Multiplicities of Pfaffian Intersections, and the Łojasiewicz Inequality

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Abstract: An effective estimate for the local multiplicity of a complete intersection of complex algebraic and Pfaffian varieties is given, based on a local complex analog of the Rolle-Khovanskii theorem. The estimate is valid also for the properly defined multiplicity of a non-isolated intersection. It implies, in particular, effective estimates for the exponents of the polar curves, and the exponents in the Łojasiewicz inequalities for Pfaffian functions. For the intersections defined by sparse polynomials, the multiplicities outside the coordinate hyperplanes can be estimated in terms of the number of non-zero monomials, independent of degrees of the monomials.

Introduction. The theory of Pfaffian manifolds, i.e. analytic manifolds defined by systems of Pfaffian equations with polynomial coefficients, was developed by Khovanskii [1, 2], see [2] for additional references. In the real domain, using a generalization of the Rolle theorem, Khovanskii showed that the number of isolated solutions of any system of Pfaffian equations can be effectively estimated in terms of the complexity of the Pfaffian functions involved. This allows also to estimate effectively global topological invariants of real varieties defined by Pfaffian equations.

The sparse polynomials (fewnomials) constitute an important class of Pfaffian functions. Fewnomials are defined as polynomials with a few non-zero monomials of arbitrary degree. Outside the coordinate hyperplanes, these polynomials can be defined as Pfaffian functions of complexity depending on the number of non-zero monomials only. This representation allows to estimate the topological complexity of a set of real solutions of a system of fewnomial equations in terms of the number of non-zero terms, independent of the degrees of these terms.

In the present paper, we develop the local complex analog of the Khovanskii theory, in order to give effective estimates for the multiplicities of the intersections of varieties defined by Pfaffian functions in the complex domain.

One difficulty that arises on this way, also in the real domain, is the emergence of non-isolated intersections in the process of reduction from Pfaffian to polynomial equations, even when the original intersection is isolated. To overcome this, we have to work with non-isolated intersections, and define the multiplicity of a one-parameter deformation of the intersection at the origin as the number of isolated solutions converging to the origin as the parameter of the deformation tends to 0. For a deformation of an isolated intersection, this number does not depend on the deformation and coincides with the usual multiplicity. For a non-isolated Pfaffian intersection, our method allows to estimate the maximum of the multiplicities over all deformations preserving complexity of the Pfaffian functions.

The principal technical tool, the estimate of the multiplicity of a non-isolated intersection of zeroes of a complex analytic function with an analytic curve, in terms of the number of zeroes of the differential of the function on the curve, is developed in section 1.

Applying this to the Pfaffian functions in section 2, we use a technique similar to that of Khovanskii [2] to estimate the multiplicity of a Pfaffian intersection through the multiplicity of a properly chosen polynomial intersection, the latter being estimated with the Bezout theorem. This gives our main result: the multiplicity of a Pfaffian intersection is effectively estimated in terms of the complexity of the Pfaffian functions involved.

This main result allows to estimate different geometric and analytic characteristics of the sets of solutions of Pfaffian equations, in complex and real domains. One of such applications is presented in section 3 where we give effective estimates of the exponents of the polar curve of a pair of Pfaffian functions, and of the exponent in the Łojasiewicz inequality for a Pfaffian function in the real domain.

The important special cases of exponential and sparse polynomials are considered at the end of the paper (section 4). In particular, for fewnomials in m complex variables with

r non-zero monomials, the multiplicity of any solution of a system of m equations outside the coordinate hyperplanes does not exceed $2^{r(r-1)/2}(m+1)^r$.

More sophisticated applications, including the complexity of elimination of universal quantifiers from semi-Pfaffian expressions and the complexity of the resolution of singularities of analytic sets defined by Pfaffian equations, will appear in separate papers.

It is known [3-9] that any expression containing real analytic functions, equalities, inequalities, arithmetic and logical operations, and universal and existential quantifiers, is equivalent to an expression of the same kind without universal (or without existential) quantifiers, as soon as all the eliminated variables remain bounded. The algorithm for the elimination of universal quantifiers suggested in [9] reduces the problem to certain finiteness properties of semi-analytic sets. If the original expression contains only Pfaffian functions, the algorithm in [9] allows to find an equivalent existential (i.e. without universal quantifiers) expression containing also only Pfaffian functions. Combined with the estimates for Pfaffian functions given in this paper, it allows to derive an explicit bound on the complexity of an equivalent existential expression, in terms of the complexity of the original expression and the degrees of the polynomials involved in the definition of the Pfaffian functions in the original expression. The first step in this direction, the estimate of complexity of a stratification of a semi-Pfaffian set, is presented in [10].

Note that this works only for the Pfaffian functions in a bounded domain, although the elimination of universal quantifiers from Pfaffian expressions is probably possible in an unbounded domain as well. In a special case of exponential polynomials this was shown by Wilkie [11] (see also [12]).

Finally, the algorithm of resolution of singularities suggested by Bierstone and Milman [7,13,14] allows, in the case when all the equations are Pfaffian, to produce the resolution of singularities where all the centers of the necessary blowing-ups are Pfaffian manifolds, and the complexity of these manifolds, as well as the number of the necessary blowings up can be effectively estimated.

1. Multiplicities of non-isolated intersections.

Definition 1.1. Let M be an n-dimensional complex analytic manifold, $\mathbf{0} \in M$, and let $\phi_0(x)$ be a germ of an analytic function in $(M, \mathbf{0})$. A germ of an analytic function $\phi(x, \epsilon)$ in $(M \times \mathbf{C},\ \mathbf{0} \times 0)$, with $\phi(x, 0) = \phi_0(x)$, is called a *deformation* of $\phi_0(x)$. We denote by $\phi_\epsilon(x)$ the function $\phi(x, \epsilon)$ for a fixed value of ϵ.

Let $\tilde{Z}$ be a germ of a reduced analytic subspace of $M \times \mathbf{C}$ without component imbedded in $\epsilon = 0$. The space $\tilde{Z}$ is called a *deformation* of $Z_0 = \tilde{Z} \cap \{\epsilon = 0\}$. As before, we define $Z_\epsilon = \tilde{Z} \cap \{\epsilon = \text{const}\}$. Note that the spaces Z_ϵ are reduced, for small $\epsilon \neq 0$, while Z_0 is not necessarily reduced.

Definition 1.2. Let $\dim \tilde{Z} = 2$, hence $\dim Z_\epsilon = 1$, for small ϵ, and let $\phi(x, \epsilon)$ be a deformation of an analytic function $\phi_0(x)$. We define the *multiplicity* $\#(\phi, \tilde{Z})$ of the intersection $\{\phi = 0\} \cap \tilde{Z}$ at $\mathbf{0}$ as the number of isolated zeroes, counted with their multiplicities, of $\phi|_{Z_\epsilon}$, $\epsilon \neq 0$, converging to $\mathbf{0}$ as $\epsilon \to 0$. For a meromorphic function $\tau(x, \epsilon) = \phi(x, \epsilon)/\psi(x, \epsilon)$, we define the multiplicity $\#(\tau, \tilde{Z}) = \#(\phi, \tilde{Z}) - \#(\psi, \tilde{Z})$.

Lemma 1.1. *Let ϕ and ψ be two analytic functions, with $\{\phi = 0\} \cap Z_\epsilon$ discrete, for small $\epsilon \neq 0$, and $\{\psi = 0\} \cap Z_0 = \mathbf{0}$. For small $\psi \neq 0$, let $Z_j = Z_j(\psi)$ be the decomposition of the germ of the one-dimensional set $\tilde{Z} \cap \{\psi = \text{const}\}$ at $\epsilon = 0$ into irreducible components, and let ν_j be the degree of $\pi|_{Z_j}$. Let*

$$\phi|_{Z_j} = u_j(\psi)\epsilon^{k_j} + o(\epsilon^{k_j}) \tag{1}$$

be the Puiseux expansion at $\epsilon = 0$, with $u_j \not\equiv 0$ and rational $k_j \geq 0$. Then

$$u_j(\psi) = \psi^{\mu_j} f_j(\psi), \quad f_j(0) \neq 0, \tag{2}$$

with rational μ_j, and the multiplicity $\#(\phi, \tilde{Z})$ is equal to $\sum_j \nu_j \mu_j$.

Proof. As $\{\psi = 0\} \cap Z_0 = \mathbf{0}$, the map $\pi = (\psi, \epsilon) : \tilde{Z} \to \mathbf{C}^2$ is finite, of the degree $\nu = \#(\psi_0, Z_0)$. Due to the Weierstrass preparation theorem, the function $\phi|_{\tilde{Z}}$ satisfies an equation $P(\phi,\psi,\epsilon)=0$ where P is a distinguished pseudopolynomial in ϕ of the degree ν, with coefficients analytic in (ψ,ϵ).

The function $\phi|_{Z_j}$ does not vanish identically because the intersection $\{\phi=0\}\cap Z_\epsilon$ is discrete, for $\epsilon\neq 0$. Hence $\phi|_{Z_j}$ has the Puiseux expansion (1) at $\epsilon=0$. Here $o(\epsilon^{k_j})$ is also a root of a distinguished pseudopolynomial with coefficients analytic in ψ and ϵ.

Let $k_j=p/q$, with integer p and q. Let us define $Q_j(u,\psi,\delta)$ as a result of reduction of the common power of δ in the coefficients of $P(\delta^p u,\psi,\delta^q)$. We find that $u_j(\psi,\delta)=\phi|_{Z_j}/\delta^p$ is a root of the pseudopolynomial Q_j, and $u_j(\psi)=\lim_{\delta\to 0}u_j(\psi,\delta)$ is a root of a non-zero (not distinguished) pseudopolynomial $Q_j(u,\psi)=Q_j(u,\psi,0)$ of degree $\nu'\leq\nu$ in u, with the coefficients analytic in ψ in the vicinity of $\psi=0$. Let the term $u^{\nu'}$ appear in $Q_j(u,\psi)$ with a coefficient that has a zero of the order κ_j at $\psi=0$. Then $\psi^{\kappa_j}u_j(\psi)$ is a root of the monic pseudopolynomial $\psi^{(\nu'-1)\kappa_j}Q(u/\psi^{\kappa_j},\psi)$, and has a Puiseux expansion (2) at 0, with a rational exponent $\mu_j\geq -\kappa_j$ and a multi-valued analytic function $f_j(\psi)$, i.e. $f_j(\psi)$ is a root of a monic pseudopolynomial in f with analytic coefficients in ψ.

Let $\zeta(\psi,\epsilon)=P(0,\psi,\epsilon)$ be the product of $\phi(x,\epsilon)$ over $x\in\pi^{-1}(\psi,\epsilon)$, with the proper multiplicities. The function ζ is analytic in ψ and ϵ, and the multiplicity $\mu=\#(\phi,\tilde{Z})$ is equal to the number of zeroes (with multiplicities) of $\zeta|_{\epsilon=\mathrm{const}}$ converging to 0 as $\epsilon\to 0$.

Let D be a small disk in $\mathbf{C}_\psi$ centered at 0 and $\Gamma=\partial D$, a circle. For small enough $\epsilon\neq 0$, the multiplicity μ is equal to the degree of the map $\psi\mapsto\zeta(\psi,\epsilon)/|\zeta(\psi,\epsilon)|:\Gamma\to S^1$. Due to the asymptotics (1) and (2) of $\phi|_{Z_j}$, we have $\mu=\sum_j\nu_j\mu_j$, q.e.d.

Theorem 1.1. *Let $\tilde{Z}$ be a deformation of a 1-dimensional space $Z=Z_0$. Let $\phi(x,\epsilon)$ and $\psi(x,\epsilon)$ be deformations of analytic functions ϕ_0 and ψ_0 such that the intersection $\{\phi=0\}\cap Z_\epsilon$ is discrete, for small $\epsilon\neq 0$, and $\{\psi_0=0\}\cap Z=\mathbf{0}$. Then*

$$\#(\phi,\tilde{Z})=\#(\theta,\tilde{Z}) \tag{3}$$

where the meromorphic on $\tilde{Z}$ function θ is defined as

$$\theta=\frac{\big(\omega\wedge(\psi d\epsilon+c\epsilon d\psi)\big)\big|_{\tilde{Z}}}{(d\psi\wedge d\epsilon)|_{\tilde{Z}}} \tag{4}$$

with an analytic 1-form $\omega = d\phi + \phi\big(\omega_0(x,\epsilon)d\epsilon + \sum_i \omega_i(x,\epsilon)dx_i\big)$ *and* $\mu_j \neq ck_j$*, for all* j*. Here the numbers* k_j *and* μ_j *are defined in the lemma 2.1.*

Remark. Note that, for $k_j = 0$, we have always $\mu_j > 0$. This means that all but finitely many values of c satisfy the condition in the theorem 1.1. For an isolated intersection, we have $k_j = 0$, for all j, and the statement of the theorem 1.1 is valid for all c.

Proof. Let $x = (x_1, \ldots, x_n)$. For $i = 1, \ldots, n$, let $x_i|_{Z_j} = v_{ij}(\psi) + o(1)$ where v_{ij} are multi-valued analytic functions in ψ, $v_{ij}(0) = 0$, and $o(1)$ is a multi-valued analytic function in ϵ and ψ, identically zero for $\epsilon = 0$. Let $\mathbf{v}_j(\psi) = \big(v_{1j}(\psi), \ldots, v_{nj}(\psi)\big)$, so that $x|_{Z_j} = \mathbf{v}_j(\psi) + o(1)$ as $\epsilon \to 0$.

Consider the asymptotics of the 1-forms $\omega|_{Z_j}$ when $\epsilon \to 0$. Due to the lemma 1.1,

$$\omega|_{Z_j} = \left[\epsilon^{k_j}\psi^{\mu_j}\left(\frac{\mu_j f_j(\psi)}{\psi} + \frac{df_j(\psi)}{d\psi} + f_j(\psi)\sum_{i=1}^{n}\omega_i\big(\mathbf{v}_j(\psi)\big)\frac{dv_{ij}(\psi)}{d\psi}\right) + o(\epsilon^{k_j})\right]d\psi$$

$$+\left[k_j\epsilon^{k_j-1}\psi^{\mu_j}f_j(\psi) + o(\epsilon^{k_j-1})\right]d\epsilon.$$

Deriving this, we have taken into account that the derivative $\partial/\partial\psi$ of a multi-valued analytic function does not change the order in ϵ at $\epsilon = 0$, and the derivative $\partial/\partial\epsilon$ decreases this order at most by 1. This implies

$$\theta|_{Z_j} = \epsilon^{k_j}\psi^{\mu_j}\left((\mu_j - ck_j)f_j(\psi) + \psi\frac{df_j(\psi)}{d\psi} + \psi f_j(\psi)\sum_{i=1}^{n}\omega_i\big(\mathbf{v}_j(\psi)\big)\frac{dv_{ij}(\psi)}{d\psi}\right) + o(\epsilon^{k_j}).$$

Note that all the terms in this expression except the first one vanish at $\psi = 0$. Hence

$$\theta|_{Z_j} = \epsilon^{k_j}(\psi_j^{\mu}(\mu_j - ck_j)f_j(0) + o(\psi^{\mu_j}) + o(\epsilon^{k_j}).$$

The same arguments as in the lemma 1.1 show that $\#(\theta, \tilde{Z}) = \sum_j \nu_j\mu_j = \#(\phi, \tilde{Z})$, as long as $\mu_j \neq ck_j$, for all j.

Example. It is easy to show that, for an exceptional value of c in the theorem 1.1, we can have $\#(\phi, \tilde{Z}) < \#(\theta, \tilde{Z})$. The following example shows that the opposite inequality is also possible.

Let $n=2,\ x=(y,z),\ \phi=z,\ \psi=y,\ \omega=dz,\ \tilde{Z}=\{z^2-2yz+\epsilon y=0\}$. Here $\mu=1,\ \nu=2$. We have

$$Z_1=\left\{z=y+\sqrt{y^2-\epsilon y}=2y+O(\epsilon)\right\},\quad Z_2=\left\{z=y-\sqrt{y^2-\epsilon y}=\frac{\epsilon}{2}+\frac{\epsilon^2}{8y}+O(\epsilon^3)\right\}.$$

Hence $k_1=0,\ \mu_1=1,\ \nu_1=1,\ k_2=1,\ \mu_2=0,\ \nu_2=1$. Next,

$$\theta(y,\epsilon)|_{Z_{1,2}}=y\pm\frac{2y^2+(c-1)\epsilon y}{2\sqrt{y^2-\epsilon y}}.$$

Hence, for $c\neq 0$,

$$\theta|_{Z_1}=2y+O(\epsilon),\quad \theta|_{Z_2}=-\frac{c\epsilon}{2}+O(\epsilon^2),$$

and $\#(\phi,\tilde{Z})=\#(\theta,\tilde{Z})=1$. The function $\theta|_{Z_\epsilon}$ has 2 zeroes and 1 pole converging to 0 as $\epsilon\to 0$.

For an exceptional value $c=0$,

$$\theta|_{Z_1}=2y+O(\epsilon),\quad \theta|_{Z_2}=-\frac{\epsilon^2}{8y}+O(\epsilon^3),$$

and $1=\#(\phi,\tilde{Z})>\#(\theta,\tilde{Z})=0$. The function $\theta|_{Z_\epsilon}$ has 1 zero and 1 pole converging to 0 as $\epsilon\to 0$.

At the end of this section, we formulate several results concerning the multiplicities of non-isolated intersections, which can be considered as local complex analogues of the Rolle-Khovanskii theorem ([2], p.43). We do not use these results in the following sections, although they can be used to estimate Pfaffian multiplicities in the same way as the theorem 1.1. Originally, the estimate for the Pfaffian multiplicities was done with the theorem 1.2 below. The modification (4) was suggested to the author by A. Khovanskii.

Theorem 1.2. *Let $x=(x_1,\ldots,x_n)$, and let $Z_0,\ \tilde{Z},\ \phi_0,\ \psi_0,\ \phi(x,\epsilon)$, and $\psi(x,\epsilon)$ be the same as in the theorem 1.1. Then*

$$\#(\Theta,\tilde{Z})\le\#(\phi,\tilde{Z})\le\#(\Theta,\tilde{Z})+\#(\psi_0,Z_0)\tag{5}$$

where the meromorphic on $\tilde{Z}$ function

$$\Theta = \frac{(\omega \wedge \gamma)|_{\tilde{Z}}}{(d\psi \wedge d\epsilon)|_{\tilde{Z}}} \tag{6}$$

is defined by an analytic 1-form $\omega = d\phi + \phi\big(\omega_0(x,\epsilon)d\epsilon + \sum_i \omega_i(x,\epsilon)dx_i\big)$ and by an analytic 1-form γ satisfying the following condition: $\gamma = d\epsilon + \epsilon \sum_i \gamma_i(x,\epsilon)dx_i$ and the vector $\big(\gamma_1(\mathbf{0},0),\dots,\gamma_n(\mathbf{0},0)\big)$ does not belong to a subset $\Sigma \subset \mathbf{C}^n$, independent of ψ, which is a union of at most $\#(\psi_0, Z_0)$ affine hyperplanes.

For an isolated intersection $\{\phi = 0\} \cap Z = \mathbf{0}$, the set Σ is empty and

$$\#(\phi, \tilde{Z}) = \#(\phi_0, Z) = \#(\Theta, \tilde{Z}) + \#(\psi_0, Z). \tag{7}$$

This theorem can be proved with the same arguments as the theorem 1.1, based on the lemma 1.1.

Suppose now that $\tilde{Z}$ is a non-singular deformation of Z_0, i.e. Z_ϵ is non-singular, for small $\epsilon \neq 0$. For an analytic 1-form ω, we define the zeroes of $\omega|_{Z_\epsilon}$, for $\epsilon \neq 0$, as the zeroes of a function $\omega|_{Z_\epsilon}/dz$ where z is any local parameter on Z_ϵ. We define $\#(\omega, \tilde{Z})$ as the number of isolated zeroes of $\omega|_{Z_\epsilon}$, counted with their multiplicities, converging to $\mathbf{0}$ as $\epsilon \to 0$. For an analytic 2-form Ω on $\tilde{Z}$, we can define the zeroes of Ω at Z_ϵ, for $\epsilon \neq 0$, as the zeroes of a function

$$\frac{\Omega}{dz \wedge d\epsilon|_{\tilde{Z}}},$$

where z is any local parameter on Z_ϵ. We denote $\#(\Omega, \tilde{Z})$ the number of the isolated zeroes of Ω at Z_ϵ, counted with their multiplicities, converging to $\mathbf{0}$ as $\epsilon \to 0$.

Theorem 1.3. *Let $\tilde{Z}$ be a non-singular deformation of Z, and let ψ be an analytic function such that $\{\psi = 0\} \cap Z = \mathbf{0}$. Then*

$$\#(\psi, Z) - \#(d\psi, \tilde{Z}) = \chi(Z_\epsilon),$$

the Euler characteristics of the non-singular fiber Z_ϵ of the deformation $\tilde{Z}$ in a small open ball centered at $\mathbf{0}$.

Proof. Consider the mapping $\psi : Z_\epsilon \to D$ where D is a small open disk in $\mathbf{C}$. The number $\#(\psi, Z)$ is equal to the degree of this mapping, and the number $\#(d\psi, \tilde{Z})$ is equal to the number of ramification points of this mapping, each counted with the multiplicity of the ramification order in it minus 1. Standard Riemann-Hurwitz type arguments show that $\chi(Z_\epsilon)$ is equal to the difference of these two numbers.

Remark. For $n = 2$, this problem was considered in [15].

Theorem 1.4. *In the conditions of the theorem 1.2, Let ϕ, ω, and γ be the same as in the theorem 1.2. Let $\tilde{Z}$ be a non-singular deformation of Z_0. Then*

$$\#(\omega \wedge \gamma, \tilde{Z}) + \chi(Z_\epsilon) - \nu \le \#(\phi, \tilde{Z}) \le \#(\omega \wedge \gamma, \tilde{Z}) + \chi(Z_\epsilon).$$

Here ν is the multiplicity of the intersection of Z_0 with a generic non-singular hypersurface through $\mathbf{0}$ in M.

Proof. The statement follows from the theorem 1.2 applied to a generic function $\psi(x)$ with $d\psi(\mathbf{0}) \neq 0$, and from the theorem 1.3 for isolated intersections. In this case, it is easy to check that $\nu = \#(\psi, Z_0)$ and $\#(\Theta, \tilde{Z}) = \#(\omega \wedge \gamma, \tilde{Z}) - \#(d\psi, Z_0) = \#(\omega \wedge \gamma, \tilde{Z}) - \nu + \chi(Z_\epsilon)$.

2. Pfaffian multiplicities.

Definition 2.1. (Cf. [1, 2].) A *Pfaffian chain* at $\mathbf{0} \in \mathbf{C}^{m+r}_x$ is defined by a sequence of differential 1-forms $\omega_1, \ldots, \omega_r$ with polynomial coefficients of degrees $\alpha_1, \ldots, \alpha_r$ in x such that $\omega_1 \wedge \ldots \wedge \omega_r \neq 0$ at $\mathbf{0}$, and by a sequence $S_1 \supset \ldots \supset S_r \ni \mathbf{0}$ of integral manifolds for $\omega_1, \ldots, \omega_r$ at $\mathbf{0}$, i.e. S_j is a germ at $\mathbf{0}$ of an analytic manifold of codimension j and $\omega_j|_{S_j} \equiv 0$, for $j = 1, \ldots, r$. The number r is called the *rank* of the Pfaffian chain.

A *special* Pfaffian chain is a Pfaffian chain with the forms

$$\omega_j = dx_j + \sum_{i=1}^{m} g_{ij}(x) dx_{r+i}, \quad j = 1, \ldots, r, \tag{8}$$

where g_{ij} is a polynomial in x of degree not exceeding α.

Theorem 2.1. *Let the polynomial 1-forms $\omega_1,\dots,\omega_r$ of degrees $\alpha_1,\dots,\alpha_r$ and the manifolds $S_1,\dots,S_r$ define a Pfaffian chain at $\mathbf{0}\in\mathbf{C}_x^{m+r}$. Let $\phi_1(x),\dots,\phi_m(x)$ be polynomials of degrees $\beta_1,\dots,\beta_m$, and let $\phi_1(x,\epsilon),\dots,\phi_m(x,\epsilon)$ be an arbitrary deformation of $\phi_1(x),\dots,\phi_m(x)$ such that $\phi_j(x,\epsilon)$ is a polynomial in x of degree β_j, for $j=1,\dots,m$. Then the multiplicity μ of the deformation $\phi_1,\dots,\phi_m$ in $(S_r,\mathbf{0})$ does not exceed*

$$\beta_1\cdots\beta_m\beta_{m+1}\cdots\beta_{m+k}$$

where

$$\beta_{m+j+1}=2^j(\alpha_1+\ldots+\alpha_r+\beta_1+\ldots+\beta_m-m)+1-\sum_{i=1}^{j}2^{i-1}\alpha_{r-j+i},$$

for $j=0,\dots,r-1$. In particular,

$$\mu\le 2^{r(r-1)/2}\beta_1\cdots\beta_m(\alpha_1+\ldots+\alpha_r+\beta_1+\ldots+\beta_m-m+1)^r.$$

Proof. Adding $c_j\epsilon^N$, with generic c_j and large enough N, to $\phi_j(x,\epsilon)$, for $j=1,\dots,m$, we reduce the problem to the case when, for small $\epsilon\neq 0$, the intersection

$$Z_\epsilon=S_{r-1}\cap\{\phi_1(x,\epsilon)=\ldots=\phi_m(x,\epsilon)=0\}\cap\{\epsilon=\text{const}\}$$

is a non-singular one-dimensional set transversal to S_r. Let $\tilde{Z}$ be the Zariski closure of $\cup_{\epsilon\neq 0}Z_\epsilon$ and $Z_0=\tilde{Z}\cap\{\epsilon=0\}$. Let $\psi(x)$ be a linear function in $\mathbf{C}^{m+r}$ such that $\{\psi=0\}$ is transversal to S_r at $\mathbf{0}$ and $Z_0\cap\{\psi=0\}=\mathbf{0}$. Let $n=m+1$. Let us choose an analytic function $\phi(x)$ such that $S_r=\{\phi(x)=0\}$ and $d\phi(\mathbf{0})\neq 0$. Then the deformations $\tilde{Z}$, $\phi(x,\epsilon)\equiv\phi(x)$ and $\psi(x,\epsilon)\equiv\psi(x)$ satisfy the conditions of the theorem 1.1.

Let us define a function

$$\phi_{m+1}(x,\epsilon)=\frac{[\psi(x)d\epsilon+c\epsilon\, d\psi(x)]\wedge\omega_1\wedge\ldots\wedge\omega_r\wedge d\phi_1(x,\epsilon)\wedge\ldots\wedge d\phi_m(x,\epsilon)}{dx_1\wedge\ldots\wedge dx_{m+r}\wedge d\epsilon}, \qquad (9)$$

which is a polynomial in x of degree not greater than $\beta_{m+1} = \alpha_1 + \ldots + \alpha_r + \beta_1 + \ldots + \beta_m - m + 1$. We want to show that, for a generic $a \in \mathbf{C}$, the multiplicity of the deformation $\phi_1(x, \epsilon), \ldots, \phi_m(x, \epsilon), \phi_{m+1}(x, \epsilon)$ in $(S_{r-1}, \mathbf{0})$ is not less than μ. It is easy to check that the zeroes of the function ϕ_{m+1} coincide with the zeroes of the function θ from (4), for $\omega = \omega_r$. For a generic $c \in \mathbf{C}$, the necessary estimate follows from (3).

Applying this inductively in $j = 1, \ldots, r$ and taking into account the relation

$$\beta_{m+j+1} = \alpha_1 + \ldots + \alpha_{r-j} + \beta_1 + \ldots + \beta_{m+j} - m - j + 1 = 2\beta_{m+j} - \alpha_{r-j+1} - 1$$

valid for $j = 1, \ldots, r-1$, we reduce the statement of the theorem 2.1 to the Bezout theorem for the polynomial intersection $\phi_1(x, \epsilon) = \ldots = \phi_{m+r}(x, \epsilon) = 0$, which is discrete, for a fixed small $\epsilon \neq 0$.

Theorem 2.2. *Let the polynomial 1-forms $\omega_1, \ldots, \omega_r$ of degrees not exceeding α and the manifolds $S_1, \ldots, S_r$ define a special Pfaffian chain at $\mathbf{0} \in \mathbf{C}_x^{m+r}$. Let $\phi_1(x), \ldots, \phi_m(x)$ be polynomials of degrees $\beta_1, \ldots, \beta_m$, and let $\phi_1(x, \epsilon), \ldots, \phi_m(x, \epsilon)$ be an arbitrary deformation of $\phi_1(x), \ldots, \phi_m(x)$ such that $\phi_j(x, \epsilon)$ is a polynomial in x of degree β_j, for $j = 1, \ldots, m$. Then the multiplicity μ of the deformation $\phi_1, \ldots, \phi_m$ in $(S_r, \mathbf{0})$ does not exceed*

$$\beta_1 \cdots \beta_m \beta_{m+1} \cdots \beta_{m+k}$$

where

$$\beta_{m+j+1} = 2^j [\min(m, r)\alpha + \beta_1 + \ldots + \beta_m - m] + 1,$$

for $j = 0, \ldots, r - 1$. In particular,

$$\mu \leq 2^{r(r-1)/2} \beta_1 \cdots \beta_m [\min(m, r)\alpha + \beta_1 + \ldots + \beta_m - m + 1]^r.$$

Proof. The proof is similar to the proof of the theorem 2.1. For a special Pfaffian chain, the degree of the polynomial (9) does not exceed

$$\beta_{m+1} = \min(m, r)\alpha + \beta_1 + \ldots + \beta_m + 1 - m$$

because only the terms containing $dx_{r+1}, \ldots, dx_{m+r}$ appear with coefficients of degree α in ω_j, and the external product of more than m such terms is always zero. Applying this procedure inductively in $j = 1, \ldots, r$ and taking into account the relation

$$\beta_{m+j+1} = m(\alpha - 1) - j + \beta_1 + \ldots + \beta_{m+j} + 2 = 2\beta_{m+j} - 1$$

valid for $j = 1, \ldots, r-1$, we reduce the statement of the theorem 2.2 to the Bezout theorem for the polynomial intersection $\phi_1(x, \epsilon) = \ldots = \phi_{m+r}(x, \epsilon) = 0$, for a fixed $\epsilon \neq 0$.

3. Polar curves and the Łojasiewicz inequality.

Definition 3.1. Let M be an analytic manifold, $\mathbf{0} \in M$, and let $f(x)$ and $g(x)$ be germs of analytic functions on M at $\mathbf{0}$, $f(\mathbf{0}) = g(\mathbf{0}) = 0$. The set $\Delta \subset \mathbf{C}^2$ of the critical values of the mapping $(f, g) : (M, \mathbf{0}) \to (\mathbf{C}^2, 0)$ is called the *polar curve* of f relative to g [16,17]. *

Theorem 3.1. *Let the polynomial 1-forms $\omega_1, \ldots, \omega_r$ of degrees $\alpha_1, \ldots, \alpha_r$, and the manifolds $S_1 \supset \ldots \supset S_r$ define a Pfaffian chain at $\mathbf{0} \in \mathbf{C}_x^{m+r}$. Let $f(x)$ and $g(x)$ be polynomials of degrees β and γ respectively, $f(\mathbf{0}) = g(\mathbf{0}) = 0$. Let Δ be the polar curve of $f|_{S_r}$ relative to $g|_{S_r}$, and let $\Delta' \neq \{f \equiv 0\}$ be an irreducible component of Δ. Let $f = \sum_{i \geq 1} c_i g^{\lambda_i}$ be the Puiseux expansion of Δ', with $c_1 \neq 0$, $\lambda_1 < \lambda_2 < \ldots$, and let $\lambda_1 = p/q$ where q is the least common denominator of the exponents λ_i. Then*

$$p \leq 2^{r(r-1)/2} \beta(\alpha_1 + \ldots + \alpha_r + \beta + \gamma - 2)^{m-1} [m(\alpha_1 + \ldots + \alpha_r + \beta + \gamma - 3) - \gamma + 3]^r, \quad (10)$$

$$q \leq 2^{r(r-1)/2} \gamma(\alpha_1 + \ldots + \alpha_r + \beta + \gamma - 2)^{m-1} [m(\alpha_1 + \ldots + \alpha_r + \beta + \gamma - 3) - \beta + 3]^r. \quad (11)$$

For a special Pfaffian chain with coefficients of degree α,

$$p \leq 2^{r(r-1)/2} \beta(2\alpha + \beta + \gamma - 2)^{m-1} [\min(m, r)\alpha + (m-1)(2\alpha + \beta + \gamma - 3) + \beta]^r, \quad (12)$$

$$q \leq 2^{r(r-1)/2} \gamma(2\alpha + \beta + \gamma - 2)^{m-1} [\min(m, r)\alpha + (m-1)(2\alpha + \beta + \gamma - 3) + \gamma]^r. \quad (13)$$

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* Usually a non-zero linear function is taken as g.

Proof. Let Σ' be an irreducible component of the critical set $\Sigma = \{df \wedge dg|_{S_r} = 0\}$ of $(f,g)|_{S_r}$ such that the image of Σ' under (f,g) is Δ'. Then, for small $\epsilon \neq 0$, the number of solutions of an equation $f(x) = \epsilon,\ x \in \Sigma'$, converging to $\mathbf{0}$ as $\epsilon \to 0$ is not less than p, and the number of solutions of an equation $g(x) = \epsilon,\ x \in \Sigma'$, converging to $\mathbf{0}$ as $\epsilon \to 0$ is not less than q. We can suppose these solutions to be isolated, for a fixed $\epsilon \neq 0$. Otherwise, after restriction of the forms ω_i and the functions f and g to a generic linear hyperplane L, Δ' remains a component of the polar curve of $f|_{S_r \cap L}$ relative to $g|_{S_r \cap L}$, and the problem can be reduced to the same problem in a lower dimension, with a better estimate for p and q.

As $\{f = \epsilon\} \cap S_r$ is non-singular for small $\epsilon \neq 0$, we can choose linear functions $l_1(x), \ldots, l_{m-1}(x)$ so that, for small $\epsilon \neq 0$,

$$w_1(x) \wedge \ldots \wedge \omega_r(x) \wedge df(x) \wedge dl_1(x) \wedge \ldots \wedge dl_{m-1}(x) \neq 0 \text{ when } x \in \Sigma'. \tag{14}$$

Let

$$\phi_j = \frac{\omega_1 \wedge \ldots \wedge \omega_r \wedge df \wedge dg \wedge dl_1 \wedge \ldots \wedge dl_{j-1} \wedge dl_{j+1} \wedge \ldots \wedge dl_{m-1}}{dx_1 \wedge \ldots \wedge dx_{m+r}},$$

a polynomial of degree not greater than $\alpha_1 + \ldots + \alpha_r + \beta + \gamma - 2$.

Due to (14), the points of $\Sigma' \cap \{f = \epsilon\}$ are isolated roots of the system of equations

$$x \in S_r,\ f(x) = \epsilon,\ \phi_1(x) = \ldots = \phi_{m-1}(x) = 0, \tag{15}$$

and the points of $\Sigma' \cap \{g = \epsilon\}$ are isolated roots of the system of equations

$$x \in S_r,\ g(x) = \epsilon,\ \phi_1(x) = \ldots = \phi_{m-1}(x) = 0, \tag{16}$$

Hence (15) and (16) have not less than, respectively, p and q isolated roots converging to $\mathbf{0}$ as $\epsilon \to 0$. Applying the theorem 2.1, we get the estimates (10) and (11).

For a special Pfaffian chain, the functions $l_1, \ldots, l_{m-1}$ can be chosen as generic linear combinations of $x_{r+1}, \ldots, x_{m+r}$ because the set $x_{r+1} = \ldots = x_{m+r} = 0$ is transversal to S_r. As only the terms containing $dx_{r+1}, \ldots, dx_{m+r}$ appear with coefficients of degree α in ω_j, and the product of more than m such terms is always zero, $\phi_j(x, \epsilon)$ is a polynomial in x of degree not greater than $2\alpha+\beta+\gamma-2$. Applying the theorem 2.2, we get the estimates (12) and (13).

Theorem 3.2. (Łojasiewicz inequality.) *Let the real polynomial 1-forms $\omega_1,\dots,\omega_r$ of degrees $\alpha_1,\dots,\alpha_r$ and the real manifolds $S_1\dots,S_r$ define a Pfaffian chain at $\mathbf{0}\in\mathbf{R}^{m+r}$. Let $f(x)$ and $g(x)$ be real polynomials of degrees β and γ respectively, $f(\mathbf{0})=g(\mathbf{0})=0$. Let $C\subset S_r$ be a connected component of $\{g>0\}\cap S_r$ such that $\mathbf{0}$ belongs to the closure of C. Suppose that $f(x)>0$, for small $x\in C$. Then, for small $\delta>0$,*

$$\min_{x\in C,|x|\le\delta,g(x)=\epsilon} f(x)=c\epsilon^{p/q}+o(\epsilon^{p/q}),$$

with $c>0$, where p and q do not exceed the right sides of (10) and (11) respectively.

For a special Pfaffian chain with the coefficients of degree α, the same is true with p and q not exceeding the right sides of (12) and (13) respectively.

Proof. Let Σ_δ be the set where f achieves its minimum over $\{g=\delta\}\cap C$. We can suppose that the closure of the union of the sets Σ_δ over $\delta>0$ contains $\mathbf{0}$, otherwise the problem reduces to the same problem in smaller dimension, and the estimate improves. In this case, the image of Σ_δ under (f,g) belongs to the polar curve of f relative to g. The statement of the theorem 3.2 follows now from the theorem 3.1.

Applying the theorem 3.2 to $f=\big(\mathrm{grad}\,(g|_{S_r})\big)^2$, we have the following variant of the Łojasiewicz inequality.

Theorem 3.3. *Let the real polynomial 1-forms $\omega_1,\dots,\omega_r$ of degrees $\alpha_1,\dots,\alpha_r$ and the real manifolds $S_1\dots,S_r$ define a Pfaffian chain at $\mathbf{0}\in\mathbf{R}^{m+r}$. Let $g(x)$ be a real polynomial of degree γ such that $g(\mathbf{0})=0$. Then, for small $\delta>0$,*

$$\min_{x\in S_r,|x|\le\delta,g(x)=\epsilon}\big|\mathrm{grad}\,(g|_{S_r})(x)\big|=c\epsilon^{p/q}+o(\epsilon^{p/q})\quad\text{as}\quad\epsilon\to 0,$$

with $c>0$, where $p<q$ and q does not exceed

$$2^{1+r(r-1)/2}3^{m-1}\gamma(\alpha_1+\ldots+\alpha_r+\gamma-1)^{m-1}[(3m-2)(\alpha_1+\ldots+\alpha_r+\gamma-2)+m+1]^r. \quad (17)$$

For a special Pfaffian chain with the coefficients of degree α, the same is true with q not exceeding

$$2^{1+r(r-1)/2}3^{m-1}\gamma(\alpha+\gamma-1)^{m-1}[\min(m,r)\alpha+(m-1)(3\alpha+3\gamma-5)+\gamma]^r. \qquad (18)$$

Proof. The estimates (17) and (18) follow from the estimates (11) and (13) after we represent $f = \big(\mathrm{grad}\,(g|_{S_r})\big)^2$ as a Pfaffian function of degree $\beta = 2(\alpha_1+\ldots+\alpha_r+\gamma-1)$ or, in the case of a special Pfaffian chain, of degree $\beta = 2(\alpha+\gamma-1)$. To do this, we represent the m components of $\mathrm{grad}\,(g|_{S_r})$ as

$$\frac{\omega_1\wedge\ldots\wedge\omega_r\wedge dl_1\wedge\ldots\wedge dl_{i-1}\wedge dl_{i+1}\wedge\ldots dl_m\wedge dg}{dx_1\wedge\ldots\wedge dx_{m+r}},$$

with the properly chosen linear functions $l_1(x),\ldots,l_m(x)$. For a special Pfaffian chain, we can take $l_i = x_{r+i}$.

To show that $p<q$, choose a germ Γ of an analytic curve adjacent to $\mathbf{0}$ in the set where $\mathrm{grad}\,(g|_{S_r})|_{g=\mathrm{const}}$ is minimal. If such a curve does not exist, the statement can be reduced to the same statement in a smaller dimension. We can suppose that $g|_{S_r}$ has a critical point at $\mathbf{0}$. Then, for $x\in\Gamma$, the function $g(x)$ is equivalent to $|x|^\nu$, with $\nu>1$. Hence the derivative of g along Γ is equivalent to $|x|^{\nu-1}$. As this derivative does not exceed $|\mathrm{grad}\,(g|_{S_r})|$, we have $p/q\le(\nu-1)/\nu<1$.

Simple analytic arguments (see [18] and [14, sect. 2]) show that an estimate $|\mathrm{grad}\,g(x)| \ge cg(x)^\kappa$, with $c>0$ and $0\le\kappa<1$, yields, for any C^1 function g, an estimate $|g(x)|\ge a\big(\mathrm{dist}(x,\{g=0\})\big)^{1/(1-\kappa)}$, with $a^{1-\kappa}=c(1-\kappa)$. Combining this with the theorem 3.3, we have the Łojasiewicz inequality in its standard form.

Theorem 3.4. *Let the real polynomial 1-forms $\omega_1,\ldots,\omega_r$ of degrees $\alpha_1,\ldots,\alpha_r$ and the real manifolds $S_1\ldots,S_r$ define a Pfaffian chain at $\mathbf{0}\in\mathbf{R}^{m+r}$. Let $g(x)$ be a real polynomial of degree γ such that $g(\mathbf{0})=0$. Then, for small $x\in S_r$,*

$$|g(x)|\ge a\big(\mathrm{dist}(x,\{g=0\}\cap S_r)\big)^q,$$

with $a > 0$ and q not exceeding (17). For a special Pfaffian chain, the same is true for q not exceeding (18).

4. Exponential and sparse polynomials.

Definition 4.1. Let $\mathcal{K}$ be a set of r vectors $\mathbf{a}_j = (a_{j1}, \ldots, a_{jm}) \in \mathbf{C}^m$. A *pseudopolynomial*, or exponential polynomial, of *pseudodegree* β with the support $\mathcal{K}$ is a polynomial of degree β in $x_i,\ i = 1, \ldots, m$, and $y_j(x) = \exp(\mathbf{a}_j x) = \exp(a_{j1}x_1 + \ldots + a_{jm}x_m)$, for $\mathbf{a}_j \in \mathcal{K},\ j = 1, \ldots, r$.

Let now all the components of the vectors $\mathbf{a} \in \mathcal{K}$ be non-negative integers. A *fewnomial*, with the support $\mathcal{K}$ is a polynomial in m variables $u = (u_1, \ldots, u_m)$ with monomials $u^{\mathbf{a}}$ present with non-zero coefficients only when $\mathbf{a} \in \mathcal{K}$.

A polynomial $P(u_1, \ldots, u_m, y_1(u), \ldots, y_r(u))$ of degree β in u_i and y_j where $y_j(u) = u^{\mathbf{a}_j},\ \mathbf{a}_j \in \mathcal{K}$, is called a *sparse polynomial* of pseudodegree β with the support $\mathcal{K}$. Note that β is not equal to the actual degree of the polynomial P after substitution $y_j = y_j(u)$.

Theorem 4.1. *The multiplicity of any solution of a system of m pseudopolynomial equations in $\mathbf{C}^m$ of degrees $\beta_1, \ldots, \beta_m$, with a common support $\mathcal{K}$ with $|\mathcal{K}| = r$, does not exceed*

$$2^{r(r-1)/2}\beta_1 \cdots \beta_m[\min(0, r - m) + \beta_1 + \ldots + \beta_m + 1]^r. \tag{19}$$

The multiplicity of any solution in $\mathbf{C}^m \setminus \{x_1 \cdots x_m = 0\}$ of a system of m equations with sparse polynomials of pseudodegrees $\beta_1, \ldots, \beta_m$, with a common support $\mathcal{K}$, does not exceed (19). In particular, the multiplicity of any solution in $\mathbf{C}^m \setminus \{x_1 \cdots x_m = 0\}$ of a system of m fewnomial equations with a common support $\mathcal{K}$ does not exceed

$$2^{r(r-1)/2}[\min(m, r) + 1]^r.$$

Here the multiplicity of a non-isolated solution of a system of pseudopolynomial (sparse polynomial) equations is defined as the maximum of the multiplicities of the one-parameter *analytic deformations of the original system of equation within the class of the pseudopolynomial (sparse polynomial) equations of the same pseudodegree and with the same support.*

Proof. The statement for pseudopolynomials follows from the theorem 2.2, because the functions $y_j(x) = \exp(\mathbf{a}_j x)$, $\mathbf{a}_j \in \mathcal{K}$, $j = 1, \ldots, r$, define a special Pfaffian chain of rank r with the polynomial 1-forms $\omega_j = dy_j - y_j(x)(\mathbf{a}_j, dx)$ of degree $\alpha = 1$ and the manifolds $S_j = \{y_1 = \exp(\mathbf{a}_1 x), \ldots, y_j = \exp(\mathbf{a}_j x)\}$.

The statement for sparse polynomials follows from the statement for pseudopolynomials after substitution $u_i = \exp(x_i)$, for $i = 1, \ldots, m$.

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References.

[1] A.G. Khovanskii, *On a class of systems of transcendental equations*, Soviet Math. Dokl., **22**, 762-765, 1980.

[2] A.G. Khovanskii, *Fewnomials*, AMS Translation of mathematical monographs, v.88, AMS, Providence, RI, 1991. (Russian original: Malochleny, Moscow, 1987)

[3] A. Gabrielov, *Projections of semi-analytic sets*, Functional Anal. Appl., v.2, n.4, p.282-291, 1968.

[4] H. Hironaka, *Subanalytic sets*, in: Number Theory, Algebraic Geometry and Commutative Algebra (in honor of Y. Akizuki), Kinokunya, Tokyo, p.453-493, 1973.

[5] R.M. Hardt. *Stratification of real analytic mappings and images*, Invent. Math., **28**, 193-208, 1975.

[6] Z. Denkowska, S. Łojasiewicz, and J. Stasica, *Certaines propriétés élémentaires des ensembles sous-analytiques*, Bull. Polish Acad. Sci. Math. **27**, 529-536, 1979.

[7] E. Bierstone and P.D. Milman. *Semianalytic and subanalytic sets*, Inst. Hautes Etudes Sci. Publ. Math., **67**, 5-42, 1988.

[8] J. Denef and L. van den Dries, *P-adic and real subanalytic sets*, Ann. Math., **128**, 79-138, 1988.

[9] A. Gabrielov, *Existential formulas for analytic functions*, preprint Cornell MSI, 1993.

[10] A. Gabrielov and N. Vorobjov, *Complexity of stratifications of semi-Pfaffian sets*, preprint Cornell MSI, 1993. To appear in J. Discr. Comput. Geometry, 1995.

[11] A.J. Wilkie, *Smooth o-minimal theories and the model completeness of the real exponential field*, preprint, 1992.

[12] L. van den Dries, A. Macintyre, and D. Marker. *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math., **140**, n.1, 183-205, 1994.

[13] E. Bierstone, P.D. Milman. *A simple constructive proof of canonical resolution of singularities*, In: Effective Methods in Algebraic Geometry, Progress in Math., **94**, Birkhäuser, Boston, p.11-30, 1991.

[14] E. Bierstone, P.D. Milman, *Canonical desingularization in characteristic zero: an elementary proof*, preprint, 1994.

[15] A. Gabrielov, J.-M. Lion, R. Moussu, *Ordre de contact de courbes intégrales du plan*, CR Ac. Sci. Paris, **319**, 219-221, 1994.

[16] B. Teissier, *Variétés polaires I*, Inv. Math., **40**, n.3, 267-292, 1977.

[17] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, in: Singularités à Cargèse, Astérisque **7-8**, 285-362, 1972.