4 has one (dotted) I-structure at the 1-junction v, one (dotted) I-structure at the 4-junction u, one (dashed) V-structure at the 2-junction v and one (dotdashed) Y-structure at the 5-junction u.
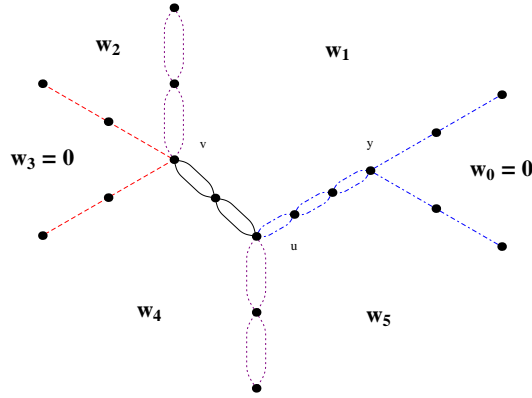
Figure 4: Graph Γ with (dotted) I-structures, a (dashed) Y-structure and a (dotdashed) Y-structure.

Note that the Y-structure is the only kind of structure that contains an additional junction. We refer to such junctions as *Y-junctions*. For example, the junction marked y in Fig. 4 is a Y-junction.

2.5. **Describing trees and junctions.** Let Γ be a graph with n branches, and Λ be the associated tree with all non-junction vertices removed. The dual graph $\hat{\Lambda}$ of Λ, is an n-gon where some non-intersecting chords are present. The junctions of Λ is in one-to-one correspondence with faces of $\hat{\Lambda}$ and vice versa. Two vertices are connected with an edge in $\hat{\Lambda}$ if and only if the corresponding faces are adjacent in Λ.

The extra condition that subdominant faces do not share an edge, implies that there are no chords connecting vertices in $\hat{\Lambda}$ corresponding to subdominant faces. For trees without this condition, we have the following lemma:

Lemma 11. *The number of $n + 1$-gons with non-intersecting chords is equal to the number of bracketings of a string with n letters, such that each bracket pair contains at least two symbols.*

Proof. See Theorem 1 in [SS00]. □

The sequence $s(n)$ of bracketings of a string with $n+1$ symbols are called the small Schröder numbers, see [SS00]. The first entries are $s(n)_{n\geq 0} = 1, 1, 3, 11, 45, 197, \ldots$.

The condition that chords should not connect vertices corresponding to subdominant faces, translates into a condition on the first and last symbol in some bracket pair.

3. ACTIONS ON GRAPHS

3.1. **Definitions.** Let us now return to the cell decomposition Ψ_0 in Fig. 1a. Let w_j be a non-zero asymptotic value of f. Choose non-intersecting paths $\beta_j(t)$ and $\beta_{j_+}(t)$ in $\bar{\mathbf{C}}_w$ with $\beta_j(0) = w_j$, $\beta_j(1) = w_{j_+}$ and $\beta_{j_+}(0) = w_{j_+}$,

$\beta_{j_+}(1) = w_j$ so that they do not intersect γ_k for $k \neq j, j_+$ and such that the union of these paths is a simple contractible loop oriented counterclockwise. These paths define a continuous deformation of the loops γ_j and γ_{j_+} such that the two deformed loops contain $\beta_j(t)$ and $\beta_{j_+}(t)$, respectively, and do not intersect any other loops during the deformation (except at ∞). We denote the action on Ψ_0 given by $\beta_j(t)$ and $\beta_{j_+}(t)$ by A_j. Basic properties of the fundamental group of a punctured plane, allows one to express the new loops in terms of the old ones:

$$A_j(\gamma_k) = \begin{cases} \gamma_j \gamma_{j_+} \gamma_j^{-1} \text{ if } k = j \\ \gamma_j \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases} , \quad A_j^{-1}(\gamma_k) = \begin{cases} \gamma_{j_+} \text{ if } k = j \\ \gamma_{j_+}^{-1} \gamma_j \gamma_{j_+} \text{ if } k = j_+ \\ \gamma_k \text{ otherwise} \end{cases}$$

Let f_t be a deformation of f. Since a continuous deformation does not change the graph, the deformed graph corresponding to $f_1^{-1}(A_j(\Psi_0))$ is the same as Γ. Let Γ' be this deformed graph with labels j and j_+ exchanged. Then the j-edges of Γ' are $f_1^{-1}(A_j(\gamma_{j_+})) = f_1^{-1}(\gamma_j)$, hence they are the same as the j-edges of $A_j(\Gamma)$. The j_+-edges of Γ' are $f_1^{-1}(A_j(\gamma_j))$. Since $\gamma_{j_+} = \gamma_j^{-1} A_j(\gamma_j)\gamma_j$, (reading left to right) this means that a j_+-edge of $A_j(\Gamma)$ is obtained by moving backwards along a j-edge of Γ', then along a j_+-edge of Γ', followed by a j-edge of Γ'.

These actions, together with their inverses, generate the Hurwitz (or sphere) braid group $\mathcal{H}_m$, where m is the number of non-zero asymptotic values. For a definition of this group, see [LZ04]. The action A_j on the loops in Ψ_0 is presented in Fig. 1b.

The property (V) of the eigenfunctions implies that each A_j induces a monodromy transformation of the cell decomposition Φ_0, and of the associated directed graph Γ.

Reading the action *right to left* gives the new edges in terms of the old ones, as follows:

Applying A_j to Γ can be realized by first interchanging the labels j and j_+. This gives an intermediate graph Γ'. A j-edge of $A_j(\Gamma)$ starting at the vertex v ends at a vertex obtained by moving from v following first the j-edge of Γ' backwards, then the j_+-edge of Γ', and finally the j-edge of Γ'. If any of these edges does not exist, we just do not move. If we end up at the same vertex v, there is no j-edge of $A_j(\Gamma)$ starting at v. All k-edges of $A_j(\Gamma)$ for $k \neq j$ are the same as k-edges of Γ'.

An example of the action A_1 is presented in Fig. 5. Note that A_j^2 preserves the standard cyclic order.

3.2. **Properties of the actions.**

Lemma 12. *Let Γ be a standard graph with no j-junction. Then $A_j^2(\Gamma) = \Gamma$.*

Proof. Since we assume $d > 2$, lemma 8 implies that the boundaries of the faces of Γ labeled j and j_+ do not have a common vertex. From the definition of the actions in subsection 3, the graphs Γ and $A_j(\Gamma)$ are the same, except that the labels j and j_+ are permuted. Applying the same argument again gives $A_j^2(\Gamma) = \Gamma$. $\square$
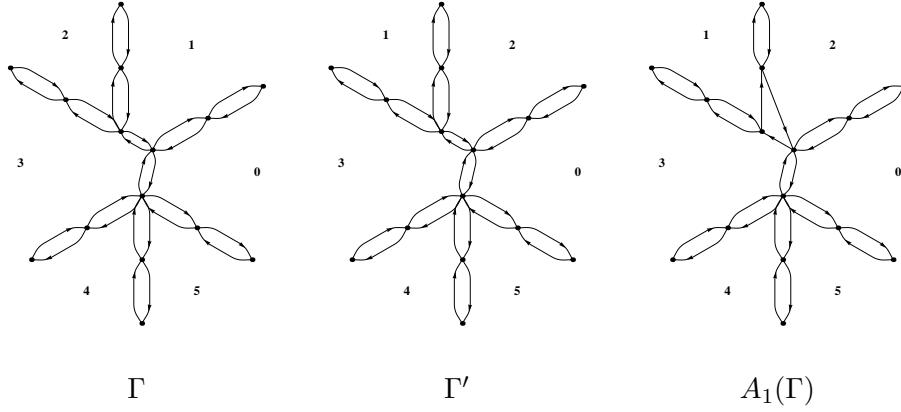
Figure 5: The action A_1. All sectors are dominant.

Theorem 13. *Let* Γ *be a standard graph with a* j-*junction* u. *Then* $A_j^2(\Gamma) \neq \Gamma$, *and the structure at the* j-*junction is moved one step in the direction of the* j-*edges under* A_j^2. *The inverse of* A_j^2 *moves the structure at the* j-*junction one step backwards along the* j_+-*edges.*

Proof. There are three cases to consider, namely I-structures, V-structures and Y-structures resp.

Case 1: The structure at the j-junction is an I-structure and Γ is as in Fig. 6a. The action A_j first permutes the asymptotic values w_j and w_{j_+}, then transforms the new j- and j_+-edges, as defined in subsection 3. The resulting graph $A_j(\Gamma)$ is shown in Fig. 6b. Applying A_j to $A_j(\Gamma)$ yields the graph shown in Fig. 6c.

Case 2: The structure at the j-junction is a V-structure and Γ is as in Fig. 7a. The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Fig. 7b and in Fig. 7c respectively.

Case 3: The structure at the j-junction is a Y-structure and Γ is as in Fig. 8a. The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Fig. 8b and in Fig. 8c respectively. The statement for A_j^{-2} is proved similarly. $\square$

Examples of the actions are given in Appendix, Figs. 16, 17 and 18.

3.3. **Contraction theorems.**

Definition 14. *Let* Γ *be a standard graph and let* u_0 *be a junction of* Γ. *The* u_0-*metric of* Γ, *denoted* $|\Gamma|_{u_0}$ *is defined as*

$$|\Gamma|_{u_0} = \sum_v (\deg(v) - 2) \, |v - u_0|$$

where the sum is taken over all vertices v *of* T_Γ. *Here* $deg(v)$ *is the total degree of the vertex* v *in* T_Γ *and* $|v - u_0|$ *is the length of the shortest path from* v *to* u_0 *in* T_Γ. *(Note that the sum in the right hand side is finite, since only junctions make non-zero contributions.)*

Definition 15. *A standard graph* Γ *is in* ivy form *if* Γ *is the union of the structures connected to a junction* u. *Such junction is called a* root *junction.*
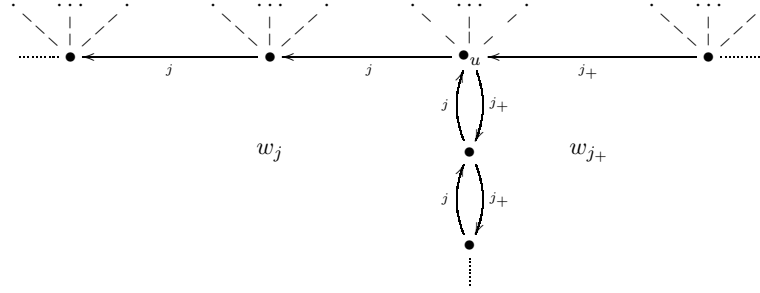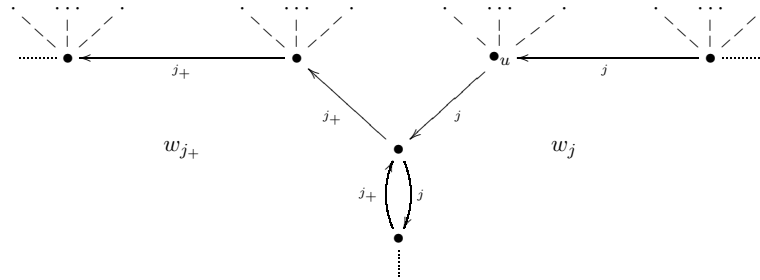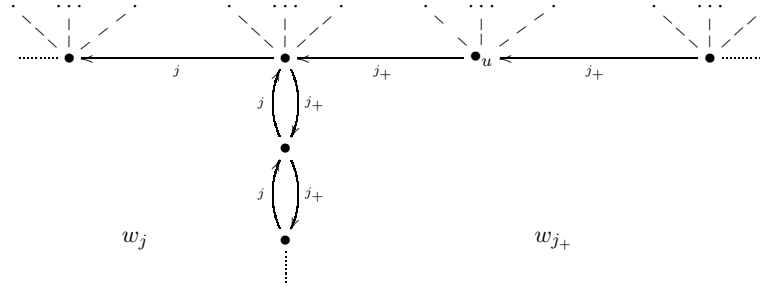
(a) Graph Γ with an I-structure


(b) Graph $A_j(\Gamma)$

(c) Graph $A_j^2(\Gamma)$

Figure 6: Case 1, moving an I-structure.

Lemma 16. *The graph Γ is in ivy form if and only if all but one of its junctions are Y-junctions.*

Proof. This follows from the definitions of the structures. $\square$

Theorem 17. *Let Γ be a standard graph. Then there is a sequence of actions $A^* = A_{j_1}^{\pm 2}, A_{j_2}^{\pm 2}, \ldots$, such that $A^*(\Gamma)$ is in ivy form.*

Proof. Assume that Γ is not in ivy form. Let U be the set of junctions in Γ that are not Y-junctions. Since Γ is not in ivy form, $|U| \geq 2$. Let $u_0 \neq u_1$ be two junctions in U such that $|u_0 - u_1|$ is maximal. Let p be the path from u_0 to u_1 in T_Γ. It is unique since T_Γ is a tree. Let v be the vertex immediately preceeding u_1 on the path p. The edge from v to u_1 in T_Γ is adjacent to at least one dominant face with label j such that $w_j \neq 0$. Therefore, there exists a j-edge between v and u_1 in Γ. Suppose first that this j-edge is directed

(a) Graph Γ with a V-structure
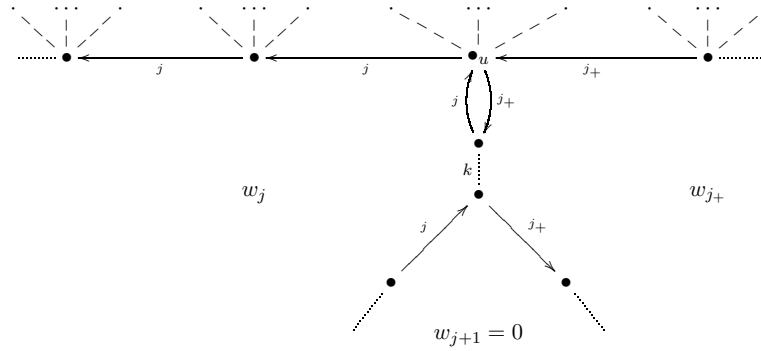

(b) Graph $A_j(\Gamma)$

(c) Graph $A_j^2(\Gamma)$
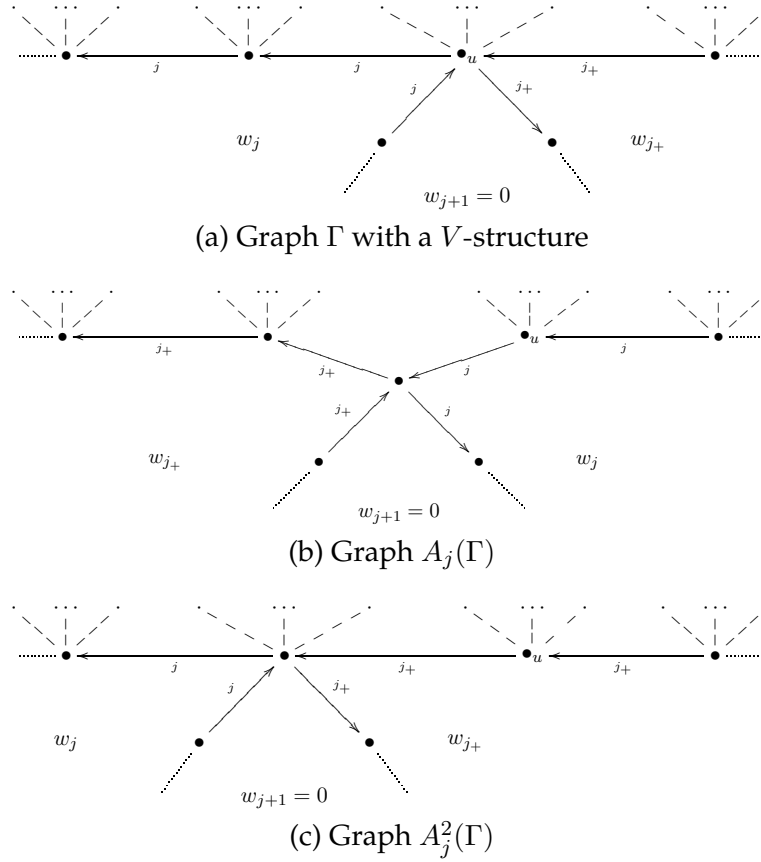
Figure 7: Case 2, moving a V-structure.

from u_1 to v. Let us show that in this case u_1 must be a j-junction, i.e., the dominant face labeled j_+ is adjacent to u_1.

Since u_1 is not a Y-junction, there is a dominant face adjacent to u_1 with a label $k \neq j, j_+$. Hence no vertices of p, except possibly u_1 may be adjacent to j_+-edges. If u_1 is not a j-junction, there are no j_+-edges adjacent to u_1. This implies that any vertex of Γ adjacent to a j_+-edge is further away from u_0 that u_1.

Let u_2 be the closest to u_1 vertex of Γ adjacent to a j_+-edge. Then u_2 should be a junction of T_Γ, since there are two j_+-edges adjacent to u_2 in Γ and at least one more vertex (on the path from u_1 to u_2) which is connected to u_2 by edges with labels other than j_+. Since u_2 is further away from u_0 than u_1 and the path p is maximal, u_2 must be a Y-junction. If the j-edges and j_+-edges would meet at u_2, u_1 would be a j-junction. Otherwise, a subdominant face labeled $j + 1$ would be adjacent to both u_1 and u_2, while a subdominant face adjacent to a Y-junction cannot be adjacent to any other junctions.
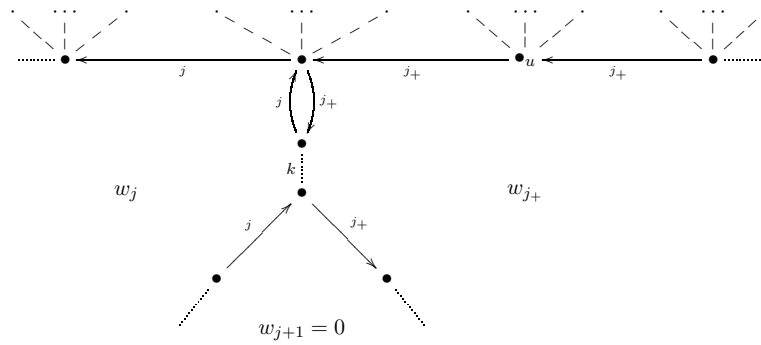
Hence u_1 must be a j-junction. By Theorem 13, the action A_j^2 moves the structure at the j-junction u_1 one step closer to u_0 along the path p, decreasing $|\Gamma|_{u_0}$ at least by 1.

(a) Graph Γ with a Y-structure
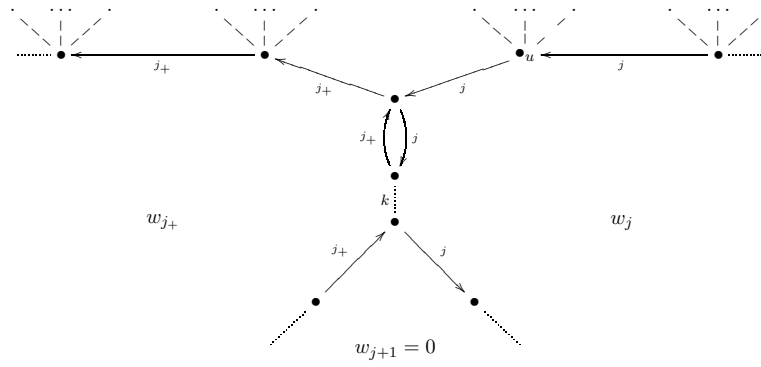
(b) Graph $A_j(\Gamma)$

(c) Graph $A_j^2(\Gamma)$

Figure 8: Case 3, moving a Y-structure.

The case when the j-edge is directed from v to u_1 is treated similarly. In that case, u_1 must be a j_--junction, and the action $A_{j_-}^{-2}$ moves the structure at the j_--junction u_1 one step closer to u_0 along the path p.

We have proved that if $|U| > 1$ then $|\Gamma|_{u_0}$ can be reduced. Since it is a non-negative integer, after finitely many steps we must reach a stage where $|U| = 1$, hence the graph is in ivy form. $\square$
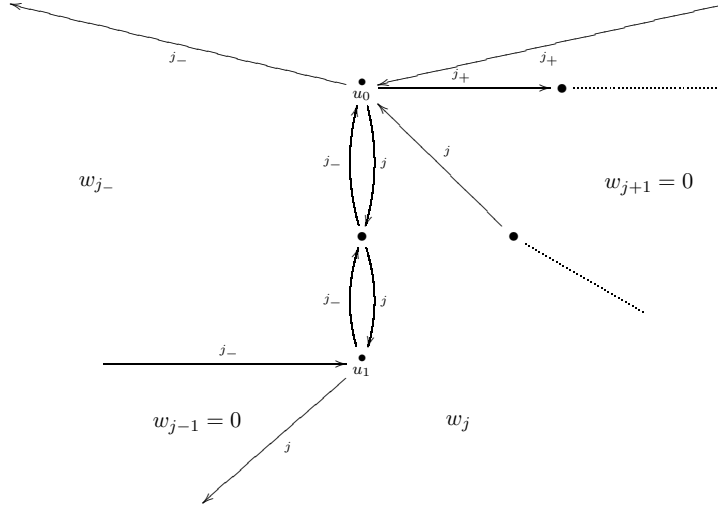
Figure 9: Adjacent Y- and V-structures.

Remark 18. *The outcome of the algorithm is in general non-unique, and might yield different final values of $|A^*(\Gamma)|_{u_0}$.*

Lemma 19. *Let Γ be a standard graph with a junction u_0 such that u_0 is both a j_--junction and a j-junction. Assume that the corresponding structures are of types Y and V, in any order. Then there is a sequence of actions from the set $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ that interchanges the Y-structure and the V-structure.*

Proof. We may assume that the Y- and V-structures are attached to u_0 counterclockwise around u_0, as in Fig. 9, otherwise we reverse the actions. By Theorem 13, the action A_j^{2k} moves the V-structure k steps in the direction of the j-edges. Choose k so that the V-structure is moved all the way to u_1, as in Fig. 10. Then u_1 becomes both a j_--junction and j-junction, with two V-structures attached. Proceed by applying $A_{j_-}^{2k}$ to move the V-structure at the j_--junction u_1 up to u_0, as in Fig. 11. $\qquad\square$

Lemma 20. *Let Γ be a standard graph with a junction u_0, such that u_0 is both a j_--junction and a j-junction, with the corresponding structures of type I and Y, in any order. Then there is a sequence of actions from the set $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ converting the Y-structures to a V-structure.*

Proof. We may assume that the I- and Y-structures are attached to u_0 counterclockwise around u_0, as in Fig. 12, otherwise, we just reverse the actions. By Theorem 13, we can apply $A_{j_-}^{-2}$ several times to move the I-structure down to u_1. *(For example, in Fig. 12, we need to do this twice. This gives the configuration shown in Fig. 13.)* Now u_1 becomes a j_--junction and a j-structure, with the I- and V-structures attached. Applying A_j^{2k}, we can move the V-structure at u_1 up to u_0. *(In our example, this final configuration is presented in Fig. 14.)* Thus the Y-structure has been transformed to a V-structure. $\quad\square$
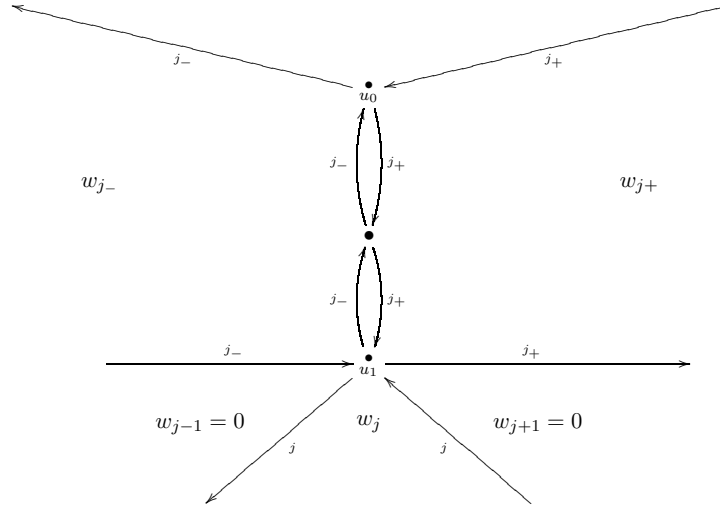
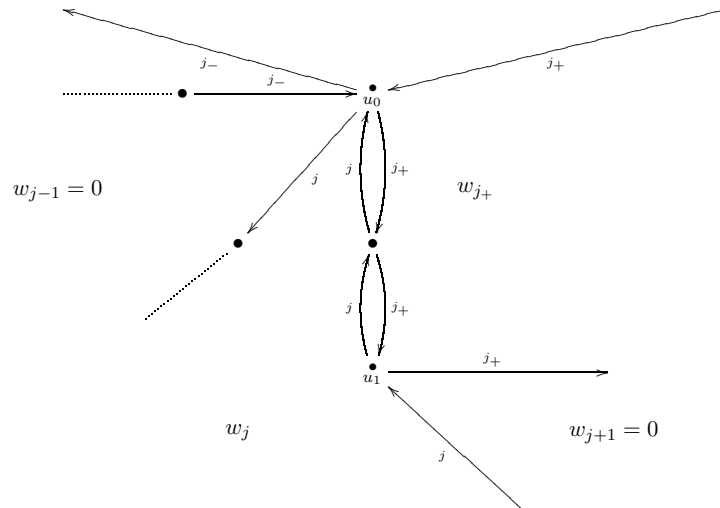Figure 10: Intermediate configuration: two adjacent V-structures.

Figure 11: Y- and V-structures exchanged.

Theorem 21. *Let Γ be a standard graph with at least two adjacent dominant faces. Then there exists a sequence of actions $A^* = A_{j_1}^{\pm 2} A_{j_2}^{\pm 2} \ldots$ such that $A^*(\Gamma)$ have only* one *junction.*

Proof. By Theorem 17 we may assume that Γ is a graph in ivy form with the root junction u_0. The existence of two adjacent dominant faces implies the existence of an I-structure. If there are only I-structures and V-structures, then u_0 is the only junction of Γ. Assume that there is at least one Y-structure. By Lemma 19, we may move a Y-structure so that it is counterclockwise next to an I-structure. By Lemma 20, the Y-structure can be transformed to a V-structure, and the Y-junction removed. This can be repeated, eventually removing all junctions of Γ except u_0. $\qquad\square$
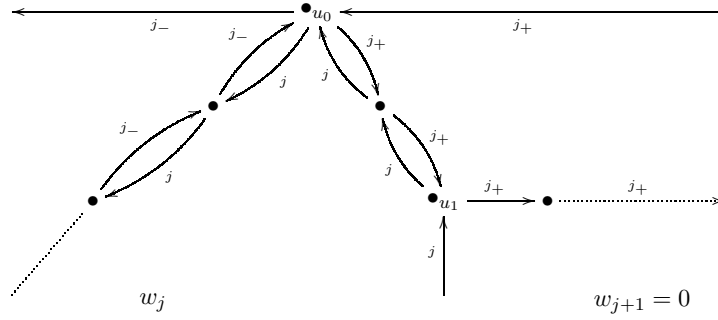
Figure 12: Adjacent I- and Y-structures
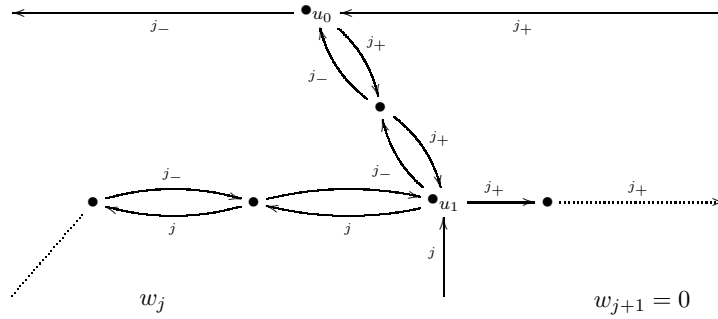
Figure 13: Moving the I-structure to u_1
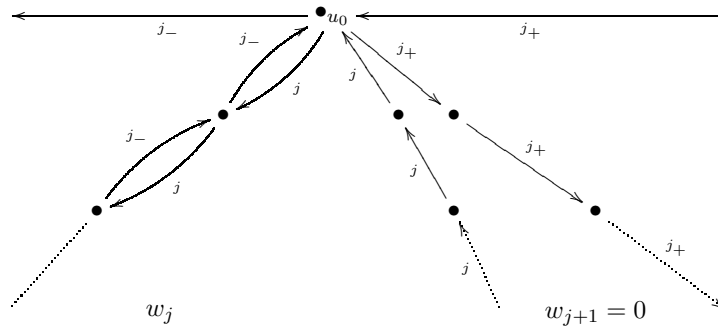
Figure 14: Moving the V-structure to u_0

Lemma 22. *Let Γ be a standard graph with a junction u_0, such that u_0 is both a j_--junction and a j-junction, with two adjacent Y-structures attached. Then there is a sequence of actions from the set $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ converting one of the Y-structures to a V-structure.*

Proof. This can be proved by the arguments similar to those in the proof of Theorem 21. □

Theorem 23. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then there exists a sequence of actions $A^* = A_{j_1}^{\pm 2}, A_{j_2}^{\pm 2}, \ldots$, such that $A^*(\Gamma)$ is in ivy form, with at most one Y-structure.*

Proof. One may assume by Theorem 17 that Γ is in ivy form, with the root junction u_0. Since no two dominant faces are adjacent, there are only V- and Y-structures attached to u_0. If there are at least two Y-structures, we may assume, by Lemma 19, that two Y-structures are adjacent. By Lemma 22, two adjacent Y-structures can be converted to a V-structure and a Y-structure. This can be repeated until at most one Y-structure remains in Γ. $\qquad\square$

Lemma 24. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then the number of bounded faces of Γ is finite and does not change after any action A_j^2.*

Proof. The bounded faces of Γ correspond to the edges of T_Γ separating two dominant faces. Since no two dominant faces are adjacent, any two dominant faces have a finite common boundary in T_Γ. Hence the number of bounded faces of Γ is finite. Lemma 12 and Theorem 13 imply that this number does not change after any action A_j^2. $\qquad\square$

4. IRREDUCIBILITY AND CONNECTIVITY OF THE SPECTRAL LOCUS

In this section, we prove the main results stated in the introduction. We start with the following statements.

Lemma 25. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$ such that equation (1) admits a solution subdominant in non-adjacent Stokes sectors $S_{j_1}, \ldots, S_{j_k}$, $k \leq (d+2)/2$. Then Σ is a smooth complex analytic submanifold of $\mathbf{C}^d$ of the codimension $k - 1$.*

Proof. Let f be a ratio of two linearly independent solutions of (1), and let $w = (w_0, \ldots, w_{d+1})$ be the set of asymptotic values of f in the Stokes sectors $S_0, \ldots, S_{d+1}$. Then w belongs to the subset Z of $\bar{\mathbf{C}}^{d+2}$ where the values w_j in adjacent Stokes sectors are distinct and there are at least three distinct values among w_j. The group G of fractional-linear transformations of $\bar{\mathbf{C}}$ acts on Z diagonally, and the quotient Z/G is a $(d - 1)$-dimensional complex manifold.

Theorem 7.2, [Bak77] implies that the mapping $W : \mathbf{C}^d \to Z/G$ assigning to (α, λ) the equivalence class of w is submersive. More precisely, W is locally invertible on the subset $\{\alpha_{d-1} = 0\}$ of $\mathbf{C}^d$ and constant on the orbits of the group $\mathbf{C}$ acting on $\mathbf{C}^d$ by translations of the independent variable z. In particular, the preimage $W^{-1}(Y)$ of any smooth submanifold $Y \subset Z/G$ is a smooth submanifold of $\mathbf{C}^d$ of the same codimension as Y.

The set Σ is the preimage of the set $Y \subset Z/G$ defined by the $k - 1$ conditions $w_{j_1} = \cdots = w_{j_k}$. Hence Σ is a smooth manifold of codimension $k - 1$ in $\mathbf{C}^d$. $\qquad\square$

Proposition 26. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$ such that equation (1) admits a solution subdominant in the non-adjacent Stokes sectors $S_{j_1}, \ldots, S_{j_k}$. If at least two remaining Stokes sectors are adjacent, then Σ is an irreducible complex analytic manifold.*

Proof. Let Σ_0 be the intersection of Σ with the subspace $\mathbf{C}^{d-1} = \{\alpha_{d-1} = 0\} \subset \mathbf{C}^d$. Then Σ has the structure of a product of Σ_0 and $\mathbf{C}$ induced by translation of the independent variable z. In particular, Σ is irreducible if and only if Σ_0 is irreducible.

Let us choose a point $w = (w_0, \ldots, w_{d+1})$ so that $w_{j_1} = \cdots = w_{j_k} = 0$, with all other values w_j distinct, non-zero and finite. Let Ψ_0 be a cell decomposition of $\bar{\mathbf{C}} \setminus \{0\}$ defined by the loops γ_j starting and ending at ∞ and containing non-zero values w_j, as in Section 2.1.

Nevanlinna theory (see [Nev32, Nev53]), implies that, for each standard graph Γ with the properties listed in Lemma 8, there exists $(\alpha, \lambda) \in \mathbf{C}^d$ and a meromorphic function $f(z)$ such that f is the ratio of two linearly independent solutions of (1) with the asymptotic values w_j in the Stokes sectors S_j, and Γ is the graph corresponding to the cell decomposition $\Phi_0 = f^{-1}(\Psi_0)$. This function, and the corresponding point (α, λ) is defined uniquely up to translation of the variable z. We can choose f uniquely if we require that $\alpha_{d-1} = 0$ in (α, λ). Conditions on the asymptotic values w_j imply then that $(\alpha, \lambda) \in \Sigma'$. Let f_Γ be this uniquely selected function, and $(\alpha_\Gamma, \lambda_\Gamma)$ the corresponding point of Σ'.

Let $W : \Sigma' \to Y \subset Z/G$ be as in the proof of Lemma 25. Then Σ' is an unramified covering of Y. Its fiber over the equivalence class of w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ. Each action A_j^2 corresponds to a closed loop in Y starting and ending at w. Since for a given list of subdominant sectors a standard graph with one vertex is unique, Theorem 21 implies that the monodromy action is transitive. Hence Σ' is irreducible as a covering with a transitive monodromy group (see, e.g., [Kho04, §5]). $\square$

This immediately implies Theorem 2, and we may also state the following corollary equivalent to Theorem 1:

Corollary 27. *For every potential P_α of even degree, with $\deg P_\alpha \geq 4$ and with the boundary conditions $y \to 0$ for $z \to \pm\infty$, $z \in \mathbf{R}$, there is an analytic continuation from any eigenvalue λ_m to any other eigenvalue λ_n in the α-plane.*

Proposition 28. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbf{C}^d$, for even d, such that equation (1) admits a solution subdominant in the $(d+2)/2$ Stokes sectors $S_0, S_2, \ldots, S_d$. Then irreducible components Σ_k, $k = 0, 1, \ldots$ of Σ, which are also its connected components, are in one-to-one correspondence with the sets of standard graphs with k bounded faces. The corresponding solution of (1) has k zeros and can be represented as $Q(z)e^{\phi(z)}$ where Q is a polynomial of degree k and ϕ a polynomial of degree $(d+2)/2$.*

Proof. Let us choose w and Ψ_0 as in the proof of Proposition 26. Repeating the arguments in the proof of Proposition 26, we obtain an unramified covering $W : \Sigma' \to Y$ such that its fiber over w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ with the properties listed in Lemma 8. Since we have no adjacent dominant sectors, Theorem 23 implies that any standard graph Γ can be transformed by the monodromy action to a graph Γ_0 in ivy form with at most one Y-structure attached at its j-junction, where j is any index such that S_j is a dominant sector. Lemma 24 implies that Γ and Γ_0

have the same number k of bounded faces. If $k = 0$, the graph Γ_0 is unique. If $k > 0$, the graph Γ_0 is completely determined by k and j. Hence for each $k = 0, 1, \ldots$ there is a unique orbit of the monodromy group action on the fiber of W over w consisting of all standard graphs Γ with k bounded faces. This implies that Σ' (and Σ) has one irreducible component for each k.

Since Σ is smooth by Lemma 25, its irreducible components are also its connected components.

Finally, let $f_\Gamma = y/y_1$ where y is a solution of (1) subdominant in the Stokes sectors $S_0, S_2, \ldots, S_d$. Then the zeros of f and y are the same, each such zero belongs to a bounded domain of Γ, and each bounded domain of Γ contains a single zero. Hence y has exactly k simple zeros. Let Q be a polynomial of degree k with the same zeros as y. Then y/Q is an entire function of finite order without zeros, hence $y/Q = e^\phi$ where ϕ is a polynomial. Since y/Q is subdominant in $(d + 2)/2$ sectors, $\deg \phi = (d + 2)/2$. □

The above propisition immediately implies Theorem 3.

5. ALTERNATIVE VIEWPOINT

In this section, we provide an example of the correspondence between the actions on cell decompositions with some subdominant sectors and actions on cell decompositions with no subdominant sectors. This correspondence can be used to simplify calculations with cell decompositions. We will illustrate our results on a cell decomposition with 6 sectors, the general case follows immediately.

Let C_6 be the set of cell decompositions with 6 sectors, none of them subdominant. Let $C_6^{03} \subset C_6$ be the set of cell decompositions such that for any $\Gamma \in C_6^{03}$, the sectors S_0 and S_3 do not share a common edge in the associated undirected graph T_Γ. Define D_6^{03} to be the set of cell decompositions with 6 sectors where S_0 and S_3 are subdominant.

Lemma 29. *There is a bijection between C_6^{03} and D_6^{03}.*

Proof. Let $\Gamma \in C_6^{03}$ be a cell decomposition, and let T_Γ be the associated undirected graph, see section 2.2. Then consider T_Γ as the (unique) undirected graph associated with some cell decomposition $\Delta \in D_6^{03}$. This is possible since the condition that the sectors 0 and 3 do not share a common edge in Γ, ensures that the subdominant sectors in Δ do not share a common edge. Let us denote this map π. Conversely, every cell decomposition $\Delta \in D_6^{03}$ is associated with a cell decomposition $\Gamma \in C_6^{03}$ by the inverse procedure π^{-1}. □

We have previously established that $\mathcal{H}_6$ acts on C_6 and that $\mathcal{H}_4$ acts on D_6^{03}. Let $B_0, B_1, \ldots, B_5$ be the actions generating $\mathcal{H}_6$, as described in subsection 3, and let A_1, A_2, A_4, A_5 generate $\mathcal{H}_4$. Let $\mathcal{H}_6^{03} \subset \mathcal{H}_6$ be the subgroup generated by $B_1, B_2 B_3 B_2^{-1}, B_4, B_5 B_0 B_5^{-1}$, and their inverses. It is easy to see that $\mathcal{H}_6^{03}$ acts on elements in C_6^{03} and preserves this set.

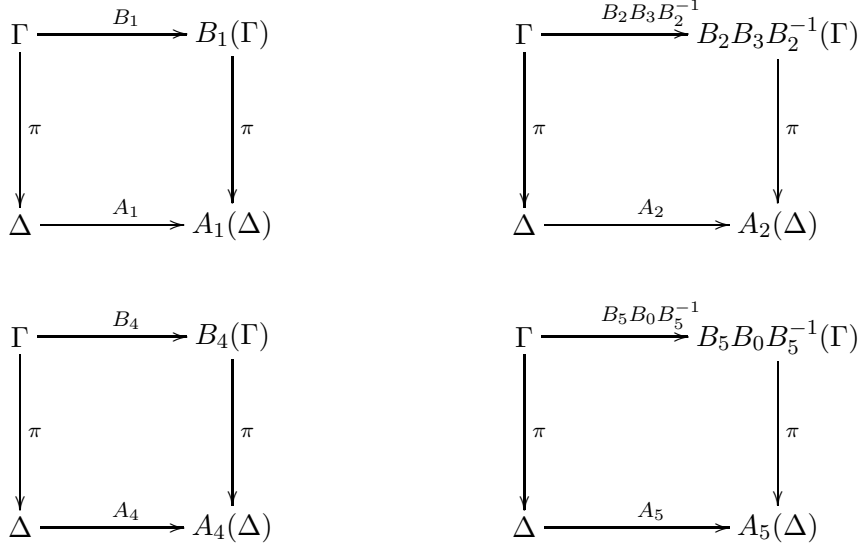**Lemma 30.** *The diagrams in Fig. 15 commute.*

$$\Gamma \xrightarrow{\;B_1\;} B_1(\Gamma)$$

$$\pi \Big\downarrow \qquad\qquad \Big\downarrow \pi$$

$$\Delta \xrightarrow{\;A_1\;} A_1(\Delta)$$

$$\Gamma \xrightarrow{\;B_2 B_3 B_2^{-1}\;} B_2 B_3 B_2^{-1}(\Gamma)$$

$$\pi \Big\downarrow \qquad\qquad \Big\downarrow \pi$$

$$\Delta \xrightarrow{\;A_2\;} A_2(\Delta)$$

$$\Gamma \xrightarrow{\;B_4\;} B_4(\Gamma)$$

$$\pi \Big\downarrow \qquad\qquad \Big\downarrow \pi$$

$$\Delta \xrightarrow{\;A_4\;} A_4(\Delta)$$

$$\Gamma \xrightarrow{\;B_5 B_0 B_5^{-1}\;} B_5 B_0 B_5^{-1}(\Gamma)$$

$$\pi \Big\downarrow \qquad\qquad \Big\downarrow \pi$$

$$\Delta \xrightarrow{\;A_5\;} A_5(\Delta)$$

Figure 15: The commuting actions

Proof. Let (a, b, c, d, e, f) be the 6 loops of a cell decomposition Ψ_0 as in Fig. 1, looping around the asymptotic values $(w_0, \ldots, w_5)$. Let Ψ_0' be the cell decomposition with the four loops (b, c, e, f), such that if $\Gamma \in C_6^{03}$ is the preimage of Ψ_0, then $\pi(\Gamma)$ is the preimage of Ψ_0'. That is, the preimages of the loops a and d in Ψ_0 are removed under π.

B_j acts on Ψ_0 and A_j acts on Ψ_0'. (See subsection 3 for the definition.) We have

(5) $\qquad A_1(b, c, e, f) = (bcb^{-1}, e, f), \; A_4(b, c, d, e) = (b, c, efe^{-1}, e).$

and

(6) $\qquad \begin{aligned} &B_1(a, b, c, d, e, f) = (a, bcb^{-1}, d, e, f), \\ &B_4(a, b, c, d, e, f) = (a, b, c, efe^{-1}, e, f). \end{aligned}$

Equation (5) and (6) shows that the left diagrams commute, since applying π to the result from (6) yields (5). We also have that

(7) $\qquad A_2(b, c, e, f) = (b, cec^{-1}, c, f), \; A_5(b, c, e, f) = (f, c, e, fbf^{-1}).$

We now compute $B_3^{-1} B_2 B_3(a, b, c, d, e, f)$. Observe that we must apply these actions *left to right*:

(8) $\qquad \begin{aligned} B_3^{-1} B_2 B_3(a, b, c, d, e, f) &= B_2 B_3(a, b, c, e, e^{-1}de, f) \\ &= B_3(a, b, cec^{-1}, c, e^{-1}de, f) \\ &= (a, b, cec^{-1}, c(e^{-1}de)c^{-1}, c, f) \end{aligned}$

A similar calculation gives

(9) $\qquad B_0^{-1} B_5 B_0(a, b, c, d, e, f) = (f(b^{-1}ab)f^{-1}, f, c, d, e, f, b, f^{-1}),$

and applying π to the results (8) and (9) give (7). $\qquad\qquad\qquad \square$

Remark 31. *Note that $B_j^{-1}B_{j-1}B_j(\Gamma) = B_{j-1}B_jB_{j-1}^{-1}(\Gamma)$ for all $\Gamma \in C_6$, which follows from basic properties of the braid group.*

The above result can be generalized as follows: Let C_n be the set of cell decompositions with n sectors such that all sectors are dominant. Let $C_n^\mathbf{l} \subset C_n$, $\mathbf{l} = \{l_1, l_2, \ldots, l_k\}$ be the set of cell decompositions such that for any $\Gamma \in C_n^\mathbf{l}$, no two sectors in the set $S_{l_1}, S_{l_2}, \ldots, S_{l_k}$ have a common edge in the associated undirected graph T_Γ. Let $D_n^\mathbf{l}$ be the set of cell decompositions with n sectors such that the sectors $S_{l_1}, S_{l_2}, \ldots, S_{l_k}$ are subdominant. Let $\{A_j\}_{j \notin \mathbf{l}}$ be the $n - k$ actions acting on $C_n^\mathbf{l}$ indexed as in subsection 3. Let $\{B_j\}_{j=0}^{n-1}$ be the actions on C_n. Let $\pi : C_n^\mathbf{s} \to D_n^\mathbf{s}$ be the map similar to the bijection above, where one obtain a cell decomposition in $D_n^\mathbf{s}$ by removing edges with a label in $\mathbf{l}$ from a cell decomposition in $C_n^\mathbf{s}$. Then

$$(10) \qquad \begin{cases} \pi(B_j(\Gamma)) = A_j(\pi(\Gamma)) & \text{if } j, j+1 \notin \mathbf{l}, \\ \pi(B_j^{-1}B_{j-1}B_j(\Gamma)) = A_j(\pi(\Gamma)), & j \notin \mathbf{l}, j+1 \in \mathbf{l}. \end{cases}$$

Remark 32. *There are some advantages with cell decompositions with no subdominant sectors:*

- *An action A_j always interchanges the asymptotic values w_j and w_{j+1}.*
- *Lemma 8, item II implies T_Γ have no bounded faces iff order of the asymptotic values is a cyclic permutation of the standard order.*

6. APPENDIX

6.1. **Examples of monodromy action.** Below are some specific examples on how the different actions act on trees and non-trees.
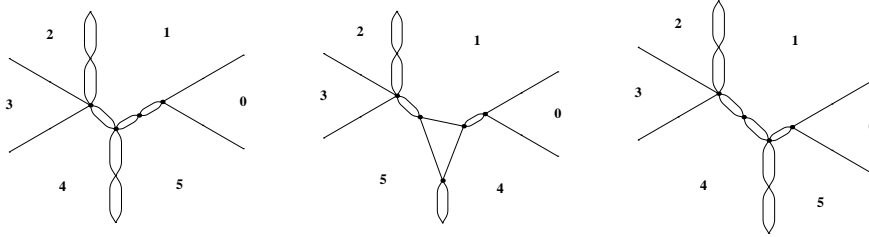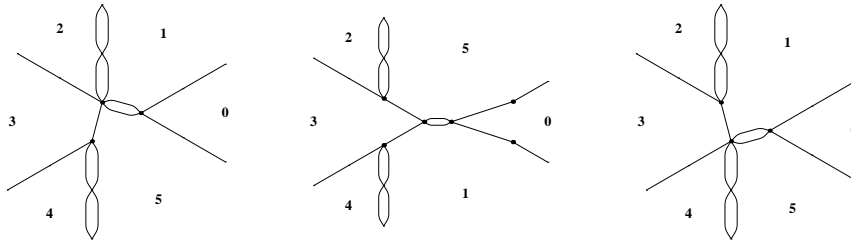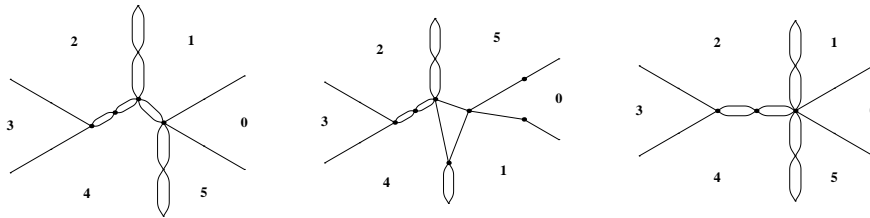

Figure 16: Example action of A_4^{-1} and A_4^{-2} in case 1.

Figure 17: Example action of A_5 and A_5^2 in case 2.

Figure 18: Example action of A_5^{-1} and A_5^{-2} in case 3.

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