CLASSIFICATION OF GENERIC SPHERICAL QUADRILATERALS

ANDREI GABRIELOV

Abstract. Generic spherical quadrilaterals are classified up to isometry. Condition of genericity consists in the requirement that the images of the sides under the developing map belong to four distinct circles which have no triple intersections. Under this condition, it is shown that the space of quadrilaterals with prescribed angles consists of finitely many open curves. Degeneration at the endpoints of these curves is also determined.

Keywords: positive curvature, conic singularities, quadrilaterals, conformal map.

MSC 2010: 30C20, 34M03.

1. Introduction

A *spherical polygon* Q is a surface homeomorphic to a closed disk equipped with a Riemannian metric of constant positive curvature 1, with n conic singularities on the boundary, labeled $a_0, \dots, a_{n-1}$ counterclockwise, and such that the boundary arcs $[a_j, a_{j+1}]$ are geodesic. The singularities a_j are the *corners* of Q, and the boundary arcs $[a_j, a_{j+1}]$ are its *sides*.

These objects appear in several areas of recent research. One of them is the problem of classification of spherical metrics with conic singularities on the sphere, see [2, 3, 4, 5, 6, 7, 8, 10, 15, 16, 18, 24, 26, 27, 28] and references there. In particular, [8] is a recent survey of the known results related to such metrics. When all singularities lie on a circle on the Riemann sphere, and the metric is symmetric with respect to that circle, the sphere is obtained by gluing of two such polygons related by an anti-conformal isometry. Thus spherical polygons provide an important class of examples of spherical metrics. In fact, conditions for existence or non-existence of spherical metrics with prescribed angles on a sphere in [27], and on tori in [4], were obtained with the help of spherical polygons.

Another related problem is description of real solutions of Painlevé VI equations with real parameters [12]. In this problem, real special

Date: January 29, 2022.

points (zeros, poles, 1-points and fixed points) of a solution correspond to circular quadrilaterals. When the monodromy of the linear equation related to the solution of Painlevé VI is unitarizable (conjugate to a subgroup of $SU(2)$), these circular quadrilaterals are spherical quadrilaterals.

Complete classification of spherical triangles is known [23], [5], [13].

The structure of the set of spherical quadrilaterals with prescribed angles strongly depends on the number of integer angles (the angles with the radian measure an integer multiple of π).

Spherical quadrilaterals with at least one integer angle have been previously classified up to isometry: when all angles are integers, in [9]; when two angles are integers in [15]; when only one angle is integer in [17].

Spherical quadrilaterals with four half-integer angles were classified in [11]. Any quadrilateral with such angles has two opposite sides mapped to the same circle by its developing map.

If S is a surface with a Riemannian metric of curvature 1, then every point of S has a neighborhood isometric to an open set of the standard unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. This isometry f is conformal, therefore it is analytic and permits an analytic continuation along any path in S. If S is simply connected, we obtain a map $\Phi : S \to \mathbf{S}^2$ which is called the *developing map*. The developing map is defined by the metric up to a composition with a rotation of $\mathbf{S}^2$.

Conversely, if Δ is a disk in the complex plane, and $\Phi : \Delta \to \mathbf{C}$ a locally univalent meromorphic function such that $\Phi(z) \sim c_j(z - a_j)^{\alpha_j}$, $0 \le j \le n{-}1$ at n boundary points a_j, and the arcs $\Phi([a_j, a_{j+1}])$ belong to great circles in $\mathbf{C}$, then Φ is a developing map of a spherical quadrilateral with angles $\pi\alpha_j$. The metric on Δ is recovered by the formula for its length element $ds = 2|\Phi'|/(1 + |\Phi|^2)$.

We call Q a *spherical quadrilateral* (resp., *triangle, digon*) if $n = 4$ (resp., $n = 3$, $n = 2$). For convenience, we often drop "spherical" and refer simply to polygons (quadrilaterals, triangles, digons).

If a spherical polygon Q has a *removable* corner with the angle 1, the metric on Q is non-singular at such a corner, thus Q is isometric to a polygon with fewer corners. However, we allow polygons with removable corners, since they may appear as building blocks of other polygons.

In this paper we consider classification of generic spherical quadrilaterals, with the sides mapped to four generic (distinct, no triple intersections) great circles of the Riemann sphere $\mathbf{S}$ (although circle configurations with triple intersections will be considered in Section 5). All angles of a generic spherical quadrilateral are non-integer. The four

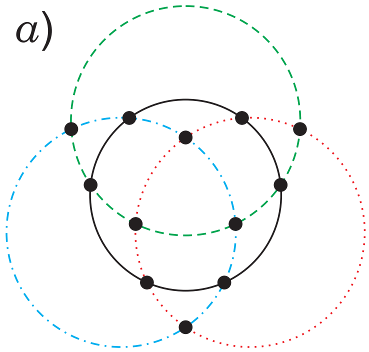

FIGURE 1. Partition $\mathcal{P}$ of the Riemann sphere $\mathbf{S}$ by four great circles.

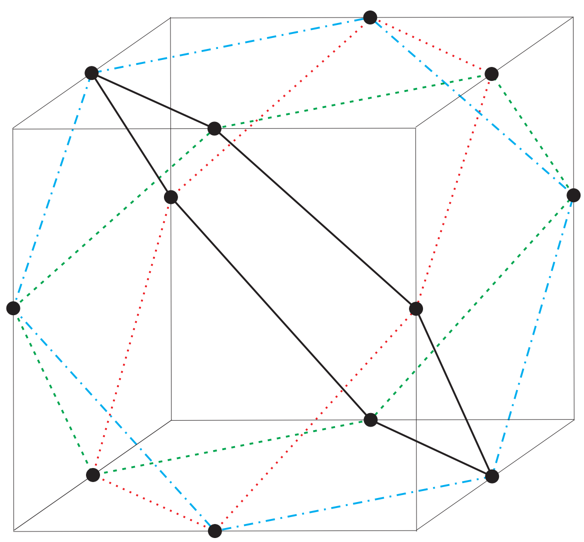

FIGURE 2. Surface of the convex hull of midpoints of the edges of a cube, combinatorially equivalent to the partition $\mathcal{P}$ of $\mathbf{S}$.

circles define a partition $\mathcal{P}$ of $\mathbf{S}$ with eight triangular faces and six quadrilateral faces, such that each edge of $\mathcal{P}$ separates a triangular face from a quadrilateral one. This partition is combinatorially equivalent to the boundary of the convex hull of midpoints of the edges of a cube in $\mathbf{R}^3$ (see Fig. 2). Two planar projections of the partition $\mathcal{P}$ are shown in Fig. 1.

Preimage of $\mathcal{P}$ defines a *net* of a quadrilateral Q: a cell decomposition of Q considered up to a label-preserving homeomorphism, so it is a combinatorial object (see Definition 2.1). Preimage of each of the circles is the subset of the net consisting of *arcs*, simple paths with the ends at vertices of the net, that may contain corners of Q only at their ends. Each side of Q is a *boundary arc* of its net. An arc which is not a side of Q is an *interior arc*. An interior arc with two ends at distinct corners of Q is a *diagonal arc*. Note that a diagonal arc of a spherical quadrilateral may have its ends at two adjacent corners. An arc is a *loop* if it has both ends at the same vertex of the net. A quadrilateral is *irreducible* if its net does not contain a diagonal arc. Note that an arc of a generic spherical quadrilateral may have both ends at its adjacent corners but not at its opposite corners (see Lemma 2.11). An irreducible quadrilateral is *primitive* if its net does not contain a loop.

For example, Fig. 1b can be interpreted as a net of a generic spherical quadrilateral Q obtained by removing from $\mathbf{S}$ an open quadrilateral face of the partition $\mathcal{P}$. Each circle in Fig. 1b consists of two arcs, a side of Q and a diagonal arc with the ends at two adjacent corners of Q. Thus Q is not irreducible. The net of Q does not have loops. A net of an irreducible but not primitive generic spherical quadrilateral P_1 is shown in Fig. 3. It contains a *pseudo-diagonal* consisting of four loops. Removing two quadrilateral faces $puqx$ and $pvqw$ from the net of P_1, we get a net of the quadrilateral Q. Both faces $puqx$ and $pvqw$ of P_1 are mapped to the same quadrilateral face of $\mathcal{P}$. The edges of the net of P_1 in Fig. 3 (and the edges of the nets of quadrilaterals shown in other figures) have four colors (styles) indicating preimages of the four circles of the partition $\mathcal{P}$.

Classification of generic quadrilaterals is done as follows: We start with four *basic* quadrilaterals (see Fig. 13), then build all primitive quadrilaterals using *side extensions* in Section 3 (see Figs. 14-17). After that, irreducible quadrilaterals are obtained by replacing the quadrilateral face of the net of a primitive quadrilateral containing two opposite corners in its boundary (at most one such face exists) with one of the quadrilaterals P_μ ($\mu = 1, 2, \dots$) (see Definition 2.5). Finally, all generic quadrilaterals are obtained in Section 4 by attaching digons to *short* (shorter than the full circle) sides of irreducible quadrilaterals (see Theorem 4.1).

Conditions on the fractional parts of the angles of spherical quadrilaterals with a given net define a pyramid Π in the unit cube of $\mathbf{R}^4$ (see Proposition 5.2) or a pyramid obtained from Π by replacing some of the angles by their complementary angles. These conditions are compatible with the *closure condition* in [27] that implies that, for the existence of a generic quadrilateral with the fractional parts $(\alpha, \beta, \gamma, \delta)$ of the angles, and with even (resp., odd) sum of the integer parts, the distance d_1 between the point $(\alpha, \beta, \gamma, \delta)$ and the odd (resp., even) integer lattice must be greater than 1. In fact, the union of Π and all pyramids obtained from Π by taking an even number of complements coincides with the 4-dimensional cross-polytope (also known as 16-cell or co-cube) which contains all points in the unit cube of $\mathbf{R}^4$ at the distance $d_1 > 1$ from the odd integer lattice, other than the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ corresponding to a spherical rectangle. It was shown in [11] that a spherical rectangle cannot be generic: two of its opposite sides are mapped to the same circle.

In Section 5 we consider chains of spherical quadrilaterals and their nets. If all four angles of a generic spherical quadrilateral Q are fixed, quadrilaterals with the same net Γ as Q constitute a one-parametric family (an open segment) I_Γ in the space of all spherical quadrilaterals. An important function on this family is the *modulus* K of the quadrilateral. Every conformal quadrilateral can be mapped conformally onto a rectangle, so that the corners $(a_0, \dots, a_3)$ correspond to the corners of the rectangle $(0, 1, 1+iK, iK)$, where $K > 0$ is the modulus. We say that a sequence (or a family) of quadrilaterals conformally degenerates if K tends to 0 or ∞.

Unlike quadrilaterals with at least one integer angle considered in [9, 15, 16, 17], different generic quadrilaterals in I_Γ are mapped to conformally non-equivalent four-circle configurations. Six angles between four great circles satisfy a single relation. Only four angles are fixed in I_Γ as fractional parts of the angles of a quadrilateral. This leaves a "fifth angle" (the angle between the circles corresponding to the opposite sides of a quadrilateral) that is not fixed in I_Γ. The main results of Section 5 are conditions on the "fixed" angles of a four-circle configuration that allow it to be deformed to a configuration with a triple intersection (Proposition 5.3) and relations between the fractional parts of the angles of a quadrilateral and the angles of the corresponding four-circle configuration, depending on the net of a quadrilateral(Propositions 5.9 and 5.10).

At the ends of the segment I_Γ, a quadrilateral may conformally degenerate, or converge to a spherical quadrilateral with a non-generic four-circle configuration containing a triple intersection, or else, after appropriate conformal transformations, converge to a non-spherical (circular) quadrilateral with a four-circle configuration which is not conformally equivalent to a spherical one. In the second case, the segment I_Γ can be extended beyond the quadrilateral with a non-generic four-circle configuration to a segment $I_{\Gamma'}$ of generic quadrilaterals with a different net Γ'. A maximal family of spherical quadrilaterals obtained by such extensions is called a *chain* of quadrilaterals, and the sequence of nets of such family is called a chain of nets (see Definition 5.13). Any chain contains quadrilaterals with finitely many nets. It is an open segment in the space of spherical quadrilaterals, and the quadrilaterals at its ends either conformally degenerate or converge to non-spherical quadrilaterals beyond which the chain cannot be extended. The chains of generic spherical quadrilaterals depend not only on their nets but also on some inequalities between the fractional parts of their angles. This makes counting the chains of generic quadrilaterals much harder than counting the chains of the quadrilaterals with at least one integer angle considered in [15, 16, 17]. As an example, Propositions 5.21 and 5.22 describe different possibilities for the chains of quadrilaterals with nets X_{kl} and X'_{pq}, depending on the integer and fractional parts of their angles.

The *length* of a chain of quadrilaterals is the non-negative number of "links" in it corresponding to four-circle configurations with triple intersections. Thus the number of nets in a chain is greater by one than its length. If the quadrilaterals conformally degenerate in the limits at both ends of a chain, the two degenerations are either of the same or of the opposite kind, depending on the parity of the length of the chain. If the length of a chain is even (for example, if all quadrilaterals in a chain of length 0 have the same net) then the modulus K of the quadrilaterals in that chain converges to distinct values 0 and ∞ at the two ends of the chain. In that case, the chain contains at least one quadrilateral with each value $K>0$ of the modulus. If the length of a chain is odd, the modulus K of the quadrilaterals in that chain converges to the same value (either 0 or ∞) at both ends of the chain. In that case, the chain contains at least two quadrilaterals with either sufficiently small or sufficiently large values of the modulus K. Thus classification of the chains of quadrilaterals allows one, in principle, to obtain lower bounds for the number of quadrilaterals with the given angles and modulus, and to count quadrilaterals with the given angles and either small enough or large enough value of the modulus. This is a hard combinatorial problem, which is not addressed in this paper. Note that, according to Lemma 3.2, if the angles at three corners of a quadrilateral Q are less than 1, then the net of Q is of one of the types X, X', $\bar{X}$, $\bar{X}'$. Thus Propositions 5.21 and 5.22 provide the answer for such quadrilaterals.

When the fractional parts of the angles of quadrilaterals with a given net Γ satisfy an additional equality (see Remark 5.4) then a family I_Γ of spherical quadrilaterals may degenerate at one end of the segment I_Γ so that the modulus converges to a finite positive value. Applying an appropriate family of linear-fractional transformations to the sphere, one can replace the family of four-circle configurations to which the family I_Γ is mapped by a conformally equivalent family of configurations of four not necessarily great circles, converging to a configuration with a single quadruple intersection (see Fig. 31), so that the corresponding family of circular quadrilaterals would be converging to a non-degenerate circular (non-spherical) quadrilateral (see Example 5.15 and Figure 35). This phenomenon was observed in [25], [11], [28] and [14]. Note that in all previously considered cases the angles were half-integer, while there is a single equality satisfied by the angles of the quadrilaterals in Example 5.15. This equality is compatible with the *non-bubbling condition* in [28] (see also [8]).

The author thanks Alexandre Eremenko who opened to him the fascinating world of spherical polygons and spherical metrics with conical singularities. He explained that, although generic spherical triangles were classified by Felix Klein in the beginning of the last Century, classification of generic spherical quadrilaterals remained an open problem, and asked whether it could be solved using the nets, a technique developed in [9]. After almost ten years after that conversation, a (partial) answer to his question is presented in this paper. The author is grateful to Prof. Eremenko and the anonymous referees for suggesting numerous improvements to this text.

2. Generic quadrilaterals and their nets

Definition 2.1. Let Q be a generic spherical quadrilateral with the sides mapped to four circles of a partition $\mathcal{P}$ of the sphere $\mathbf{S}$. Preimage of $\mathcal{P}$ defines a cell decomposition Γ of Q, called the *net* of Q. The *vertices*, *edges* and *faces* of Γ are connected components of the preimages of vertices, edges and faces of $\mathcal{P}$. For simplicity, we call them vertices, edges and faces of Q. The net Γ has the same types of faces (triangles and quadrilaterals) as the partition $\mathcal{P}$, with the same adjacency rules: if two faces of Γ have a common edge then one of them is a triangle and another one a quadrilateral.

The corners of Q are vertices of its net Γ. The *order* of a corner p of Q is the integer part of the angle of Q at p. Since the angle of Q at p is not integer and the four-circle configuration does not have triple intersections, the two sides of Q adjacent to p map to two different circles, thus the degree of a corner as a vertex of Γ is even. In addition, the net Γ of Q may have *interior vertices* of degree 4 and *lateral vertices* (on the sides but not at the corners) of degree 3.

The *order* of a side L of Q is the number of edges of Γ in L. The union of three consecutive edges on the same circle of $\mathcal{P}$ is a half-circle with both ends at the intersection of the same two circles of $\mathcal{P}$. Since the opposite sides of Q are mapped to different circles, corners at the ends of each of its side cannot be mapped to intersection of the same two circles, thus the order of a side of a generic quadrilateral Q cannot be divisible by 3. A side of Q is *short* (shorter than a full circle) if its order is less than 6, and *long* otherwise.

Two spherical polygons Q and Q' are *combinatorially equivalent* if there is an orientation preserving homeomorphism $Q \to Q'$ mapping the corners a_j of Q to the corners a'_j of Q' and the net Γ of Q to the net Γ' of Q'.

Definition 2.2. Let Q be a generic spherical quadrilateral with the sides mapped to four circles of a partition $\mathcal{P}$ and the net Γ. If C is a circle of $\mathcal{P}$ then the intersection Γ_C of Γ with the preimage of C in Q is called the *C-net* of Q. All interior vertices of Γ_C have degree two. A *C-arc* of the net of Q (or simply an arc when C is not specified) is a simple path in Γ_C with the ends at vertices of Γ, which may have corners of Q only at its ends. The *order* of an arc is the number of edges in it. An arc is a *loop* if it is a closed path. An arc γ of Q is *lateral* if it is a subset of a side of Q. Otherwise, γ is an *interior arc.* An arc is *maximal* if it cannot be extended to a larger arc.

A *diagonal* arc is an interior arc of Q with both ends at distinct corners of Q. A spherical quadrilateral Q is *irreducible* if it does not have a diagonal arc. An irreducible quadrilateral is *primitive* if it does not contain a loop.

2.1. **Quadrilaterals P_μ and pseudo-diagonals.**

Lemma 2.3. *Let Q be an irreducible spherical quadrilateral that contains a loop γ. Then γ has a vertex at a corner of Q.*

Proof. If all vertices of γ are interior vertices of Q, then γ bounds a disk D inside Q which is mapped one-to-one onto a disk in the sphere $\mathbf{S}$ bounded by a circle C of the partition $\mathcal{P}$, and γ maps one-to-one onto C. Since all vertices of γ are interior vertices of the net Γ of Q, the faces of Γ adjacent to γ map one-to-one to the faces of $\mathcal{P}$ adjacent to C. The union of D and all these faces is a spherical triangle T which maps one-to-one to the complement of a triangular face of the partition $\mathcal{P}$ (see Fig. 1a). Note that the corners of T cannot be lateral vertices of Γ, since they have degree 4. If a corner of T is an interior vertex of Γ, then Γ must have a triangular face completing the image of T to the full sphere. Since Q is not a sphere, this is possible only when the other two corners of T are corners of Q. Since any two corners of T are connected by an interior arc, Q is not irreducible, a contradiction. $\square$

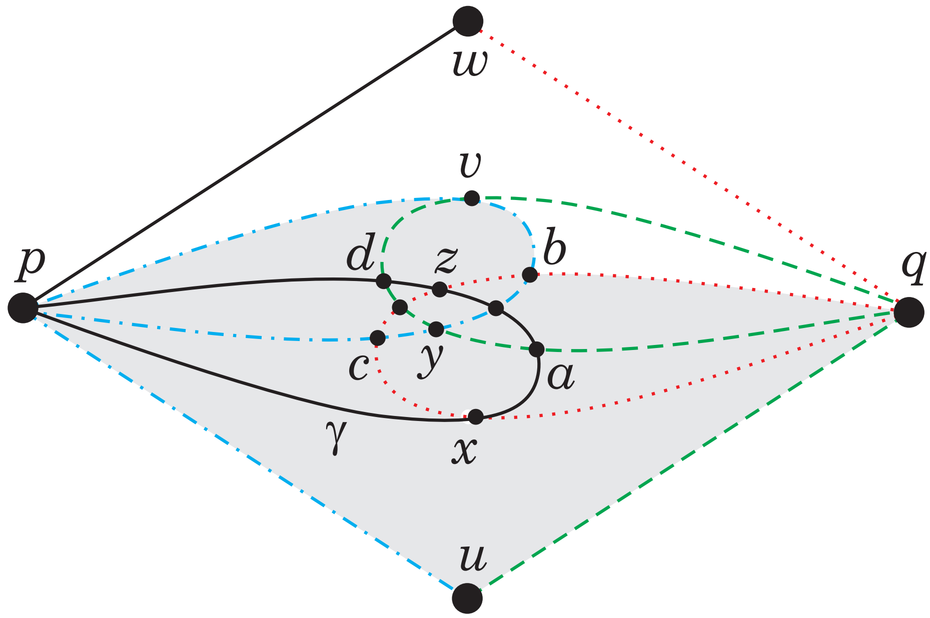

FIGURE 3. The net of the quadrilateral P_1 with corners p, u, q, w.

Lemma 2.4. *Let Q be an irreducible spherical quadrilateral that contains a loop γ. Then Q contains a quadrilateral combinatorially equivalent to the quadrilateral P_1 shown in Fig. 3.*

Proof. According to Lemma 2.3, the loop γ has a vertex p on the boundary of Q. It cannot be a lateral vertex (otherwise γ would be a side of Q) thus p is a corner of Q. Since Q is irreducible, all other vertices of γ must be interior vertices of the net Γ of Q. The loop γ bounds a disk $D \subset Q$ mapped one-to-one to a disk bounded by a circle C of $\mathcal{P}$. Thus Γ must contain the union of D and six more faces adjacent to γ outside D, which is the quadrilateral G with corners p, u, q, b shaded in Fig. 3. Since b is connected to p by an interior arc, it cannot be a corner of Q. Also, b cannot be a lateral vertex of Γ since its degree is greater than 3. Thus b is an interior vertex, and Γ contains a triangular face with vertices b, q, v. Similarly, v cannot be a corner of Q, as it is connected to p by an interior arc. Thus v is an interior vertex, and Γ contains a quadrilateral face with vertices p, v, q, w. This implies that Q contains the quadrilateral P_1 with corners p, u, q, w shown in Fig. 3, completing the proof of Lemma 2.4. Note that vertices p and q of P_1 must be opposite corners of Q. □

Definition 2.5. Starting from the quadrilateral P_1 shown in Fig. 3, we define a sequence P_μ of non-primitive irreducible quadrilaterals as follows. All sides of P_μ have order 1. Two opposite corners p and q of P_μ have order 2μ, and two other corners have order 0. The net of P_μ contains $\mu+1$ quadrilateral faces having both p and q as their opposite vertices. These faces are separated by μ "pseudo-diagonals," each consisting of four loops, two of them having a common vertex p and another two a common vertex q. For $\mu \geq 1$, the quadrilateral $P_{\mu+1}$ is obtained by replacing any one of these $\mu+1$ faces of P_μ by the quadrilateral P_1.

For convenience, we define P_0 to be a spherical quadrilateral which maps one-to-one to a single quadrilateral face of the partition $\mathcal{P}$.

2.2. **Spherical digons.** A spherical digon D has two corners with equal angles, which may be integer and even removable, and two short sides. Since the boundary of D consists of arcs of at most two circles, geometrically (up to conformal equivalence) D is completely determined by the angles at its corners. (See [15], Theorem 4.1.) However, since we need spherical digons as building blocks of spherical quadrilaterals, we define their nets as preimages of all four circles of the partition $\mathcal{P}$. The following Lemma provides classification of combinatorially distinct irreducible spherical digons.

Lemma 2.6. *There are three combinatorially distinct types of irreducible digons: D_{15}, D_{24} and D_{33} (see Figs. 4, 5 and 6). Digons D_{15} and D_{24} have integer corners of order 1. Digon D_{33} has non-integer corners of order 0. There are two sorts, D_{15}^a and D_{15}^b, of digons D_{15} (see Fig. 4). Their nets have reflection symmetry. There are two sorts, D_{24}^a and D_{24}^b, of digons D_{24} (see Fig. 5). Their nets are reflection symmetric to each other. There are two sorts, D_{33}^a and D_{33}^b, of digons D_{33} (see Fig. 6). Their nets have two reflection symmetries.*

Proof. Let D be an irreducible digon with the corners p and q, and the sides L and L'. Then L and L' map to some circles C and C' of the partition $\mathcal{P}$ (possibly, to the same circle). If the equal angles at the corners of D at p and q are non-integer then $C' \neq C$, and D is a digon of a partition of $\mathbf{S}$ by the two circles C and C'. Since D is irreducible, it is either D_{33}^a or D_{33}^b, with the face of the net of D adjacent to its corner being a quadrilateral or a triangle, respectively.

If the angles at the corners of D are integer then $C' = C$, D maps one-to-one onto a disk $\mathbf{D}$ bounded by C, and, since D is irreducible, the images of p and q are two vertices of the partition $\mathcal{P}$ on C which are not connected by an interior arc. Thus D is one of the digons D_{15}^a, D_{15}^b, D_{24}^a, D_{24}^b. $\square$

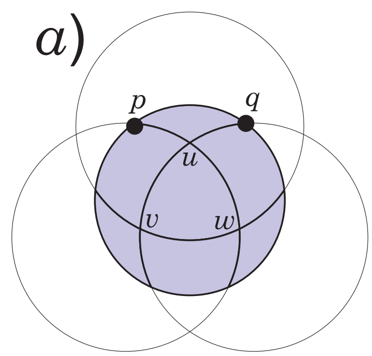

FIGURE 4. Irreducible digons of type D_{15}.

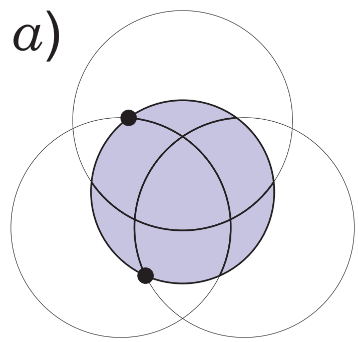

FIGURE 5. Irreducible digons of type D_{24}.

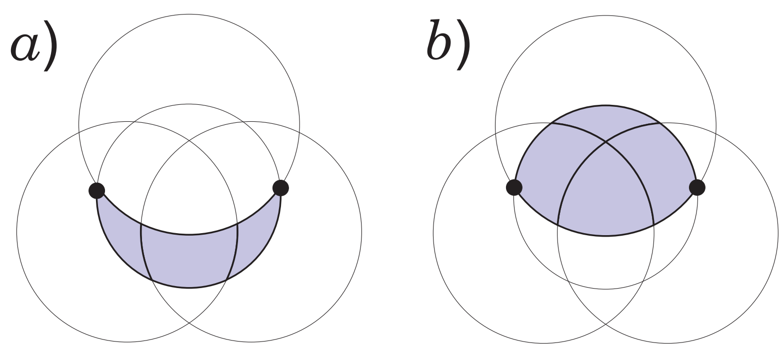

FIGURE 6. Irreducible digons of type D_{33}.

Theorem 2.7. *Any spherical digon D is a union of $k > 0$ irreducible digons of the same type with common vertices, glued together along their common sides. The type of irreducible digons in D is called the type of D. A digon D_{15} can be only glued to a digon D_{15} of a different sort. A digon D_{24} can be only glued to a digon D_{24} of the same sort. A digon D_{33} can be only glued to a digon D_{33} of a different sort. The* *corners at both vertices of D have order k if its type is D_{15} or D_{24}, and $[k/2]$ if its type is D_{33}.*

The proof is an easy exercise. Sometimes it is convenient to allow $k=0$ in Theorem 2.7, to denote an empty digon.

Remark 2.8. Note that a generic quadrilateral Q cannot contain a digon D of type D_{33} having both corners at the corners of Q, since the same two circles intersect at both corners of D.

2.3. **Spherical triangles.** A spherical triangle has its sides on at most three circles, and classification of such triangles goes back to Klein (see also [17], Section 6). Since we need spherical triangles as building blocks of generic spherical quadrilaterals, we consider only spherical triangles with all sides mapped to some circles of the partition $\mathcal{P}$ and all corners mapped to intersection points of the circles of $\mathcal{P}$. All irreducible spherical triangles are primitive, and can be classified as follows. Triangle T_n (see Fig. 7a) has an integer corner of order n and two non-integer corners of order 0. The angles at its non-integer corners are equal when n is odd and complementary (adding up to 1) when n is even. Triangle E_n (see Fig. 7b) has a non-integer corner of order n and two non-integer corners of order 0.

Remark 2.9. Fig. 7 does not show preimages of the fourth circle of $\mathcal{P}$ (which does not pass through any corners of a triangle). We'll need these preimages to understand the nets of generic quadrilaterals obtained by attaching triangles to basic quadrilaterals. In particular, the triangle T_1 geometrically is a digon D_{33} (see Fig. 6) with one of the side vertices of the net of D_{33} being the integer corner of T_1. Figs. 8 and 9 show possible nets for triangles T_1 and E_1, respectively, with the preimages of all four circles of $\mathcal{P}$ included. Note that the net of T_n always has an edge connecting its integer corner with its base (the side opposite to its integer corner). This property will be important for understanding the chains of quadrilaterals later in this paper.

2.4. **Interior arcs of generic spherical quadrilaterals.** An arc γ of a spherical quadrilateral Q was defined (see Definition 2.2) as a simple path in the preimage Γ_C in Q of a circle C of the partition $\mathcal{P}$, with the corners of Q allowed only at the ends of γ. An arc is interior if it does not belong to a side of Q. It follows from Theorem 2.12 below (see Remark 2.14) that an interior arc of a generic quadrilateral Q cannot have both ends at the lateral vertices on the opposite sides of Q.

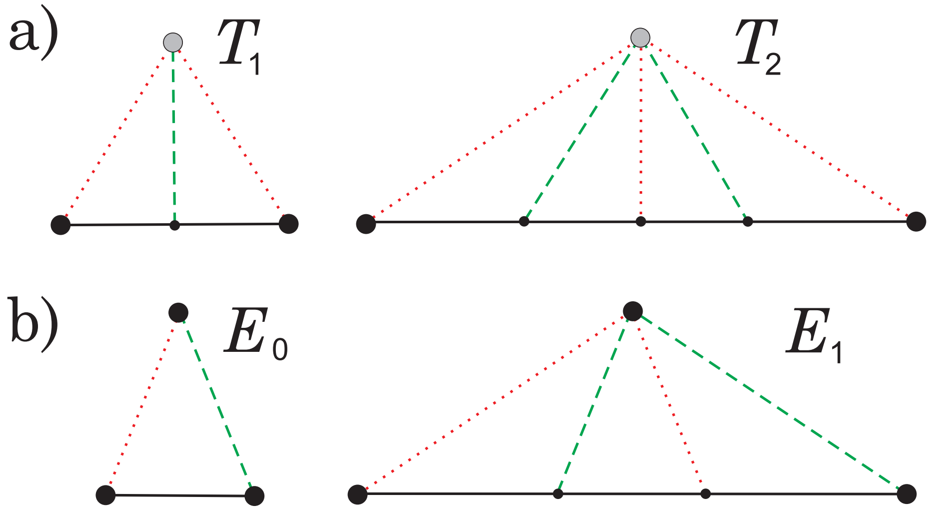

FIGURE 7. Irreducible triangles T_n and E_n.

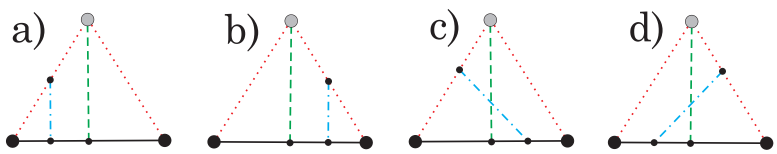

FIGURE 8. Nets for a triangle T_1.

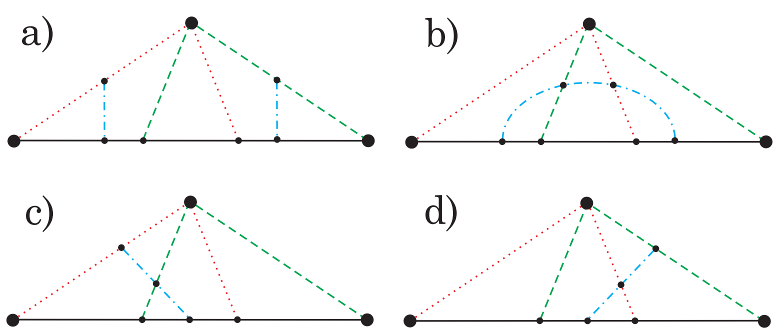

FIGURE 9. Nets for a triangle E_1.

Definition 2.10. An interior arc γ of a spherical quadrilateral Q is *one-sided* if both ends of γ are on the same side of Q, and at least one of them is not a corner of Q. If the ends of γ are lateral vertices on two adjacent sides of Q, it is *two-sided*. If one end of γ is a corner p of Q, and another end is a lateral vertex on the side of Q not adjacent to p, it is a *separator*. A separator arc partitions Q into a quadrilateral and a triangle.

Lemma 2.11. *The net of a generic quadrilateral Q does not contain an arc with the ends at two opposite corners of Q.*

Proof. Let γ be an arc of the net of Q, and let C be a circle of $\mathcal{P}$ such that γ belongs to the preimage of C. Since the sides of Q map to four distinct circles, only one of two opposite corners of Q belongs to the preimage of C. Hence γ cannot have the ends at two opposite corners of Q. $\square$

The opposite is also true: if a spherical quadrilateral Q with all non-integer corners has two opposite sides mapped to the same circle then the net of Q, defined as the preimage of the partition $\mathcal{T}$ of the Riemann sphere $\mathbf{S}$ by the three great circles to which the sides of Q are mapped, contains an interior arc with the ends at two opposite corners of Q. This follows from a more general statement about spherical polygons over a three-circle partition of the sphere (see [11], Theorem 2.2).

Theorem 2.12. *Let $\mathcal{T}$ be a partition of the Riemann sphere $\mathbf{S}$ by three distinct great circles. Let Q be a spherical n-gon having each side mapped to one of the circles of $\mathcal{T}$, and all corners mapped to vertices of $\mathcal{T}$. If $n > 3$ then the net of Q, defined as the preimage of the partition $\mathcal{T}$, contains an interior arc with the ends at two non-adjacent corners of Q.*

Remark 2.13. Note that a quadrilateral Q with all non-integer corners cannot have all sides mapped to only two circles. Due to Theorem 2.12 such a quadrilateral would be a union of two spherical triangles with all corners at the corners of Q. Each of these triangles would have an integer corner, but a spherical triangle with an integer corner p cannot have p at the intersection of the two circles to which its sides are mapped.

Remark 2.14. Theorem 2.12 implies that a generic spherical quadrilateral Q cannot have an interior arc γ with the ends at lateral vertices on opposite sides of Q. Such an arc would partition Q into two quadrilaterals, one of them having all sides mapped to three circles (since one of the sides of Q maps to the same circle of $\mathcal{P}$ as γ). According to Theorem 2.12, such a quadrilateral has an interior arc with the ends at its opposite corners. This is impossible since both corners adjacent to its side γ have order 0.

Lemma 2.15. . *Let γ be a two-sided arc of a generic irreducible quadrilateral Q with the ends a and b on the sides of Q adjacent to its corner p. Then p has order 0.*

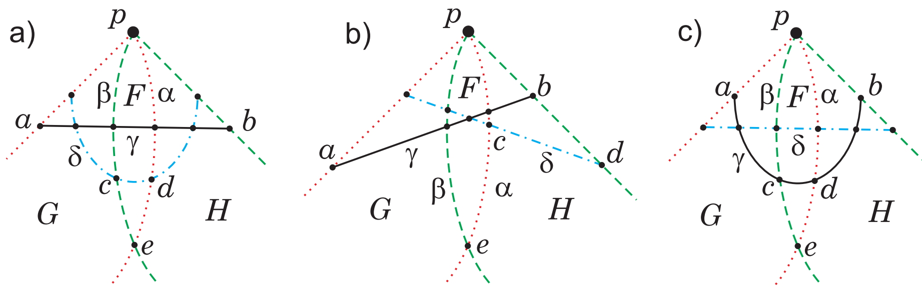

FIGURE 10. Illustration for the proof of Lemma 2.15.

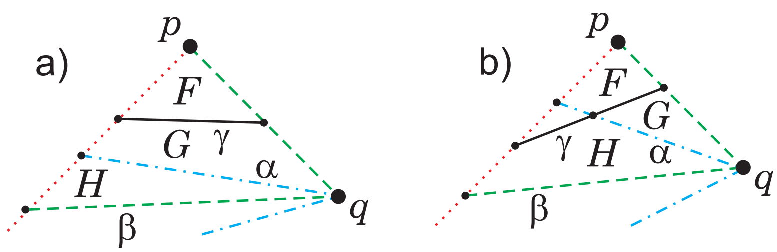

FIGURE 11. Illustration for the proof of Lemma 2.16.

Proof. We prove the statement by contradiction, assuming that the corner p has order 1 (the case of a corner with order greater than 1 can be treated similarly). Then Q contains an irreducible triangle E_1 (see Figs. 7b and 9) with the vertices p, a, b, bounded by γ and the two sides of Q adjacent to p. There are two interior arcs of Q, say α and β, both with one end at p, such that α is mapped to the same circle of $\mathcal{P}$ as the side ap of E_1, and β is mapped to the same circle as its side bp. Fig. 9 shows four possibilities for the net of E_1. We consider two cases (see Figs. 10a and 10b) corresponding to the nets in Figs. 9a and 9c. Note that the net in Fig. 9d is reflection symmetric to that in Fig. 9c, and the net in Fig. 9b is obtained from Fig. 10a by exchanging the arcs γ and δ (see Fig. 10c).

Let F be the face of the net Γ of Q adjacent to p and bounded by the arcs α and β. In Fig. 10a the face F is a triangle and the faces of Γ adjacent to F are quadrilaterals, so there should be an interior arc δ at the boundary of each of these faces, mapped to a circle of $\mathcal{P}$ other than those for α, β and γ. Note that vertices c and d of δ are interior vertices of Γ: they cannot be corners of Q since they are connected to p by interior arcs. For the same reason the intersection point e of α and β should be an interior vertex of Γ. Thus Γ should contain a face G bounded by the side of Q extending ap and segments of γ, δ, β, α, and a face H bounded by the side of Q extending bp and segments of

γ, δ, α, β (see Fig. 10a). This is a contradiction: a side of Q and an interior arc mapped to the same circle cannot belong to the boundary of the same face of Γ. The same arguments apply to the case shown in Fig. 10c where the arcs γ and δ are exchanged. Note that at least one end of the arc δ in Fig. 10c is not a corner of Q, since Q is irreducible.

In Fig. 10b the face F is a quadrilateral. The same arguments as before show that Γ should contain a face G bounded by the side of Q extending ap and segments of γ, δ, β and α. This is a contradiction: a side of Q and an interior arc α mapped to the same circle cannot belong to the boundary of the same face G. □

Lemma 2.16. . *Let γ be a two-sided arc of a generic irreducible quadrilateral Q, with the ends on the sides of Q adjacent to its corner p. Then one of the sides of Q adjacent to p has another end at a corner q of Q of order at least 1. The net Γ of Q has a face H adjacent to q, such that the boundary of H contains segments of two separator arcs with a common end at q .*

Proof. According to Lemma 2.15, the corner p of Q has order 0, thus there is a unique face F of Γ adjacent to p, which may be either triangular or quadrilateral.

If F is a triangle (see Fig. 11a) then γ is a side of F. Let G be the quadrilateral face of Γ adjacent to γ from the other side. Then the boundary α of G opposite to γ has both ends on the sides of Q adjacent to p. The same arguments as in the proof of Lemma 2.15 show that α cannot be a two-sided arc, thus α has one end at a corner q of Q. Since Q is irreducible, the other end of α cannot be a corner of Q, thus α is a separator arc. Note that q cannot have order 0, otherwise α would be a side of Q, and Q would be a triangle. Thus the order of q is at least 1, and Γ contains a triangular face H adjacent to q and bounded by a side of Q and two separator arcs α and β.

If F is a quadrilateral (see Fig. 11b) then one of its boundary edges belongs to γ, and the triangular face G of Γ on the other side of γ is bounded by a side of Q and the edges of arcs γ and α. Note that α cannot be a two-sided arc of Γ. If it were a two-sided arc, the quadrilateral face H of Γ on the other side of α would have two boundary arcs (other than γ and α) belonging to two sides of Q adjacent to p, intersecting at a vertex of H. This is impossible, since the sides of Q adjacent to p do not intersect at any other point. Thus the end q of α is a corner of Q, and both arcs α and β at the boundary of H are separator arcs. □

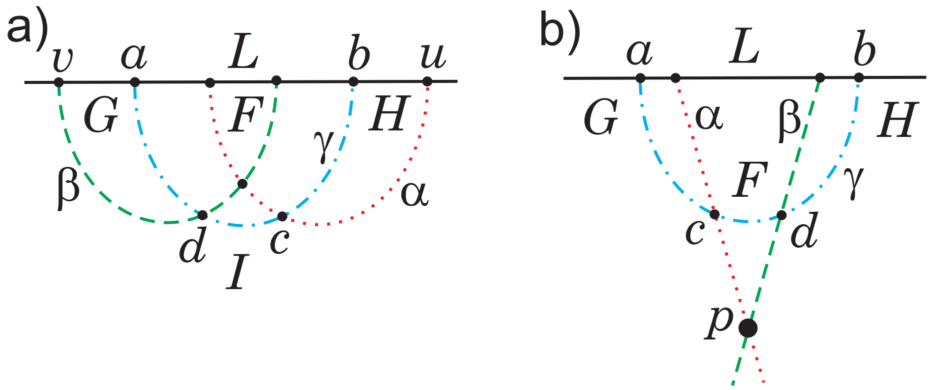

FIGURE 12. Illustration for the proof of Lemma 2.17.

Lemma 2.17. *Let γ be a one-sided arc of a generic primitive quadrilateral Q, with the ends a and b on a side L of Q. Then there are two separator arcs of the net Γ of Q with a common end at a corner p of Q and the other ends inside the segment (a,b) of L. If both a and b are not corners of Q then the order of p is greater than 1.*

Proof. Since the endpoints a and b of γ belong to the intersection of the same two circles, there are two lateral vertices of Γ inside the segment (a,b) of L, corresponding to the intersections of L with preimages of two circles of $\mathcal{P}$ other than those to which L and γ are mapped. Let α and β be interior arcs of Γ with endpoints at the two vertices inside (a,b). The face F of Γ bounded by the arcs α and β is either a triangle (see Fig. 12a) or a quadrilateral (see Fig. 12b).

If F is a triangle then the faces G and H of Γ adjacent to a and b outside γ are triangular, thus the edges of arcs α and β outside γ (with endpoints c and d, respectively) must have other ends mapped to the same circle as L. This is impossible since the face I of Γ (see Fig. 12a) cannot have two disjoint segments mapped to the same circle in its boundary. This argument holds also when either a or b is a corner of Q.

If F is a quadrilateral then the arcs α and β intersect at a point p outside γ. Note that p cannot be a lateral vertex since both cp and dp are interior edges of Γ. If p were an interior vertex of Γ, the edges of arcs α and β beyond p would have their other ends mapped to the same circle as L, which is impossible for the same reason as when F a triangle. Thus p must be a corner of Q. If both a and b are not corners of Q then p cannot have order 1. Otherwise, the boundary edges of G and H opposite γ would be two sides of Q, each of them

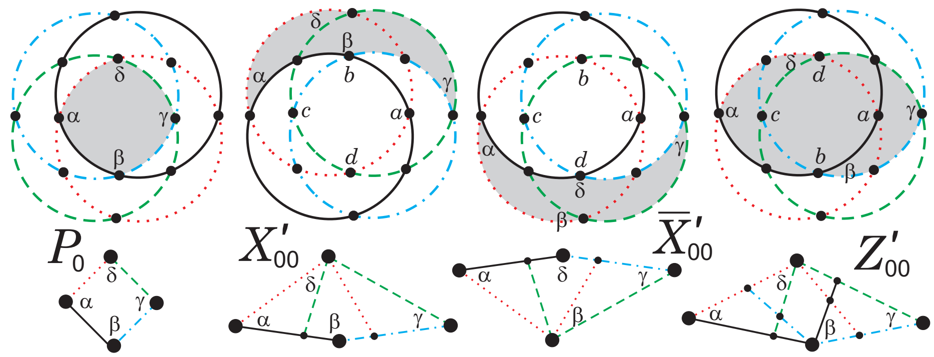

FIGURE 13. Basic primitive quadrilaterals and their nets.

having one end at p and the other end on the side L of Q, thus Q would be a triangle, a contradiction. □

Corollary 2.18. *Let Q be a generic quadrilateral with all four angles of order 0. Then the net of Q does not have interior arcs.*

Proof. Note that Q is primitive and cannot have separator arcs. It follows from Lemmas 2.16 and 2.17 that Q cannot have two-sided or one-sided arcs. □

3. Classification of nets of primitive and irreducible quadrilaterals

In this section, Q is a primitive (see Definition 2.2) generic spherical quadrilateral. It is shown below that Q has at least two corners of order 0. We order corners $(a_0, \dots, a_3)$ of Q so that the order of a_0 is 0, and the sum of orders of a_0 and a_2 does not exceed the sum of orders of a_1 and a_3.

3.1. **Extension of a side.** Let p be a corner of Q of order 0, with the angle $\alpha < 1$. Let L and M be two sides of Q adjacent to p. Suppose that M has order at most 2, and let q be the corner of Q at the other end of M. Then we can attach to Q a spherical triangle T_n with an integer corner at q and two other corners at p and p', so that the side $[p,q]$ of T_n is common with the side M of Q, and the base $[p,p']$ of T_n is extending L beyond p. The union of Q and T_n is a primitive spherical quadrilateral Q' with the side $L' = L \cup [p,p']$, and the angle at its corner p' equal α if n is even and $1-\alpha$ if n is odd. We call this operation *extension of the side* L of Q beyond its corner p. Note that extending L' beyond p' by attaching a triangle T_m to Q' is the same as a single extension attaching a triangle T_{n+m} to Q.

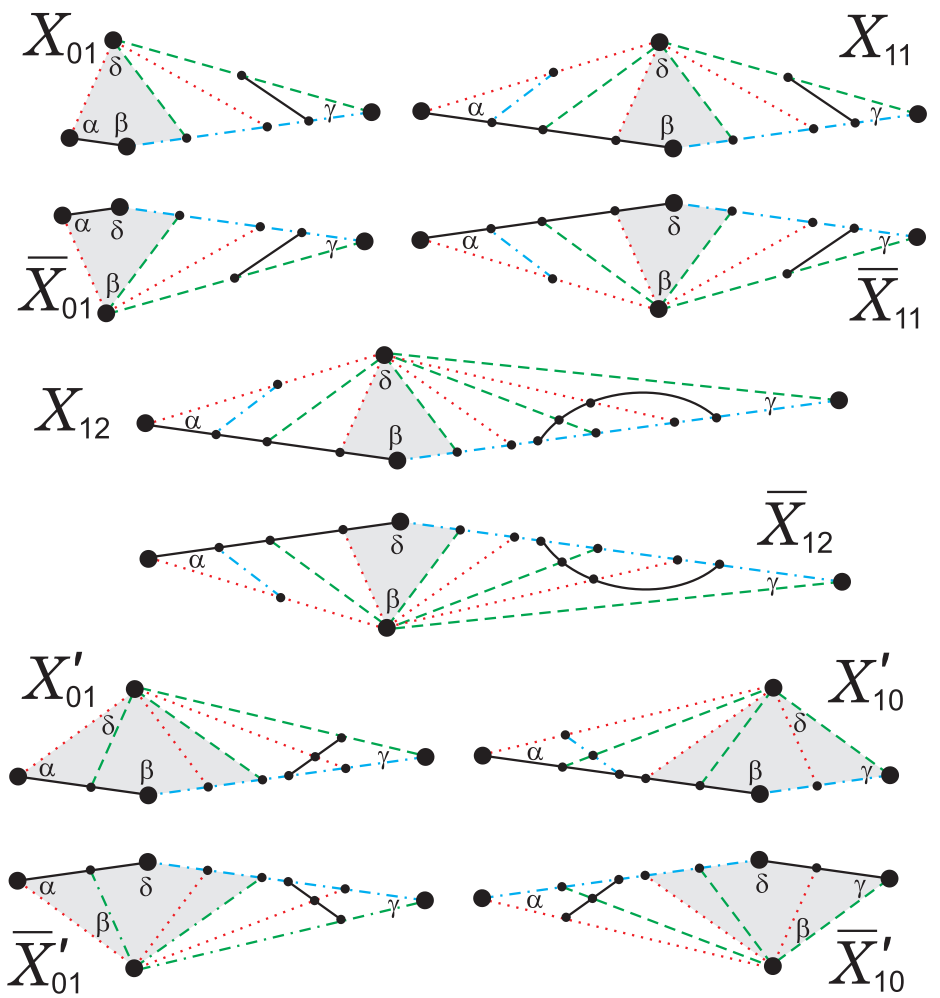

FIGURE 14. Quadrilaterals of types X_{kl}, $\bar{X}_{kl}$, X'_{kl} and $\bar{X}'_{kl}$.

Theorem 3.1. *Every generic primitive quadrilateral Q can be obtained from one of the basic quadrilaterals P_0, X'_{00}, $\bar{X}'_{00}$ and Z'_{00} (see Fig. 13) by at most two extension operations.*

Proof of Theorem 3.1 will be given at the end of this section. It implies that the nets of all primitive quadrilaterals belong to the following list (see Figs. 14, 15, 16 and 17 where the basic quadrilateral is shaded).

Type X. A quadrilateral X_{kl} for $k, l \geq 0$ and $k + l \geq 1$ (see Fig. 14) can be obtained by attaching triangles T_k and T_l to adjacent sides of the basic quadrilateral P_0. When either $k = 0$ or $l = 0$, only one triangle is attached. The quadrilateral X_{kl} has one corner of order

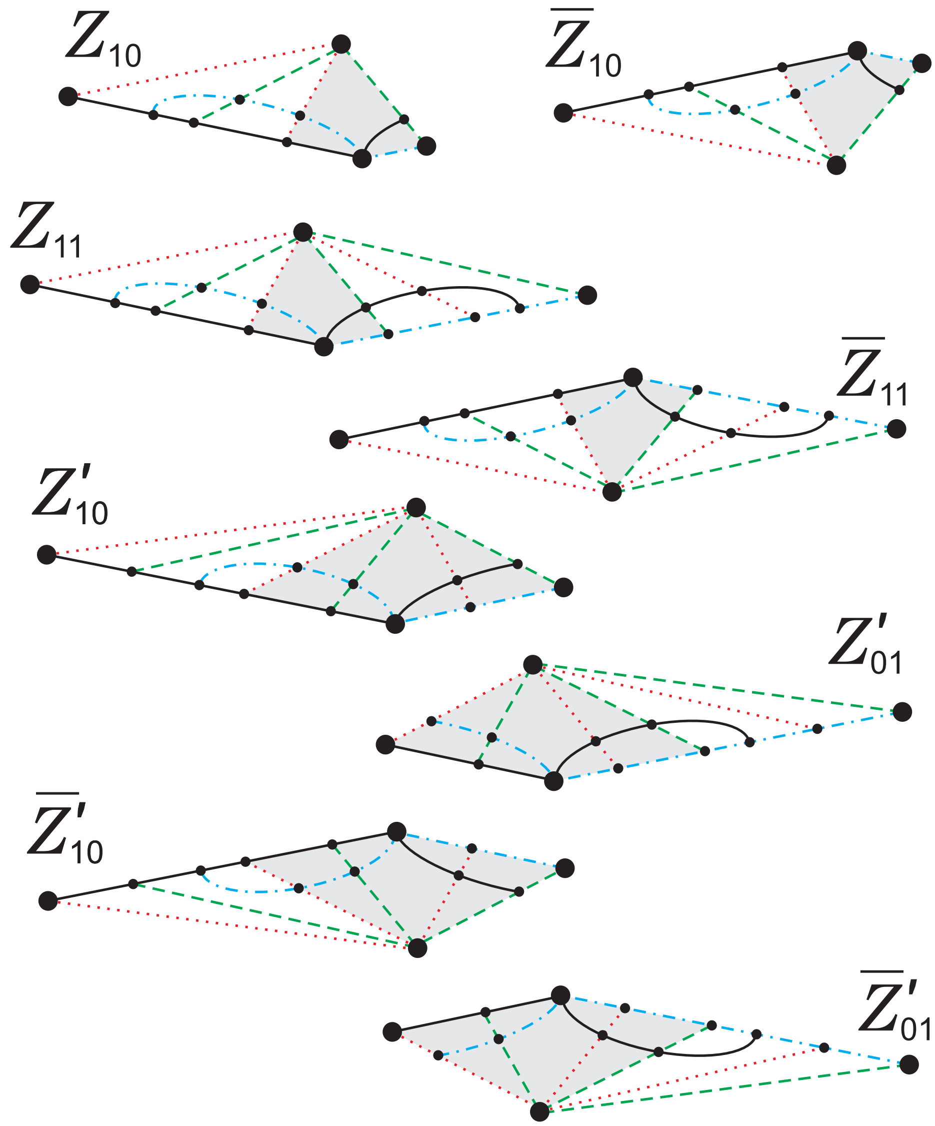

FIGURE 15. Quadrilaterals of types Z_{kl}, $\bar{Z}_{kl}$, Z'_{kl} and $\bar{Z}'_{kl}$.

$k+l$, the other three corners being of order 0. A quadrilateral $\bar{X}_{kl}$ is reflection-symmetric (preserving opposite corners of order 0) to X_{kl}.

Type X'. A quadrilateral X'_{kl} for $k,l \geq 0$ (see Fig. 14) can be obtained by attaching triangles T_k and T_l to adjacent sides of the basic quadrilateral X'_{00} so that T_k and T_l have a common vertex at the corner of X'_{00} of order 1. The quadrilateral X'_{kl} has one corner of order $k+l+1$, the other three corners being of order 0. A quadrilateral

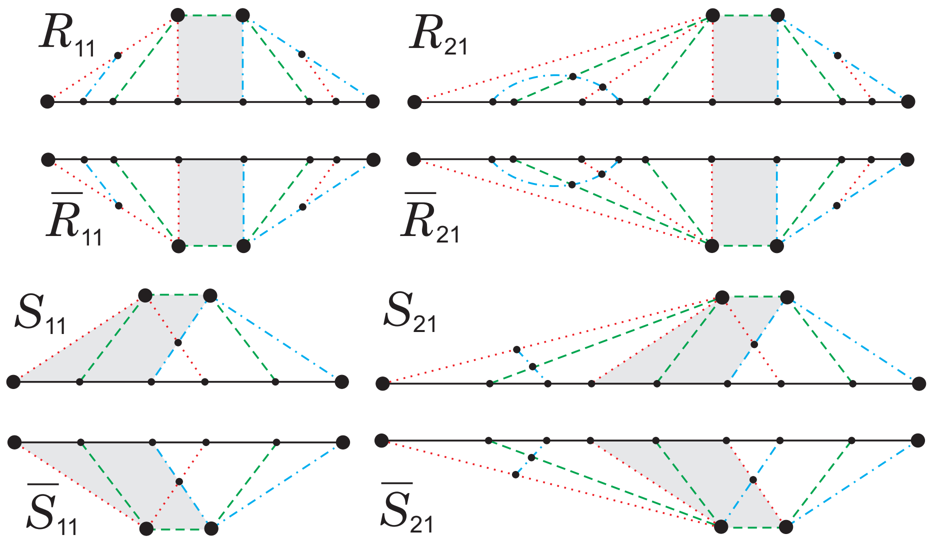

FIGURE 16. Quadrilaterals of types R_{kl}, $\bar{R}_{kl}$, S_{kl} and $\bar{S}_{kl}$.

$\bar{X}'_{kl}$ is reflection-symmetric (preserving opposite corners of order 0) to X'_{kl}.

Type Z. A quadrilateral Z_{kl} for $k,l \geq 0$ and $k+l \geq 1$ (see Fig. 15) can be obtained by attaching triangles T_k and T_l to adjacent sides of the basic quadrilateral $\bar{X}'_{00}$ so that T_k and T_l have a common vertex at the corner of $\bar{X}'_{00}$ opposite to its corner of order 1. When either $k=0$ or $l=0$, only one triangle is attached. The quadrilateral Z_{kl} has opposite corners of orders $k+l$ and 1, the other two opposite corners being of order 0. A quadrilateral $\bar{Z}_{kl}$ is reflection-symmetric (preserving the corners of order 0) to Z_{kl}.

Type Z'. A quadrilateral Z'_{kl} for $k,l \geq 0$ (see Fig. 15) can be obtained by attaching triangles T_k and T_l to adjacent sides of the basic quadrilateral Z'_{00} so that T_k and T_l have a common vertex at a corner of Z'_{00} of order 1. The quadrilateral Z'_{kl} has opposite corners of orders $k+l+1$ and 1, the other two corners being of order 0. A quadrilateral $\bar{Z}'_{kl}$ is reflection-symmetric (preserving the corners of order 0) to Z'_{kl}.

Type R. A quadrilateral R_{kl} for $k \geq l \geq 1$ (see Fig. 16) can be obtained by attaching triangles T_k and T_l to opposite sides of the basic quadrilateral P_0 so that both triangles extend the same side of P_0. The quadrilateral R_{kl} has adjacent corners of orders k and l, the other two corners being of order 0. A quadrilateral $\bar{R}_{kl}$ is reflection-symmetric (preserving the corners of order 0) to R_{kl}.

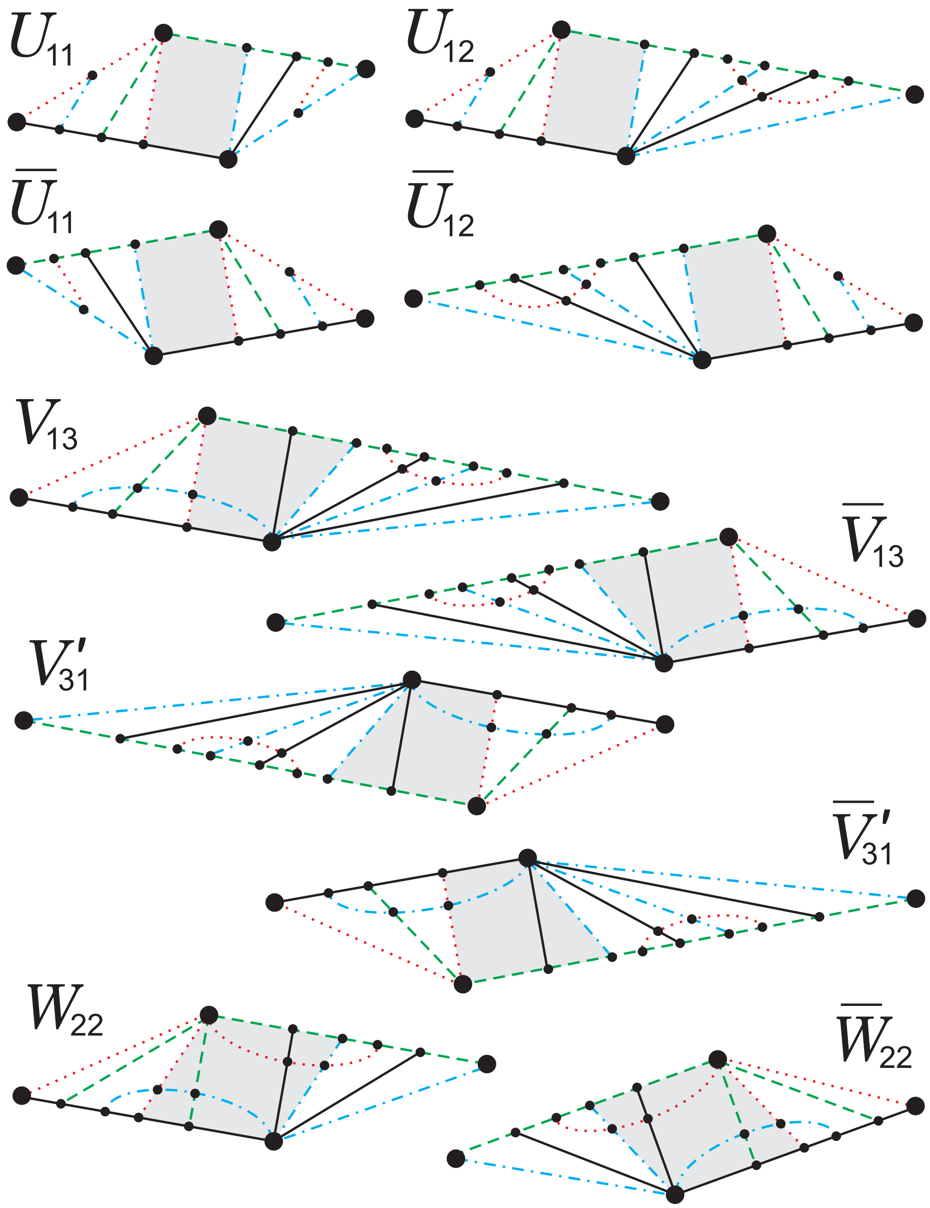

FIGURE 17. Quadrilaterals of types U_{kl}, $\bar{U}_{kl}$, V_{kl}, $\bar{V}_{kl}$, V'_{kl}, $\bar{V}'_{kl}$, W_{kl}, $\bar{W}_{kl}$.

Type S**.** A quadrilateral S_{kl} for $k \geq l \geq 1$ (see Fig. 16) can be obtained by attaching triangles T_{k-1} and T_l (or T_k and T_{l-1}) to opposite sides of the basic quadrilateral X'_{00}, so that both triangles extend the same side of order 2 of X'_{00}. The quadrilateral S_{kl} has adjacent corners of orders k and l, the other two corners being of order 0. A quadrilateral $\bar{S}_{kl}$ is reflection-symmetric(preserving the corners of order 0) to S_{kl}.

Type U. A quadrilateral U_{kl} for $k, l \geq 1$ (see Fig. 17) can be obtained by attaching triangles T_k and T_l to opposite sides of the basic quadrilateral P_0, so that T_k and T_l have vertices at the opposite corners of the quadrilateral P_0 and extend its opposite sides. The quadrilateral U_{kl} has opposite corners of orders k and l, the other two corners being of order 0. A quadrilateral $\bar{U}_{kl}$ is reflection symmetric (exchanging the opposite corners of order 0) to U_{kl}.

Type V. A quadrilateral V_{kl} for $k \geq 1$, $l \geq 2$ (see Fig. 17) can be obtained by attaching triangles T_k and T_{l-1} to opposite sides of the basic quadrilateral $\bar{X}'_{00}$, so that T_{l-1} has its vertex at the corner of order 1 of $\bar{X}'_{00}$, and T_k has its vertex at the opposite corner. The quadrilateral V_{kl} has opposite corners of orders k and l, the other two corners being of order 0. A quadrilateral $\bar{V}_{kl}$ is reflection symmetric (exchanging the opposite corners of order 0) to V_{kl}. A quadrilateral V'_{kl} is rotation symmetric to V_{lk}. A quadrilateral $\bar{V}'_{kl}$ is reflection symmetric (exchanging the opposite corners of order 0) to V'_{kl}.

Type W. A quadrilateral W_{kl} for $k, l \geq 2$ (see Fig. 17) can be obtained by attaching triangles T_{k-1} and T_{l-1} to opposite sides of the basic quadrilateral Z'_{00}, so that T_{k-1} and T_{l-1} have vertices at the opposite corners of order 1 of Z'_{00} and extend its opposite sides. The quadrilateral W_{kl} has two opposite corners of orders k and l, the other two corners being of order 0. A quadrilateral $\bar{W}_{kl}$ is reflection symmetric (exchanging the opposite corners of order 0) to W_{kl}.

Note that an extended side of a quadrilateral has order greater than 3. It is short (of order less than 6) only when extended by a single triangle T_1. Quadrilaterals of types R and S have one extended side. Quadrilaterals of types X, X', Z, Z' have either one extended side or two adjacent extended sides. Quadrilaterals of types U, V, W have two opposite extended sides.

Lemma 3.2. *Let Q be a generic primitive quadrilateral with one corner p of order greater than 0 and three other corners of order 0. Then the net of Q is of the type either X or X', or one of their reflection symmetric quadrilaterals $\bar{X}$ and $\bar{X}'$.*

Proof. Let q be the corner of Q opposite to p, and let F be the face of the net Γ of Q adjacent to q. It follows from Lemma 2.16 that there are no two-sided arcs of Γ with the ends on the sides of Q adjacent to q. Thus F must be a quadrilateral face of Γ. The vertex a of F

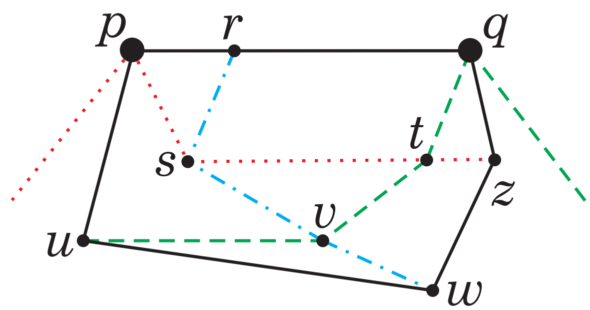

FIGURE 18. Illustration for the proof of Lemma 3.3.

opposite to its vertex q cannot be an interior vertex of Γ. Otherwise the two arcs of the boundary of F would be two-sided, in contradiction with Lemma 2.16. The vertex a cannot be a lateral vertex of Γ, since in that case one of the arcs of the boundary of F would have two ends at the opposite sides of Q, which is forbidden, or would be a one-sided arc, in contradiction with Lemma 2.17. Thus $a = p$ is the corner of Q opposite q.

It follows that Q is a union of F and either two triangles T_k and T_l with integer angles k and l at their common vertex p attached to the sides of F adjacent to p, such that $k + l > 0$ is the order of p, or two triangles E_k and E_l with non-integer corners of order k and l at their common vertex p attached to the sides of F adjacent to p, such that $k + l + 1$ is the order of p. In the first case, Q has type X_{kl} or $\bar{X}_{kl}$, in the second case Q has type X'_{kl} or $\bar{X}'_{kl}$. □

Lemma 3.3. *Let Q be a generic primitive quadrilateral with two adjacent corners p and q at the ends of its side pq. If the angles of Q at both p and q are greater than 1 then its side pq has order 1.*

Proof. Assume that pq has order 2 (larger orders can be treated similarly). Then the net of Q contains two faces, one triangular and one quadrilateral, adjacent to pq. We may assume that these two faces are psr and $qrst$ (see Fig. 18). Note that the vertex s cannot be a corner of Q since it is connected to its corner p by an interior arc. Similarly, the vertex t cannot be a corner of Q as it is connected to q. The vertex s cannot be a boundary vertex of the net of Q since ps and rs are interior arcs. Thus s is an interior vertex, and the net of Q contains a quadrilateral face $puvs$ and a triangular face svt. The vertex t cannot be a boundary vertex since st and qt are interior arcs. Thus t is an interior vertex, and the net of Q contains a quadrilateral face $tvwz$ and a triangular face qtz. The vertex v cannot be a corner

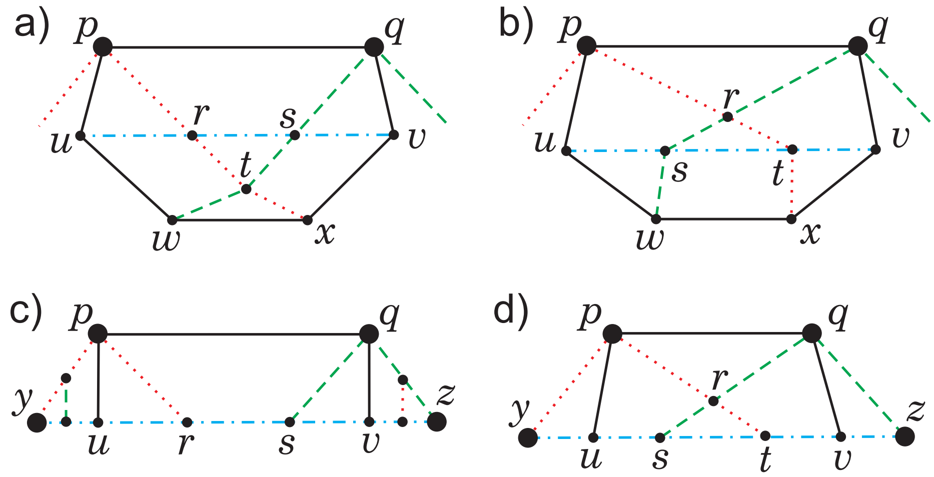

FIGURE 19. Illustration for the proof of Lemma 3.4.

of Q as it is connected to q by an interior arc. It cannot be a boundary vertex, as sv and tv are interior arcs. Thus v is an interior vertex, and the net of Q contains a triangular face vuw. The same arguments as above show that both u and w should be interior vertices. This contradicts irreducibility of Q, since p and q are connected by an interior arc $puwzq$. □

Lemma 3.4. *Let Q be a generic primitive quadrilateral with two corners p and q at the ends of its side pq, with both angles greater than 1. Let C be the circle of $\mathcal{P}$ to which the side of Q opposite to pq is mapped. Then the net of Q has no interior arcs mapped to C, and has the type either R or S, or one of their reflection symmetric quadrilaterals $\bar{R}$ and $\bar{S}$.*

Proof. According to Lemma 3.3, the side pq of Q has order 1. The face F of the net Γ of Q adjacent to pq may be either a quadrilateral $prsq$ (see Fig. 19a) or a triangle prq (see Fig. 19b).

Consider first the case $F = prsq$. Note that its arc rs is mapped to C. Neither r nor s may be a corner of Q, since r is connected to p and s is connected to q by an interior arc of Γ. If one of these vertices, say r, is an interior vertex of Γ, then Γ contains the faces pur, $uwtr$ and rts. It follows that s is also an interior vertex of Γ, since it has two interior arcs rs and qs adjacent to it. Thus Γ contains faces $ruwt$ and $stxv$. Note that t cannot be a corner of Q since the two arcs intersecting at t are preimages of the circles of $\mathcal{P}$ corresponding to two opposite sides of Q. Since rt and st are interior arcs of Γ, t must be an interior vertex of Γ, and twx is a face of Γ. This implies that

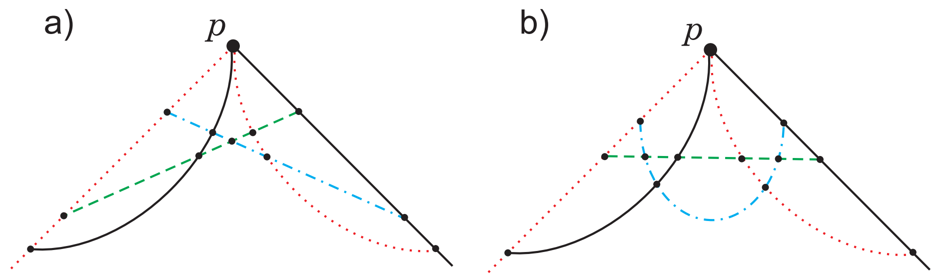

FIGURE 20. Illustration for the proof of Lemma 3.5.

the arc $puwxvq$ of Γ connects p and q, contradicting irreducibility of Q. Thus both r and s must be boundary vertices of Γ, and $ursv$ is part of the side of Q opposite pq. Extending this side till the corners y and z of Q results in a quadrilateral R_{kl} or $\bar{R}_{kl}$, the union of $prsq$ and two triangles T_k and T_l (see Fig. 19c where $k = l = 1$).

Next we consider the case when $F = prq$ is a triangle. Since r is not a corner of Q (it is connected to both p and q by interior arcs) it must be an interior vertex of Γ. Thus Γ contains the faces $pusr$, rst and $qrtv$ (see Fig. 19b). Note that neither s nor t may be corners of Q (they are connected by interior arcs to q and p, respectively). If one of these vertices, say s, is an interior vertex of Γ then suw and $swxt$ are faces of Γ, thus t is also an interior vertex of Γ (it has interior arcs rt and st adjacent to it) and txv is a face of Γ. This implies that the arc $puwxvq$ of Γ connects p and q, contradicting irreducibility of Q. Thus both s and t must be boundary vertices of Γ, and $ustv$ is part of the side of Q opposite pq. Extending this side till the corners y and z of Q results in a quadrilateral S_{kl} or $\bar{S}_{kl}$, the union of prq and two triangles T_k and T_l (see Fig. 19c where $k = l = 1$ intersecting over rst). □

Lemma 3.5. *Let Q be a generic primitive quadrilateral with two opposite corners p and q of order 1, and two other corners of order 0. Then the net Γ of Q is one of the following: Z'_{00}, Z_{01}, Z_{10}, $\bar{Z}_{01}$, $\bar{Z}_{10}$, U_{11}, $\bar{U}_{11}$.*

Proof. If Q does not have one-sided arcs, there are two separator arcs α and α' with a common end at p, and two separator arcs β and β' with a common end at q. If the other ends of α and α' are on different sides of Q then the same is true for β and β', thus the net of Q is Z'_{00}. If the other ends of α and α' are on the same side of Q then the same is true for β and β', thus the net of Q is either U_{11} or $\bar{U}_{11}$.

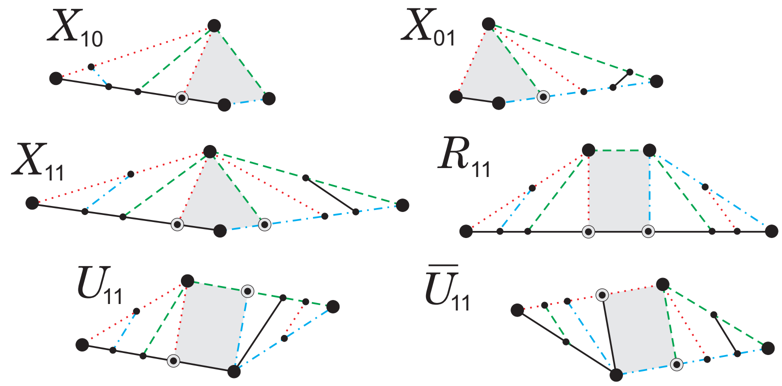

FIGURE 21. Attaching one or two triangles T_1 to the quadrilateral P_0.

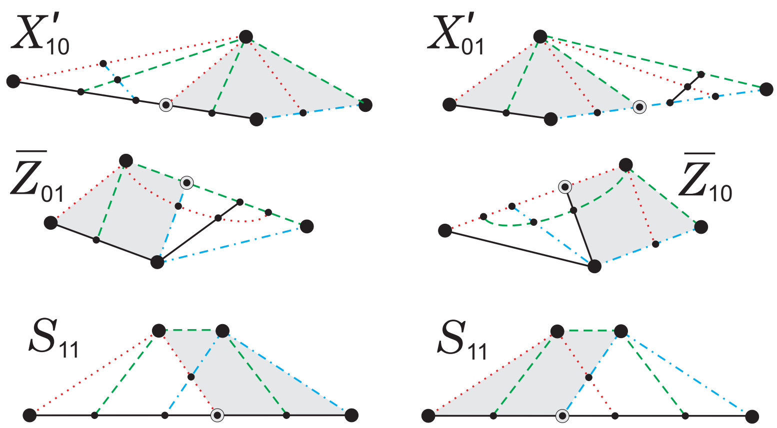

FIGURE 22. Attaching a triangle T_1 to the quadrilateral X'_{00}.

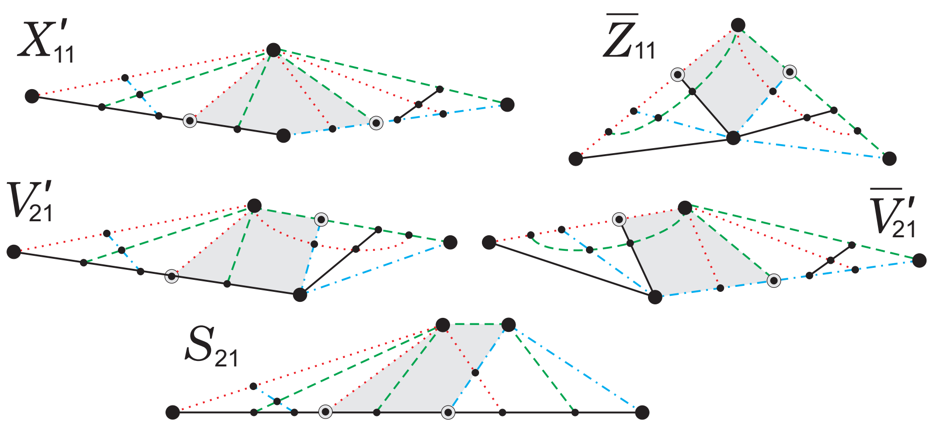

FIGURE 23. Attaching two triangles T_1 to the quadrilateral X'_{00}.

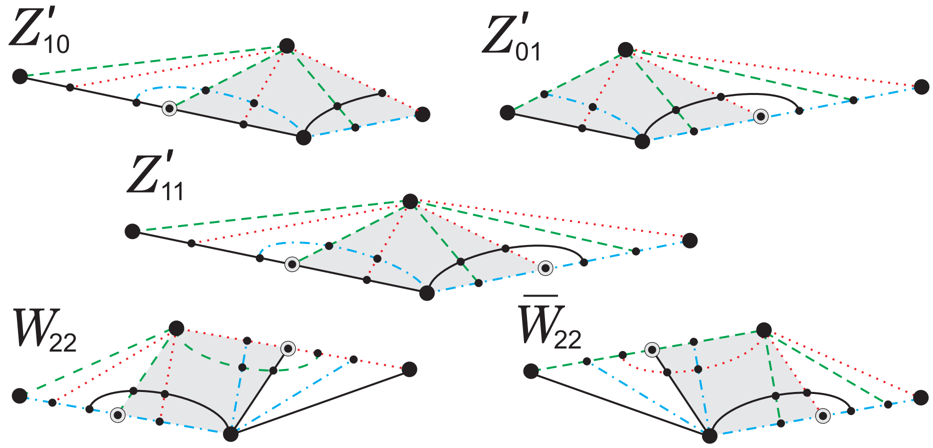

FIGURE 24. Attaching one or two triangles T_1 to the quadrilateral Z'_{00}.

Next, Q may have a one-sided arc γ with one end at a corner of Q (due to Lemma 2.17, if both ends of γ were not at corners of Q then Q would have a corner of order greater than 1, a contradiction). Then the net of Q is one of Z_{01}, Z_{10}, $\bar{Z}_{01}$, $\bar{Z}_{10}$.

If two one-sided arcs were adjacent to the same corner of Q, Lemma 2.17 would imply that there should be at least 4 separator arcs with the common end at the opposite corner, a contradiction. Finally, having two one-sided arcs with the ends at opposite corners of Q would imply that each of these corners must have, in addition to a one-sided arc, two separator arcs with a common end in it, a contradiction. This completes the proof of Lemma 3.5. □

Lemma 3.6. *Let Q be a generic primitive quadrilateral with a corner p of order greater than 1. Then there is a separator arc of the net Γ of Q partitioning Q into a generic primitive quadrilateral Q' smaller than Q and an irreducible triangle T with an integer corner.*

Proof. If the net Γ has a one-sided arc γ with both ends on the side L of Q then there are, according to Lemma 2.17, two separator arcs of Γ with a common end at a corner of Q and the other ends on L inside γ. One of these two arcs partitions Q into a quadrilateral with one integer corner and a triangle with all non-integer corners, and another one partitions Q into a generic quadrilateral Q' smaller than Q and a triangle T with an integer corner. Thus we may assume that Γ does not have any one-sided arcs. In this case, there are at least four separator arcs with a common end at the corner p of Q. At least two of these arcs must be at the boundary of the same face of Γ adjacent to

p and have their other ends on the same side of Q. Then one of these two arcs partitions Q into a quadrilateral with one integer corner and a triangle with all non-integer corners, and another one partitions Q into a generic quadrilateral Q' smaller than Q and a triangle T with an integer corner. This completes the proof of Lemma 3.6. $\square$

Proof of Theorem 3.1. According to Lemma 3.6, every generic primitive quadrilateral with a corner of order greater than 1 can be partitioned into a smaller generic quadrilateral and a triangle along one of the separator arcs of its net. This implies that every generic primitive quadrilateral Q can be obtained by attaching triangles with integer corners to a quadrilateral Q' with all corners of order at most 1. If all corners of Q' have order 0 then Q' is the quadrilateral P_0 (see Corollary 2.18). If only one corner of Q' has order 1, the other three corners having order 0, then the net of Q' is one of X_{01}, X_{10}, X'_{00}, $\bar{X}_{01}$, $\bar{X}_{10}$, $\bar{X}'_{00}$ according to Lemma 3.2. Note that each of the quadrilaterals X_{01}, X_{10}, $\bar{X}_{01}$ and $\bar{X}_{10}$ can be partitioned into the quadrilateral P_0 and a triangle T_1. If two adjacent corners of Q' have order 1 then the net of Q' is either R_{11} or S_{11} according to Lemma 3.4. If two opposite corners of Q' have order 1 then the net of Q' is either Z'_{00} or one of Z_{01}, Z_{10}, $\bar{Z}_{01}$, $\bar{Z}_{10}$, U_{11}, $\bar{U}_{11}$, according to Lemma 3.5. Note that each of the latter six quadrilaterals can be further reduced to either P_0 or X'_{00} or $\bar{X}'_{00}$.

There are eight options for attaching a triangle T with an integer corner to the quadrilateral P_0. The vertex of T can be placed at any of the four corners of the quadrilateral, and the base of T at the extension of one of its two sides opposite to that corner. Combining these options would result in quadrilaterals X_{kl}, $\bar{X}_{kl}$, R_{kl}, $\bar{R}_{kl}$, U_{kl} and $\bar{U}_{kl}$ (see Fig. 21).

There are six options for attaching a triangle T with an integer corner to the quadrilateral X'_{00} (and to $\bar{X}'_{00}$). The vertex of T can be placed at the corner p of order 1, and the base of T at the extension of one of its two sides opposite to p. Alternatively, the vertex of T can be placed at the corner q opposite p and the base of T at the extension of one of the two sides adjacent to p. Finally, the vertex of T can be placed at one of the corners u and v other than p and q, and the base of T at the extension of the side of order 2 not adjacent to the corner where the vertex of T is placed. Fig. 22 shows six options of attaching the triangle T_1 to the quadrilateral X'_{00}. Attaching T_1 to $\bar{X}'_{00}$ corresponds to replacing all quadrilaterals in Fig. 22 by their reflections preserving the corners of order 0. Combining these options would result in quadrilaterals X'_{kl}, $\bar{X}'_{kl}$, Z_{kl}, $\bar{Z}_{kl}$, V_{kl}, $\bar{V}_{kl}$, V'_{kl}, $\bar{V}'_{kl}$, S_{kl}, $\bar{S}_{kl}$ (see Fig. 23).

There are four options to attach a triangle T to the quadrilateral Z'_{00}. The vertex of T can be placed at one of the corners of order 1 of the quadrilateral, and the base of T at the extension of one of its two sides opposite to that corner. Combining these options would result in quadrilaterals Z'_{kl}, $\bar{Z}'_{kl}$, W_{kl} and $\bar{W}_{kl}$ (see Fig. 24).

Summing up, all primitive quadrilaterals that can be obtained by attaching at most two irreducible triangles, each with an integer corner, to one of the basic quadrilaterals P_0, X'_{00}, $\bar{X}'_{00}$ and Z'_{00}, appear in the list of primitive quadrilaterals in Theorem 3.1. This completes the proof of Theorem 3.1. $\square$

Corollary 3.7. *Every generic irreducible quadrilateral is either primitive or is obtained from a primitive quadrilateral Q listed in Theorem 3.1, of type other than R, S, $\bar{R}$ and $\bar{S}$, by replacing a quadrilateral face of the net of Q adjacent to its two opposite corners with the quadrilateral P_μ, for some $\mu > 0$ (see Definition 2.5).*

Proof. If Q is an irreducible quadrilateral that is not primitive then, according to Lemma 2.4, Q contains a quadrilateral P_1 (see Fig. 3). In particular, the net of Q has a quadrilateral face adjacent to two of its opposite corners. Replacing P_1 with P_0, we obtain a smaller quadrilateral. We can repeat this operation μ times until we get a primitive quadrilateral Q'. Since Q' still has a quadrilateral face adjacent to two of its opposite corners, it should belong to one of the types listed in Theorem 3.1 other than R, S, $\bar{R}$ and $\bar{S}$. The original quadrilateral Q is obtained from Q' by replacing its quadrilateral face adjacent to two of its opposite corners with the quadrilateral P_μ, as stated in Theorem 3.7. $\square$

Notation. The irreducible quadrilaterals obtained from the primitive quadrilaterals X_{kl}, $\bar{X}_{kl}$, X'_{kl}, $\bar{X}'_{kl}$, Z_{kl}, $\bar{Z}_{kl}$, Z'_{kl}, $\bar{Z}'_{kl}$, U_{kl}, $\bar{U}_{kl}$, V_{kl}, $\bar{V}_{kl}$, V'_{kl}, $\bar{V}'_{kl}$, W_{kl}, $\bar{W}_{kl}$ by replacing a quadrilateral face of their net by the quadrilateral P_μ are denoted X^μ_{kl}, $\bar{X}^\mu_{kl}$, X'^μ_{kl}, $\bar{X}'^\mu_{kl}$, Z^μ_{kl}, $\bar{Z}^\mu_{kl}$, Z'^μ_{kl}, $\bar{Z}'^\mu_{kl}$, U^μ_{kl}, $\bar{U}^\mu_{kl}$, V^μ_{kl}, $\bar{V}^\mu_{kl}$, V'^μ_{kl}, $\bar{V}'^\mu_{kl}$, W^μ_{kl}, $\bar{W}^\mu_{kl}$, respectively. The original primitive quadrilaterals are assigned the same notation with $\mu = 0$.

4. Classification of nets of generic spherical quadrilaterals

Theorem 4.1. *Any generic spherical quadrilateral can be obtained by attaching spherical digons of types either D_{15} or D_{24} to some of the short (of order less than 6) sides of an irreducible quadrilateral Q_0. The* *types of digons attached to the sides of Q_0 are completely determined by its net.*

Proof. According to Lemma 2.11, any diagonal arc γ of a reducible quadrilateral Q must have its ends at adjacent corners p and q of Q. Thus γ partitions Q into a digon D and a quadrilateral Q'. Note that D must have integer angles at its corners. Otherwise, the common side of D and Q' would be mapped to a circle C' different from the circle C to which the common side of D and Q is mapped. This would imply that p and q are mapped to the intersection of the same two circles C and C', which is not possible since Q is a generic quadrilateral. Thus D should be of type either D_{15} (when the order k of γ is odd) or D_{24} (when k is even). □

Remark 4.2. An irreducible quadrilateral Q_0 in Theorem 4.1 may be not unique. If an irreducible quadrilateral Q_0 has a short side L of order greater than 3, and the quadrilateral Q obtained by attaching a disk D to L contains a disk D' other than D, with part of the boundary of D' being at a side of Q, then D' can be removed from Q to obtain an irreducible quadrilateral Q_1. All such non-uniqueness cases are listed below. All other cases can be obtained from these by attaching digons or pseudo-diagonals. Note that a quadrilateral Q obtained by attaching a disk D to a side of order less than 3 of an irreducible quadrilateral Q_0 does not contain a disk other than D having part of its boundary on the side of Q.

4.1. **Non-uniqueness cases in Remark 4.2.** (a) The quadrilateral $S_{11} \cup D_{15}$ (see Fig. 25a) contains three more disks. Removing each of them results in a quadrilateral S_{11}.

(b) The quadrilateral $X_{01} \cup D_{24}$ (see Fig. 25b) contains one more disk D_{24}. Removing it results in a quadrilateral X_{10} (see Fig. 25c). In the opposite direction, removing a disk from $X_{10} \cup D_{24}$ may result in X_{01}.

(c) The quadrilateral $X_{1k} \cup D_{24}$ with $k > 0$ contains one more disk D_{24} (shaded in Fig. 26a). Removing it results in a quadrilateral U_{k1}. In the opposite direction, removing a disk from $U_{k1} \cup D_{24}$ results in X_{1k}. Similarly, removing a disk from $X_{k1} \cup D_{24}$ with $k > 0$ (shaded in Fig. 26b) results in $\bar{U}_{k1}$, and removing a disk from $\bar{U}_{k1} \cup D_{24}$ results in X_{k1}. Note that X_{11} allows to attach a disk D_{24} to any of its two sides of order 4. Removing a disk from $X_{11} \cup D_{24}$ results either in U_{11} or in $\bar{U}_{11}$, depending on the side of X_{11} to which the disk is attached (see Figs. 26a and 26b). The cases $\bar{X}_{1k} \cup D_{24}$ and $\bar{X}_{k1} \cup D_{24}$ are obtained by reflection symmetry preserving the opposite corners of order 0.

(d) The quadrilateral $X'_{1k} \cup D_{15}$ contains a disk D_{24} (shaded in Fig. 26c). Removing it results in a quadrilateral $V_{k+1,1}$ (a quadrilateral $\bar{Z}_{10}$ when $k = 0$, see Fig. 26d). In the opposite direction, removing a disk D_{15} from $V_{k+1,1} \cup D_{24}$ (from $\bar{Z}_{10} \cup D_{24}$ if $k = 0$) results in X'_{1k}. Similarly, removing a disk D_{24} (shaded in Fig. 26e) from $X'_{k1} \cup D_{15}$ results in $\bar{V}_{k+1,1}$ (in $\bar{Z}_{01}$ if $k = 0$, see Fig. 26f), and removing a disk D_{15} from $\bar{V}_{k+1,1} \cup D_{24}$ (from $\bar{Z}_{01}$ if $k = 0$) results in X'_{k1}. Note that X'_{11} allows to attach a disk D_{15} to any of its two sides of order 5. Removing a disk D_{24} from $X'_{11} \cup D_{15}$ results either in V_{21} or in $\bar{V}_{21}$, depending on the side of X'_{11} to which the disk is attached (see Figs. 26c and 26e). The cases $\bar{X}'_{1k} \cup D_{24}$ and $\bar{X}'_{k1} \cup D_{24}$ are obtained by reflection symmetry preserving the opposite corners of order 0.

(e) The quadrilateral $Z_{1k} \cup D_{24}$ contains disk D_{15} (shaded area in Fig. 27a). Removing it results in a quadrilateral V_{k2} (in $\bar{X}'_{10}$ if $k = 0$, see Fig. 27b). In the opposite direction, removing a disk D_{24} from $V_{k2} \cup D_{15}$ (from $\bar{X}'_{10}$ if $k = 0$) results in Z_{1k}. Similarly, removing a disk D_{15} (shaded in Fig. 27c) from $Z_{k1} \cup D_{24}$ results in $\bar{V}_{k2}$ (in $\bar{X}'_{01}$ if $k = 0$, see Fig. 27d), and removing a disk D_{24} from $\bar{V}_{k2} \cup D_{15}$ (from $\bar{X}'_{01}$ if $k = 0$) results in Z_{1k}. Note that Z_{11} allows to attach a disk D_{24} to any of its two sides of order 4 (see Figs. 27a and 27c). Removing a disk D_{15} from $Z_{11} \cup D_{24}$ results either in V_{12} or in $\bar{V}_{12}$, depending on the side of Z_{11} to which the disk is attached. Also, V_{12} (resp., $\bar{V}_{12}$) has one side of order 4 and another one of order 5. Thus either D_{24} or D_{15} can be attached to a side of V_{12} (resp., $\bar{V}_{12}$). Removing either D_{15} or D_{24} would result in either X'_{11} or Z_{11}. The cases $\bar{Z}_{1k} \cup D_{24}$ and $\bar{Z}_{k1} \cup D_{24}$ are obtained by reflection symmetry preserving the opposite corners of order 0.

(f) The quadrilateral $Z'_{01} \cup D_{15}$ contains another disk D_{15}. Removing it results in a quadrilateral $\bar{Z}'_{01}$. In the opposite direction, removing a disk D_{15} from $\bar{Z}'_{01} \cup D_{15}$ results in Z'_{01}. The cases $Z'_{10} \cup D_{15}$ and $\bar{Z}'_{10} \cup D_{15}$ are obtained by reflection symmetry exchanging the opposite corners of order 0.

(g) The quadrilateral $Z'_{1k} \cup D_{15}$ for $k > 0$ contains another disk D_{15} (shaded in Fig. 27e). Removing it results in a quadrilateral $\bar{W}_{k+1,2}$. In the opposite direction, removing a disk D_{15} from $\bar{W}_{k+1,2} \cup D_{15}$ results in Z'_{1k}. Similarly, removing a disk D_{15} from $Z'_{k1} \cup D_{15}$ for $k > 0$ results in $W_{k+1,2}$, and removing a disk D_{15} from $W_{k+1,2} \cup D_{15}$ results in Z'_{k1}. Note that Z'_{11} allows to attach a disk D_{15} to any of its two sides of order 5. Removing a disk D_{15} from $Z'_{11} \cup D_{15}$ results either in W_{22} or in $\bar{W}_{22}$, depending on the side of Z'_{11} to which the disk is attached.

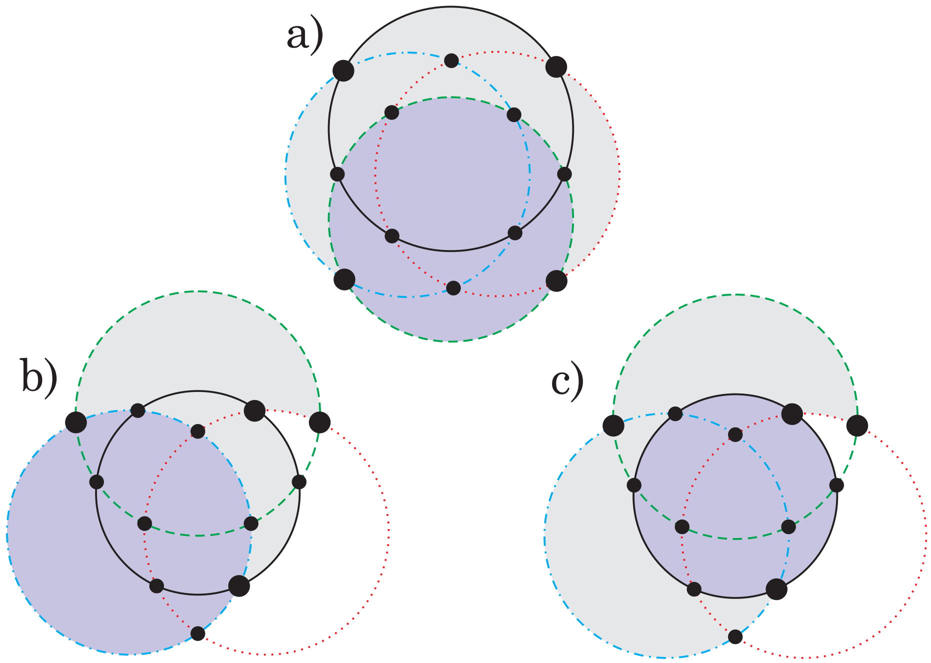

FIGURE 25. Non-uniqueness of an irreducible quadrilateral, cases (a) and (b).

The cases $\bar{Z}'_{1k} \cup D_{15}$ and $\bar{Z}'_{k1} \cup D_{15}$ are obtained by reflection symmetry preserving the opposite corners of order 0.

5. Chains of generic spherical quadrilaterals

In this section we show that generic quadrilaterals with a given net Γ and fixed four angles form an open segment I_Γ in the set of all generic quadrilaterals, parameterized by the angle between any two circles of the four-circle configurations corresponding to opposite sides of the quadrilaterals. In the limits at the ends of the interval I_Γ, a quadrilateral either conformally degenerates or converges to a non-generic spherical quadrilateral Q' (with the sides mapped to a non-generic four-circle configuration), or converges to a non-spherical quadrilateral after appropriate conformal transformations. When a non-generic spherical quadrilateral Q' is the limit of two families of generic quadrilaterals with the same angles and distinct nets Γ_- and Γ_+, we say that the two families of generic quadrilaterals belong to a chain of quadrilaterals, and their nets Γ_- and Γ_+ belong to a chain of nets (see Definition 5.13 below).

Remark 5.1. The chains of spherical quadrilaterals with at least one integer angle considered in [15] - [17] were completely determined by

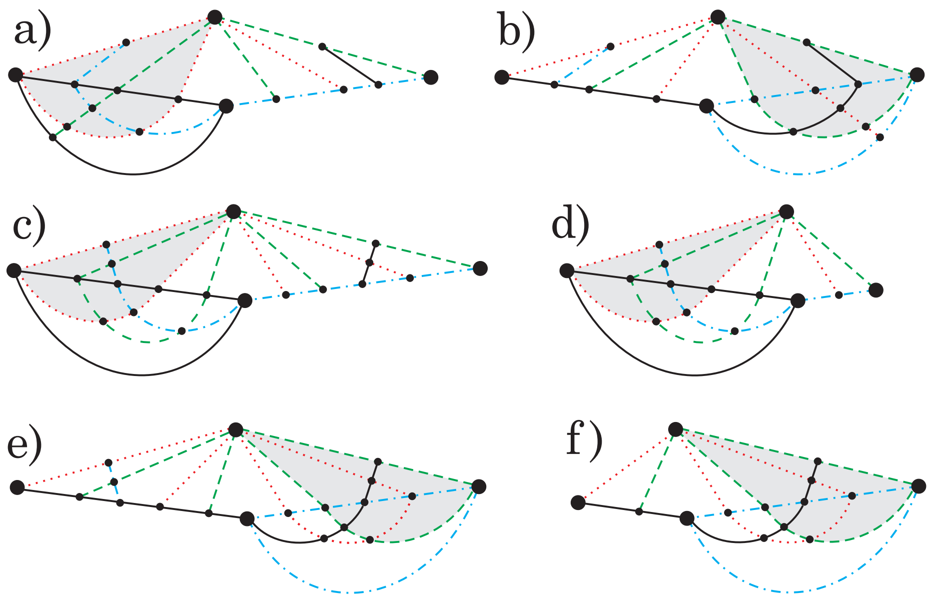

FIGURE 26. Non-uniqueness of an irreducible quadrilateral, cases (c) and (d).

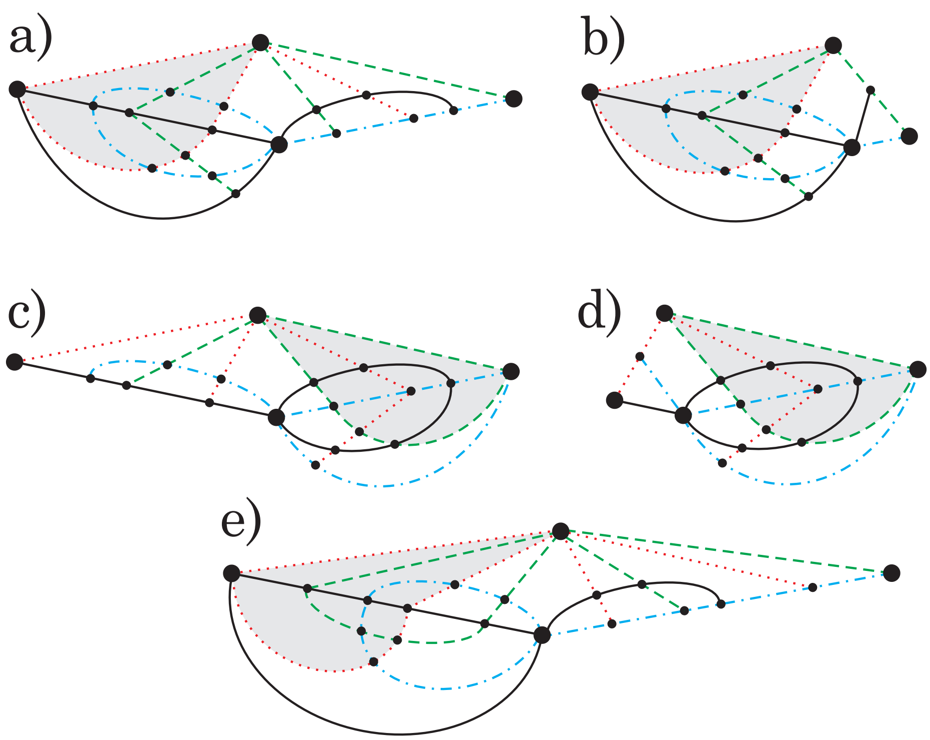

FIGURE 27. Non-uniqueness of an irreducible quadrilateral, cases (e) - (g).

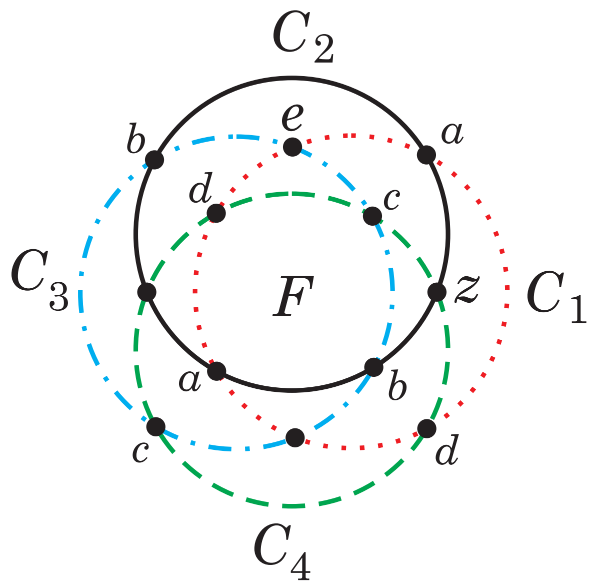

FIGURE 28. The face F of the four-circle configuration $\mathcal{P}$ with fixed angles (a, b, c, d).

the integer parts of their angles. For generic quadrilaterals, the chains depend also on the fractional parts of their angles.

Consider a generic quadrilateral Q_0 with the net Γ_0 and the sides mapped to four circles of a partition $\mathcal{P}_0$ of the sphere. We claim that the set of all generic quadrilaterals Q with the same net Γ_0 and the same angles as Q_0, obtained from Q_0 by continuous deformation $\mathcal{P}$ of the partition $\mathcal{P}_0$, constitute an open segment, with the limit at each end corresponding either to a non-generic four-circle partition $\mathcal{P}'$ of the sphere with a triple intersection of circles, or to a three-circle partition. At this limit, the quadrilateral Q may conformally degenerate, its conformal modulus converging to either 0 or ∞ (see [16, Sections 14-15] and [17, Section 7]). If it does not conformally degenerate, it usually converges to a spherical quadrilateral Q' over a non-generic partition $\mathcal{P}'$ (see Remark 5.4 below for an exception). In this case, the deformation of Q can be extended through Q' to another segment of generic quadrilaterals, with a net Γ_1 different from Γ_0. A chain of quadrilaterals is a sequence of such extensions, starting and ending with degenerate quadrilaterals. The corresponding sequence of nets is called a chain of nets (see Definition 5.13 below).

Let $C_1, \dots, C_4$ be the circles of the partition P to which the sides of Q are mapped, indexed according to the order of the sides of Q. Since the angles of Q are fixed, the angles a, b, c, d between the circles of

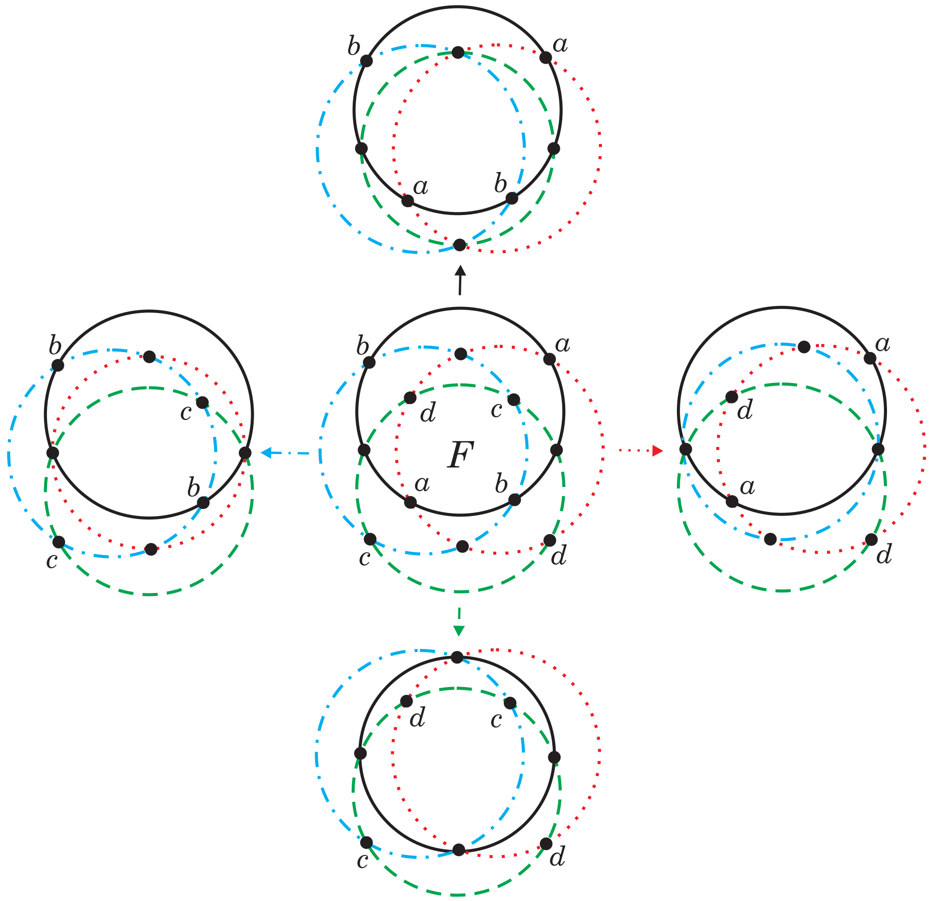

FIGURE 29. Degenerations of a family of four-circle configurations to triple intersections.

$\mathcal{P}$ are also fixed. Here a is the angle between C_1 and C_2, b is the angle between C_2 and C_3, c is the angle between C_3 and C_4, and d is the angle between C_4 and C_1 (see Fig. 28). The angle between two circles is defined up to its complement, but we can choose the values a, b, c, d uniquely by requiring that there is a quadrilateral face F of the partition $\mathcal{P}$ having exactly these angles. In fact, there are exactly two such faces of $\mathcal{P}$, having the same angles a, b, c, d (with opposite cyclic orders). The cyclic order of the circles at the sides of these faces is either the same or opposite to the cyclic order of the circles $C_1, \ldots, C_4$ to which the sides of Q are mapped.

Proposition 5.2. *Let $\mathcal{P}$ be a partition of the sphere defined by a generic configuration of four great circles, and let F be a quadrilateral face of $\mathcal{P}$ with the angles a, b, c, d. Then the following inequalities are*

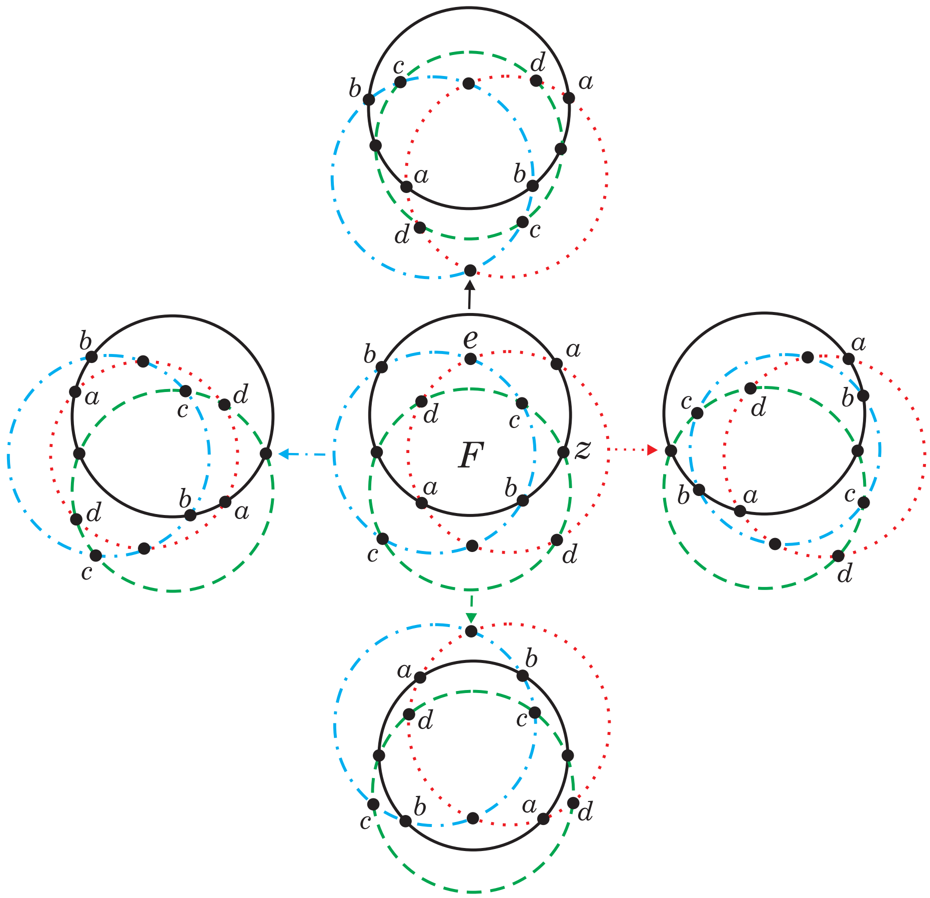

FIGURE 30. Transformations of a family of four-circle configurations beyond triple intersections.

satisfied:

$$0 < a, b, c, d < 1, \quad 0 < a + b + c + d - 2 < 2\min(a, b, c, d). \tag{1}$$

The subset of the unit cube in $\mathbf{R}^4$ *defined by these inequalities is an open convex pyramid* Π *with the vertex* $(1,1,1,1)$ *and the base an octahedron* P *in the plane* $a+b+c+d=2$ *having vertices* $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,0)$, $(1,0,0,1)$, $(1,0,1,0)$, $(1,1,0,0)$.

Proof. By definition, (a, b, c, d) is a point of the unit cube in $\mathbf{R}^4$. Since the area $A = a + b + c + d - 2$ of F is positive, we have $a + b + c + d > 2$. Since F is an intersection of the four digons with the angles a, b, c, d and the areas $2a$, $2b$, $2c$, $2d$, respectively, we have $A < 2\min(a, b, c, d)$. Note that Π is defined by linear inequalities, thus it is a convex polytope with each facet on a plane defined by one of these inequalities. The octahedron P belongs to the plane $a+b+c+d = 2$,

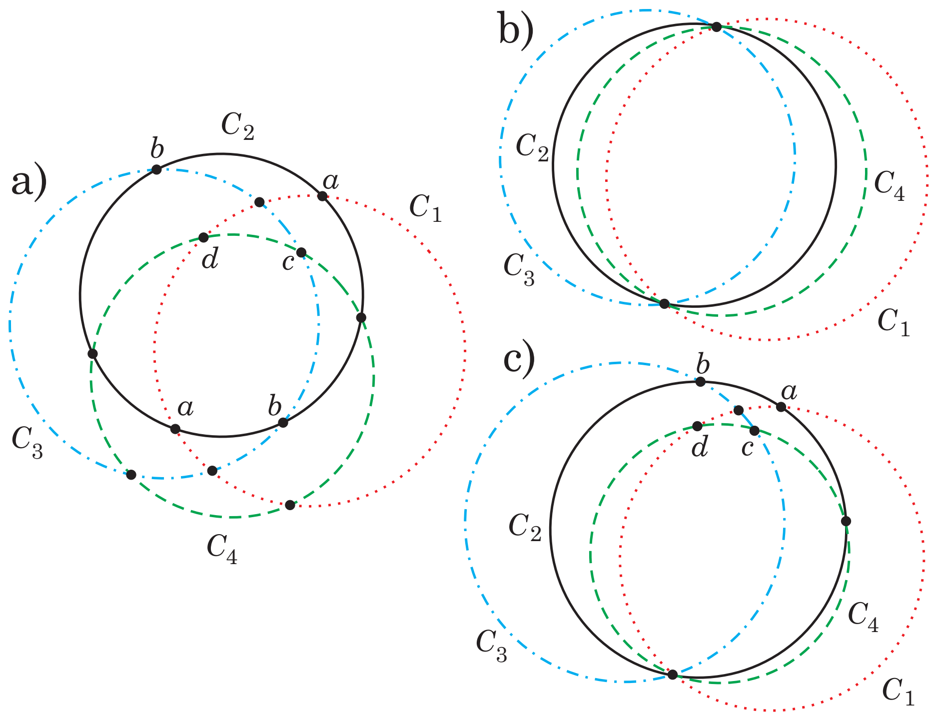

FIGURE 31. Degeneration of a family of four-circle configurations (a) to a configuration with quadruple intersections (b) and to a non-spherical four-circle configuration (c).

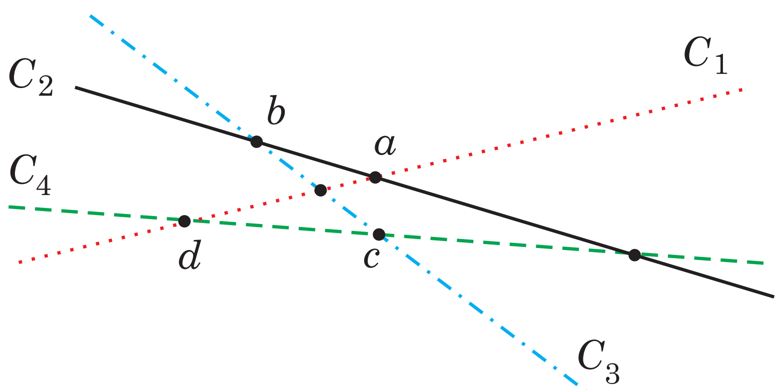

FIGURE 32. A four-line configuration conformally equivalent to the configuration in Fig. 31c.

it is the convex hull of the six vertices listed in Proposition 5.2, and each of its eight triangular facets belongs to a side of the unit cube. Each of the remaining facets of Π is a 3-simplex, the convex hull Δ

of the union of $(1, 1, 1, 1)$ and one of the facets δ of P. If one of the variables, say a, equals 1 on δ then Δ belongs to the plane $a = 1$. If $a = 0$ on δ then $a = \min(a, b, c, d)$ on Δ, and Δ belongs to the plane $a + b + c + d - 2 = 2a$. This proves that on each facet of the pyramid Π one of the inequalities in (1) becomes an equation. Since $a+b+c+d > 2$ in Π, and all other inequalities in (1) are satisfied at the center $(1/2, 1/2, 1/2, 1/2)$ of P, we conclude that the pyramid Π is indeed the set in $\mathbf{R}^4$ defined by the inequalities (1). This completes the proof. $\square$

There are four triangular faces of the four-circle configuration $\mathcal{P}$ in Fig. 28 adjacent to the face F. The areas of the bottom and top faces are $1 - a - b + e$ and $1 - c - d + e$, respectively. The areas of the left and right faces are $1 - a - d + z$ and $1 - b - c + z$, respectively. Here e is the angle between C_1 and C_3 and z is the angle between C_2 and C_4. When the configuration $\mathcal{P}$ is deformed, the areas of the top and bottom are either both decreasing or both increasing as e decreases or increases, with the same rate as e. It follows from the cosine theorem that the top and bottom sides of F are decreasing or increasing as e decreases or increases. Similarly, the areas of the left and right triangular faces adjacent to F are either both decreasing or both increasing as z decreases or increases, with the same rate as z, and the left and right sides of F are decreasing or increasing as z decreases or increases. Since the area of F is constant, e and z cannot be both increasing or both decreasing when $\mathcal{P}$ is deformed. This implies the following statement.

Proposition 5.3. *If $a+b > c+d$ (resp., $a+b < c+d$) then a generic four-circle configuration $\mathcal{P}$ can be deformed until the bottom (resp., top) triangular face adjacent to F is contracted to a point, so that the circles C_1, C_2, C_3 (resp., C_3, C_4, C_1) have a triple intersection, but cannot be deformed so that the top (resp., bottom) face is contracted to a point. Similarly, if $a + d > b + c$ (resp., $a + d < b + c$) then $\mathcal{P}$ can be deformed until the left (resp., right) triangular face adjacent to F is contracted to a point, so that the circles C_4, C_1, C_2 (resp., C_2, C_3, C_4) have a triple intersection, but cannot be deformed so that the right (resp., left) face is contracted to a point.*

Four possible degenerations of a family of four-circle configurations to non-generic configurations with triple intersections are shown in Fig. 29. The color (style) of each arrow between configurations in Fig. 29 indicates a circle that is not part of a triple intersection in the corresponding non-generic configuration. Note that, according to

Proposition 5.3, at most two of the four possible degenerations (one with a horizontal arrow and another one with a vertical arrow) can be realized for any given angles (a, b, c, d). If a generic four-circle configuration can be deformed to a non-generic configuration with a triple intersection, its deformation can be extended beyond the triple intersection to a combinatorially different generic four-circle configuration. Four possibilities for such generic configurations are shown in Fig. 30, on top, bottom, left and right of the original (central) configuration $\mathcal{P}$, depending on the four possible triple intersections to which it may degenerate. The color (style) of each arrow between configurations in Fig. 30 indicates a circle that is not part of a triple intersection in the corresponding non-generic configuration, as in Fig. 29.

Note that condition on the angles of the face F of $\mathcal{P}$ which determines whether it can be deformed through a triple intersection to a configuration $\mathcal{P}'$ (either top or bottom, left or right in Fig. 30) is exactly the inequality for the fixed angles of $\mathcal{P}'$ which guarantees that the quadrilateral face F' of $\mathcal{P}'$ with four fixed angles has positive area.

Remark 5.4. What happens if, e.g., $a + b = c + d$? Then configuration $\mathcal{P}$ (see Fig. 31a) can be deformed so that in the limit both top and bottom triangular faces adjacent to F are contracted to a point, and all four circles intersect at two opposite points in the limit (see Fig. 31b). However, combining this family of four-circle configurations with an appropriate family of linear-fractional transformations of the sphere, one can obtain a configuration with only one four-circle intersection point (see Fig. 31c). This limit configuration cannot be realized by great circles, and the family of four-circle configurations cannot be extended beyond the quadruple intersection to a four-circle family equivalent to a family of generic great circles with the net different from the net shown in Fig. 31a. Note that four-circle configuration in Fig. 31c is conformally equivalent to a four-line configuration shown in Fig. 32. A (circular but not spherical) quadrilateral mapped to such a configuration is called a "singular flat quadrilateral" (see [10, 11, 14]). Here "singular" refers to the possibility of the developing map of such quadrilateral to have simple poles inside the quadrilateral or on its sides (but not at the corners).

In Fig. 31 we assume that $a < d$, thus the angle between the circles C_2 and C_4 in Fig. 31b and Fig. 31c is $d - a = b - c$. If, in addition, $a = d$ (thus $b = c$) then the circles C_2 and C_4 converge in the limit to a single circle passing through the intersection points of C_1 and C_3. Combining this family of four-circle configurations with an appropriate family of linear-fractional transformations of the sphere, one can obtain in the limit a configuration with two tangent circles C_2 and C_4, their tangency point being at an intersection of the circles C_1 and C_3 (see [10], Section 4).

5.1. **Relations between adjacent four-circle configurations in a chain.** Consider a generic four-circle configuration $\mathcal{P}$ (see Fig. 28) with the angles (a, b, c, d) of a quadrilateral face F. When $\mathcal{P}$ is deformed beyond a triple intersection to a generic configuration $\mathcal{P}'$ (see Fig. 30) keeping the angles (a, b, c, d) fixed, the face F of $\mathcal{P}$ is replaced with a quadrilateral face F' of $\mathcal{P}'$ with different fixed angles: the angles $(1-a, 1-b, c, d)$ in the top configuration in Fig. 30, the angles $(a, b, 1-c, 1-d)$ in the bottom configuration, the angles $(a, 1-b, 1-c, d)$ in the left configuration, and the angles $(1-a, b, c, 1-d)$ in the right configuration. The general rule for the transformation through a triple intersection is that two fixed angles of F which are at the vertices not passing through a triple intersection are replaced with their complementary angles of F'. If we apply the same rule to the four configurations in Fig. 30 that can be obtained from $\mathcal{P}$, we get more generic four-circle configurations. Eight distinct generic four-circle configurations obtained this way are shown in Figs. 33a - 33h. Four more configurations shown in the bottom row of Fig. 33 are equivalent to configurations in its top row: Fig. 33i is equivalent to Fig. 33c, Fig. 33j is equivalent to Fig. 33d, Fig. 33k is equivalent to Fig. 33a, Fig. 33l is equivalent to Fig. 33b. The original configuration $\mathcal{P}$ and its four transformations (see Fig.30) are shown in Figs. 33f, 33b, 33j, 33e, and 33g, respectively.

Any two configurations in Fig. 33 adjacent either vertically or horizontally can be obtained from each other by a deformation passing through a triple intersection when certain inequalities on (a, b, c, d) are satisfied. In fact, Fig. 33 should be considered as part of a double periodic square lattice with periods $(4, 0)$ and $(2, 2)$. For example, configuration in Fig. 33a (and Fig. 33k) has a quadrilateral face with the fixed angles $(a, 1-b, c, 1-d)$. It exists when $a + c > b + d$ and $a - b + c - d < 2\min(a, 1-b, c, 1-d)$. It can be obtained, when $a + c > b + d$, either from the configuration in Fig.33b, replacing the angles $1-a$ and d by their complementary angles, or from the configuration in Fig.33e, replacing the angles $1-c$ and d by their complementary angles.

Note that configurations in Fig. 33 twice removed either horizontally or vertically have all four fixed angles of a quadrilateral face complementary, reversing the second inequality in (1). Thus at most one of them may exist for any given values of the fixed angles.

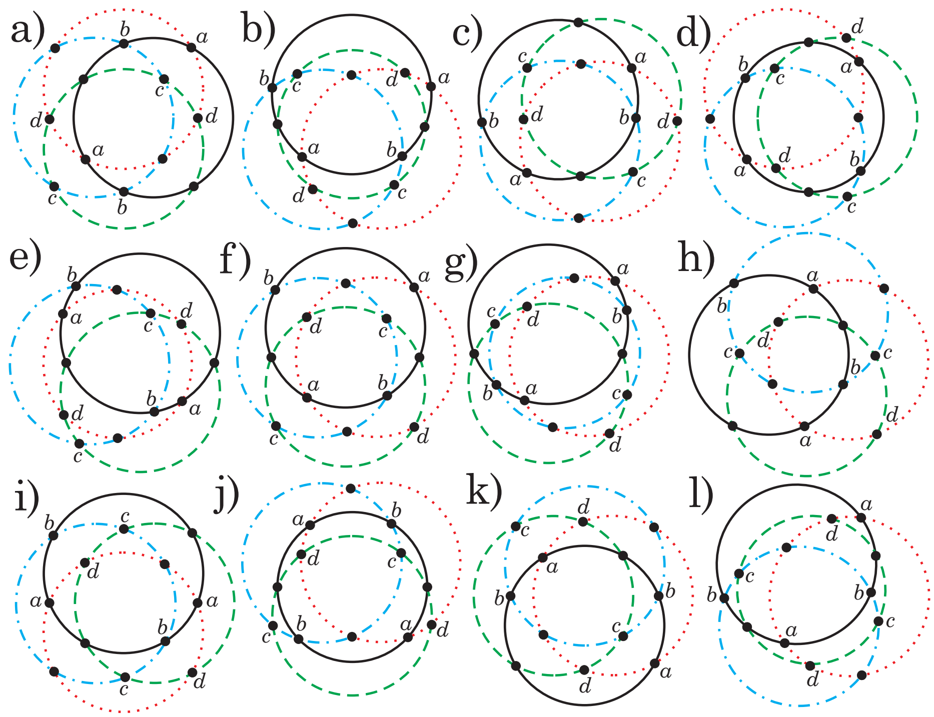

FIGURE 33. Generic "even" four-circle configurations related by deformation with fixed angles (a, b, c, d) through triple intersections.

Remark 5.5. All configurations in Fig. 33 are "even": each of them has a quadrilateral face with four fixed angles, an even number of them complementary to the angles (a, b, c, d). Replacing one of the angles by its complementary angle we get eight distinct "odd" configurations with the given angles (a, b, c, d). Any transformation through a triple intersection preserving the angles (a, b, c, d) is possible either between two even configurations or between two odd ones.

Let us return to configuration $\mathcal{P}$ in Fig. 33f and assume that

$$a+b<c+d,\ \ a+d<b+c,\ \ a+c<b+d. \tag{2}$$

Then configuration $\mathcal{P}$ can be transformed either upward, to configuration in Fig. 33b, or to the right, to configuration in Fig. 33g. Configuration in Fig. 33b has a quadrilateral face with the angles $(1-a, 1-b, c, d)$. It can be transformed either downward, back to configuration in Fig. 33f, or to the left, to configuration in Fig. 33a, since $a+c<b+d$ implies $1-a+d>1-b+c$. Configuration in Fig. 33g has a quadrilateral face with the angles $(1-a, b, c, 1-d)$. It can be transformed either to the left, back to configuration in Fig. 33f, or downward, to configuration in Fig. 33k, since $a + c < b + d$ implies $1 - a + b > c + 1 - d$. Note that configuration in Fig. 33k is equivalent to configuration in Fig. 33a. Thus any sequence of transformations of a configuration satisfying inequalities (2) is possible within the "ladder" pattern in Fig. 33.

Consider now configuration $\mathcal{P}$ in Fig. 33f, with the angles satisfying

$$a + b < c + d,\ a + d < b + c,\ a + c > b + d. \tag{3}$$

Then configuration $\mathcal{P}$ can be transformed, as before, either upward, to configuration in Fig. 33b, or to the right, to configuration in Fig. 33g, since this depends only on the first two inequalities in (2) and (3). The configuration in Fig. 33b can be transformed either downward, back to configuration in Fig. 33f, or to the right, to configuration in Fig. 33c, since $a+c > b+d$ implies $1-a+d < 1-b+c$. Similarly, configuration in Fig. 33g can be transformed either to the left, back to configuration in Fig. 33f, or upward, to configuration in Fig. 33c, since $a + c > b + d$ implies $1 - a + b < c + 1 - d$. Thus any sequence of transformations of a configuration satisfying inequalities (3) is possible within the "box" pattern in Fig. 33. This can be summarized as follows.

Proposition 5.6. *Let (a, b, c, d) be the fixed angles of a quadrilateral face F of a generic four-circle configuration $\mathcal{P}$, such that the "opposite" pairs (a, c) and (b, d) are at the opposite vertices of F. According to Proposition 5.3,exactly one of the angles remains unchanged when $\mathcal{P}$ is deformed through both permitted triple intersections. The "ladder" pattern, as in (2), appears when the sum of two angles in the opposite pair containing the "twice unchanged" angle is smaller than the sum of two angles in the other opposite pair. Otherwise, the "box" pattern, as in (3), appears.*

Remark 5.7. The conditions on the angles (a, b, c, d) related to the possibility of transforming generic configurations through triple intersections can be described as subsets of the unit cube in $\mathbf{R}^4$. According to Proposition 5.2, the point (a, b, c, d) for the partition $\mathcal{P}$ belongs to the open pyramid Π with the vertex $(1, 1, 1, 1)$ and the base an octahedron with vertices $(0, 0, 1, 1)$, $(0, 1, 0, 1)$, $(0, 1, 1, 0)$, $(1, 0, 0, 1)$, $(1, 0, 1, 0)$, $(1, 1, 0, 0)$. Adding the first inequality in (2) or (3) cuts this pyramid in half: the inequality $a + b < c + d$ removes the vertex $(1, 1, 0, 0)$ from the octahedron, leaving the convex hull of the remaining vertices of Π. Note that the inequalities $a+b+c+d-2 < 2c$ and $a+b+c+d-d < 2d$ in (1) are now automatically satisfied, so the last inequality in (1) can be replaced with $a + b + c + d - 2 < 2 \min(a, b)$. Adding the second inequality in (2) or (3) leaves a quarter of Π, which is a 4-simplex: the inequality $a+d<b+c$ removes the vertex $(1,0,0,1)$ from the octahedron, reducing Π to the convex hull of its remaining 5 vertices. The last inequality in (1) can be now replaced with $a+b+c+d-2<2a$, i.e., $b+c+d<2+a$. Adding the third inequality in (2) or (3) cuts that simplex in half, removing one more vertex of the octahedron (either $(1,0,1,0)$ for the ladder pattern or $(0,1,0,1)$ for the box pattern) and leaving the convex hull of the 4 remaining vertices of Π and the center $(1/2,1/2,1/2,1/2)$ of the octahedron. Note that in the box case the last inequality in (1) is automatically satisfied.

The fractional parts $(\alpha,\beta,\gamma,\delta)$ of the angles of a spherical quadrilateral Q are either some of the fixed angles (a,b,c,d) of a quadrilateral face of its four-circle configuration or some of their complementary angles $(1-a,\ 1-b,\ 1-c,\ 1-d)$. The choice between each angle and its complement is determined by the net of Q (see Propositions 5.9 and 5.10 below). For the basic primitive quadrilaterals (see Fig. 13) the fractional parts $(\alpha,\beta,\gamma,\delta)$ of the angles are

$$(4)\quad (a,b,c,d) \text{ for } P_0,\ (1-a,b,1-c,1-d) \text{ for } X'_{00},$$
$$(1-a,1-b,1-c,d) \text{ for } \bar{X}'_{00},\ (a,1-b,c,1-d) \text{ for } Z'_{00}.$$

Note that the quadrilateral faces with the fixed angles (a,b,c,d) of the four-circle configurations corresponding to the quadrilaterals X'_{00} and $\bar{X}'_{00}$ are outside their nets.

Attaching a triangle T_k with an integer angle to the side of a quadrilateral does not change the fractional parts of its angles when k is even, and replaces the fractional part of one of its angles with its complement when k is odd. According to Theorem 3.1, every primitive quadrilateral can be obtained by attaching one or two triangles, each with an integer angle, to the sides of one of the basic quadrilaterals. Corollary 3.7 states that each irreducible quadrilateral can be obtained from a primitive one by inserting a quadrilateral P_μ with two of its angles of order 2μ and the other two angles of order 0. Theorem 4.1 states that any generic quadrilateral can be obtained from an irreducible one by attaching some digons, with two equal integer angles each, to its sides. Note that the sum Σ of the integer parts of the angles of a quadrilateral is increased by k when a triangle T_k is attached. Note also that $\Sigma=0$ for P_0, $\Sigma=1$ for X'_{00} and $\bar{X}'_{00}$, $\Sigma=2$ for Z'_{00}. Thus relations (4) imply the following relation between the angles of a generic spherical quadrilateral Q and the fixed angles of a quadrilateral face of its underlying partition $\mathcal{P}$ of the sphere.

Proposition 5.8. *Let Q be a generic spherical quadrilateral with the sum Σ of the integer parts of its angles, and the corresponding partition*

$\mathcal{P}$ with a quadrilateral face having angles (a,b,c,d), each of them being either a fractional part of the angle of Q or its complementary angle. Then the number of the complements among (a,b,c,d) has the same parity as Σ.

Relations between the fractional parts $(\alpha,\beta,\gamma,\delta)$ of the angles of a primitive quadrilateral Q from the list in Section 3 and the fixed angles (a,b,c,d) of a quadrilateral face F of its underlying partition $\mathcal{P}$ are presented in the following two Propositions, separate for even and odd values of the sum Σ of the integer parts of the angles of Q.

Proposition 5.9. *Let Q be a primitive spherical quadrilateral with the even sum of the integer parts of its angles. Let $\mathcal{P}$ be the corresponding partition of the sphere having a quadrilateral face with the fixed angles (a,b,c,d) which are either fractional parts $(\alpha,\beta,\gamma,\delta)$ of the angles of Q or their complements. The following list describes the angles (a,b,c,d) in terms of the angles $(\alpha,\beta,\gamma,\delta)$, depending on the net of Q:*

$(\alpha,\beta,\gamma,\delta)$ for P_0, X_{kl} with k and l even, $\bar{X}_{kl}$ with k and l even, R_{kl} with k and l even, $\bar{R}_{kl}$ with k and l even, U_{kl} with k and l even, $\bar{U}_{kl}$ with k and l even;

$(1-\alpha,1-\beta,\gamma,\delta)$ for R_{kl} with k and l odd, $\bar{X}'_{kl}$ with k even and l odd, Z_{kl} with k even and l odd, V_{kl} with k and l even, $\bar{V}_{kl}$ with k and l odd;

$(1-\alpha,\beta,1-\gamma,\delta)$ for X_{kl} with k and l odd, $\bar{X}_{kl}$ with k and l odd, U_{kl} with k and l odd, $\bar{U}_{kl}$ with k and l odd;

$(1-\alpha,\beta,\gamma,1-\delta)$ for $\bar{R}_{kl}$ with k and L odd, X'_{kl} with k even and l odd, $\bar{Z}_{kl}$ with k even and l odd, V'_{kl} for k and l odd, $\bar{V}'_{kl}$ for k and l even;

$(\alpha,1-\beta,1-\gamma,\delta)$ for $\bar{X}'_{kl}$ with k odd and l even, Z_{kl} with k odd and l even, $\bar{S}_{kl}$ with k and l even, V_{kl} with k and l odd, $\bar{V}_{kl}$ with k and l even;

$(\alpha,1-\beta,\gamma,1-\delta)$ for Z'_{kl} with k and l even, $\bar{Z}'_{kl}$ with k and l even, W_{kl} with k and l odd, $\bar{W}_{kl}$ with k and l odd;

$(\alpha,\beta,1-\gamma,1-\delta)$ for S_{kl} with k and l even, X'_{kl} with k odd and l even, $\bar{Z}_{kl}$ with k odd and l even, V'_{kl} for k and l even, $\bar{V}'_{kl}$ for k and l odd;

$(1-\alpha,1-\beta,1-\gamma,1-\delta)$ for S_{kl} with k and l odd, $\bar{S}_{kl}$ with k and l odd, Z'_{kl} with k and l odd, $\bar{Z}'_{kl}$ with k and l odd, W_{kl} with k and l even, $\bar{W}_{kl}$ with k and l even.

Proposition 5.10. *Let Q be a primitive spherical quadrilateral with the odd sum of the integer parts of its angles. Let $\mathcal{P}$ be the corresponding partition of the sphere having a quadrilateral face with the fixed* *angles (a,b,c,d) which are either fractional parts $(\alpha,\beta,\gamma,\delta)$ of the angles of Q or their complements. The following list describes the angles (a,b,c,d) in terms of the angles $(\alpha,\beta,\gamma,\delta)$, depending on the net of Q.*

$(1-\alpha,\beta,\gamma,\delta)$ for X_{kl} with k odd and l even, $\bar{X}_{kl}$ with k odd and l even, R_{kl} with k odd and l even, $\bar{R}_{kl}$ with k odd and l even, U_{kl} with k odd and l even, $\bar{U}_{kl}$ with k even and l odd;

$(\alpha,1-\beta,\gamma,\delta)$ for R_{kl} with k even and l odd, $\bar{X}'_{kl}$ with k and l odd, Z_{kl} with k and l odd, V_{kl} with k odd and l even, $\bar{V}_{kl}$ with k odd and l even;

$(\alpha,\beta,1-\gamma,\delta)$ for X_{kl} with k even and l odd, $\bar{X}_{kl}$ with k even and l odd, U_{kl} with k even and l odd, $\bar{U}_{kl}$ with k odd and l even;

$(\alpha,\beta,\gamma,1-\delta)$ for $\bar{R}_{kl}$ with k even and l odd, X'_{kl} with k and l odd, $\bar{Z}_{kl}$ with k and l odd, V'_{kl} for k even and l odd, $\bar{V}'_{kl}$ with k even and l odd;

$(\alpha,1-\beta,1-\gamma,1-\delta)$ for S_{kl} with k even and l odd, $\bar{S}_{kl}$ with k even and l odd, Z'_{kl} with k even and l odd, $\bar{Z}'_{kl}$ with k odd and l even, W_{kl} with k odd and l even, $\bar{W}_{kl}$ with k even and l odd;

$(1-\alpha,\beta,1-\gamma,1-\delta)$ for X'_{kl} with k and l even, $\bar{Z}_{kl}$ with k and l even, V_{kl} with k and l even, $\bar{V}_{kl}$ with k and l even, S_{kl} with k odd and l even, V'_{kl} for k odd and l even, $\bar{V}'_{kl}$ with k odd and l even;

$(1-\alpha,1-\beta,\gamma,1-\delta)$ for Z'_{kl} with k odd and l even, $\bar{Z}'_{kl}$ with k even and l odd, W_{kl} with k even and l odd, $\bar{W}_{kl}$ with k odd and l even;

$(1-\alpha,1-\beta,1-\gamma,\delta)$ for $\bar{X}'_{kl}$ with k and l even, Z_{kl} with k and l even, $\bar{S}_{kl}$ with k odd and l even, V_{kl} with k even and l odd, $\bar{V}_{kl}$ with k even and l odd.

Remark 5.11. Note that adding pseudo-diagonals or attaching disks to the sides of a quadrilateral does not change relations between the angles $(\alpha,\beta,\gamma,\delta)$ and (a,b,c,d) in Propositions 5.9 and 5.10, so these relations hold for any generic spherical quadrilateral.

Example 5.12. If the angles of X_{kl} in Fig. 14 are $(\alpha,\ \beta,\ \gamma,\ k+l+\delta)$, then the angles (a,b,c,d) of the shaded quadrilateral face of its net are $(\alpha,\ \beta,\ 1-\gamma,\ \delta)$ for X_{01}, $(1-\alpha,\ \beta,\ 1-\gamma,\ \delta)$ for X_{11}, $(1-\alpha,\ \beta,\ \gamma,\ \delta)$ for X_{12}.

Definition 5.13. A *chain* of quadrilaterals of length $n\ge 0$ is a maximal sequence of segments I_j in the space of generic spherical quadrilaterals with given angles, corresponding to distinct nets $\Gamma_0,\dots,\Gamma_n$, so that the segments I_j and I_{j+1}, for $j=0,\dots,n-1$, have a common non-generic non-degenerate quadrilateral Q_j in the limit at their ends, its sides being mapped to a four-circle configuration with a triple intersection. A chain of length 0 is a segment I_0 of generic spherical quadrilaterals with the net Γ_0 such that the limits of quadrilaterals at both ends of I_0 degenerate.

Example 5.14. In Fig. 34a the shaded quadrilateral Q_0 with the net X_{01} and angles $(\alpha, \beta, \gamma, 1+\delta)$ is shown together with the configuration $\mathcal{P}_0$ to which its sides are mapped. Note that $\mathcal{P}_0$ has a quadrilateral face with the angles $(a, b, c, d) = (\alpha, \beta, 1-\gamma, \delta)$. According to (1), the angles of Q_0 satisfy the inequalities

$$0 < \alpha + \beta - \gamma + \delta - 1 < 2\min(\alpha, \beta, 1-\gamma, \delta). \tag{5}$$

Proposition 5.3 describes the conditions on (a, b, c, d) and $(\alpha, \beta, \gamma, \delta)$ that allow one to transform $\mathcal{P}_0$ to one of the four four-circle configurations with triple intersections. It is easy to check that only one of these four transformations results in a deformation of Q_0 that is non-degenerate in the limit: if $\alpha + \delta < 1 - \gamma + \beta$ then $\mathcal{P}_0$ can be deformed to a configuration with a triple intersection shown in Fig. 34b, and Q_0 to the quadrilateral shaded in Fig. 34b. Passing through the triple intersection, we get a generic configuration $\mathcal{P}_1$ shown in Fig. 34c, and a quadrilateral Q_1 (shaded in Fig. 34c) with the same angles as Q_0. The net of Q_1 is X'_{00}. Configuration $\mathcal{P}_1$ has a quadrilateral face with the angles $(1-\alpha, \beta, 1-\gamma, 1-\delta)$. If $1 + \alpha < \beta + \gamma + \delta$ then $\mathcal{P}_1$ can be transformed to a configuration with a triple intersection shown in Fig. 34d. The quadrilateral Q_1 shaded in Fig. 34d is non-degenerate. Passing through the triple intersection we get a generic configuration $\mathcal{P}_2$ shown in Fig. 34e, and a quadrilateral Q_2 (shaded in Fig. 34e) with the same angles as Q_0. The net of Q_2 is X_{10}, and any deformation of $\mathcal{P}_2$ to a configuration with a triple intersection, other than that shown in Fig. 34, results in a degenerate quadrilateral. Thus the chain of nets for the angles satisfying (5) and

$$\alpha + \gamma + \delta < 1 + \beta, \quad \beta + \gamma + \delta > 1 + \alpha \tag{6}$$

has length 2. Remark 5.7 implies that the last inequality in (5) can be replaced by $\beta - \gamma + \delta < 1 + \alpha$ when the inequalities in (6) are satisfied. If the first inequality in (6) is violated then the quadrilateral Q_1 with the net X'_{00} does not exist, and the chain breaks down into two chains of length 0 consisting of the quadrilaterals X_{01} and X_{10}. If the second inequality in (6) is violated then our chain of length 2 reduces to a chain $\{X_{01}, X'_{00}\}$ of length 1. If both the first and the second inequalities are violated, the chain of length 2 reduces to a chain of length 0 consisting of a single net X_{01}.

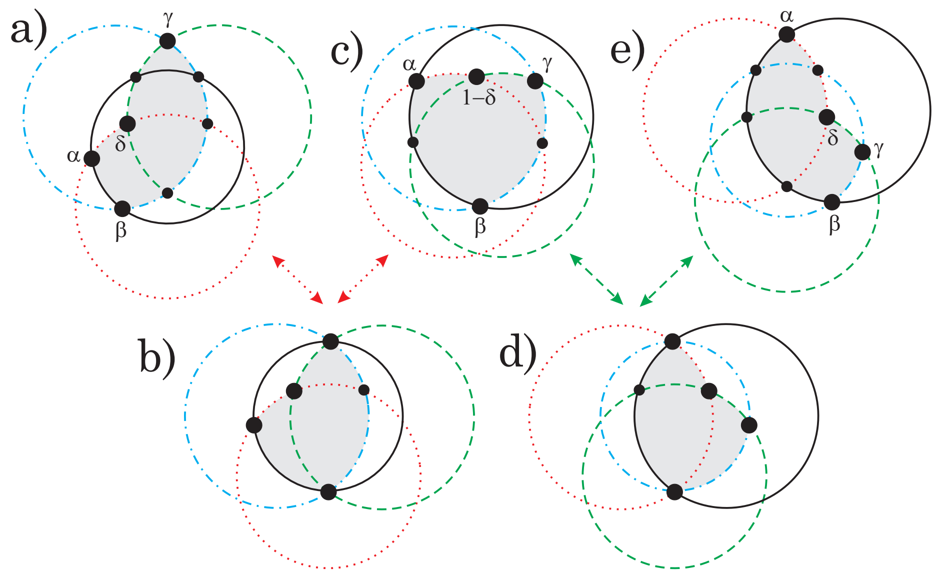

FIGURE 34. The chain of quadrilaterals X_{01}, X'_{00}, X_{10}.

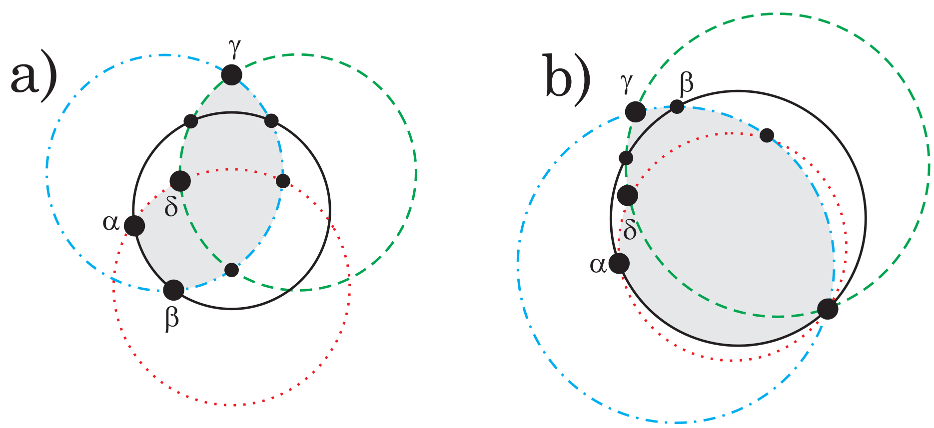

FIGURE 35. Limit of the quadrilateral X_{01} when $\alpha+\delta = \beta+1-\gamma$.

Example 5.15. If $\alpha+\gamma+\delta = 1+\beta$ in (6) then the configuration $\mathcal{P}_0$ of the quadrilateral Q_0 with the net X_{01} in Fig. 34a (shown also in Fig. 35a) can be deformed (see Remark 5.4) so that in the limit the four circles intersect at one point, and the limit quadrilateral (see Fig. 35c), with the same angles as Q_0, is not degenerate. Note that the limit quadrilateral is not spherical, and the family of quadrilaterals obtained by deformation of $\mathcal{P}_0$ cannot be extended beyond the limit configuration with quadruple intersection as a spherical configuration.

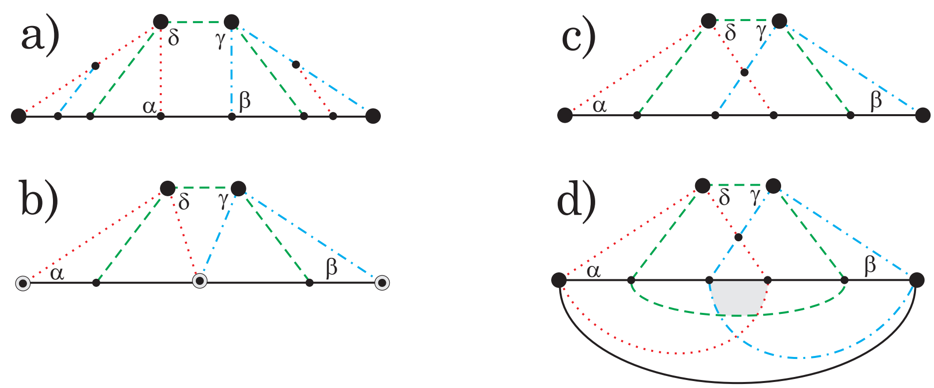

FIGURE 36. The chain of quadrilaterals R_{11} and S_{11}.

Example 5.16. In Fig. 36a, a quadrilateral Q_0 with the net R_{11} (see Fig. 16) with angles α, β, $1+\gamma$, $1+\delta$ is shown. The quadrilateral face with fixed angles of the corresponding configuration $\mathcal{P}_0$ has the angles $(a,b,c,d)=(1-\alpha,1-\beta,\gamma,\delta)$ satisfying the inequalities

$$0<\gamma+\delta-\alpha-\beta<2\min(1-\alpha,1-\beta,\gamma,\delta). \tag{7}$$

Deforming $\mathcal{P}_0$ to a configuration with a triple intersection such that one of triangular faces of Q_0 adjacent to its top side is contracted to a point results in conformal degeneration of the quadrilateral with modulus tending to 0 in the limit. If $1-\alpha+1-\beta>\gamma+\delta$, i.e.,

$$\alpha+\beta+\gamma+\delta<2 \tag{8}$$

then $\mathcal{P}_0$ can be deformed to a configuration with a triple intersection such that the corresponding quadrilateral shown in Fig. 36b is not degenerate. Passing through the triple intersection we get a generic configuration $\mathcal{P}_1$ corresponding to the quadrilateral Q_1 shown in Fig. 36c. The net of Q_1 is S_{11} (see Fig. 16). Note that $\mathcal{P}_1$ has a quadrilateral face with the angles $(1-\alpha,1-\beta,1-\gamma,1-\delta)$ which is not part of the net of Q_1 but can be seen if a disk D_{51} is attached to the side of order 5 of Q_1 (shaded area in Fig. 36d). Thus R_{11} and S_{11} form a chain of length 1 when the inequalities (7) and (8) are satisfied. Note that, according to Remark 5.7, the last inequality in (7) can be replaced with $\gamma+\delta-\alpha-\beta<2\min(\gamma,\delta)$.

If $\alpha+\beta+\gamma+\delta>2$ then $\mathcal{P}_0$ could be deformed so that the top side of Q_0 is contracted to a point. This would result in conformal degeneration of Q_0 with the modulus tending to ∞. Thus the net R_{11} would constitute a chain of length 0 in that case. Note that a quadrilateral with the net S_{11} does not exist in this case. However, when $\alpha+\beta+\gamma+\delta>2$, in addition to a quadrilateral Q_0 with the

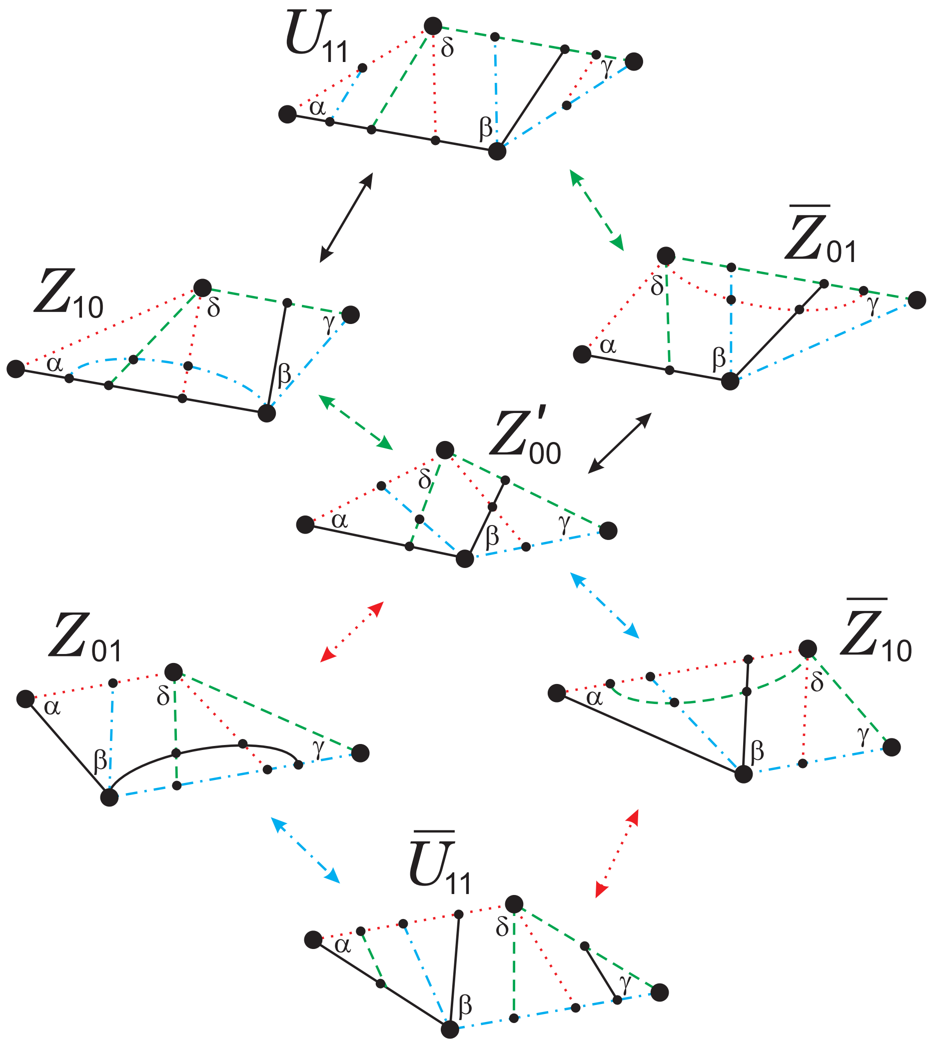

FIGURE 37. The chains of quadrilaterals associated with Z'_{00}.

net R_{11}, a quadrilateral Q_2 with the same angles as Q_0, such that its net is $P_0 \cup D_{15}$, with a disk attached to one side of a quadrilateral P_0, may exist. The chain of Q_2 consists of a single net and has length 0. Thus there may be either one chain or two chains of length 0 (either R_{11} or $P_0 \cup D_{15}$, or both) in this case.

Attaching disks to the sides of R_{11} and S_{11} does not affect the existence (or non-existence) of the chain of length 1 containing the two nets, except when a disk D_{51} is attached to the side of order 5 of S_{11} (see Fig. 36d). In this case, even when both (7) and (8) hold, the quadrilateral $S_{11} + D_{51}$ conformally degenerates (its bottom side is

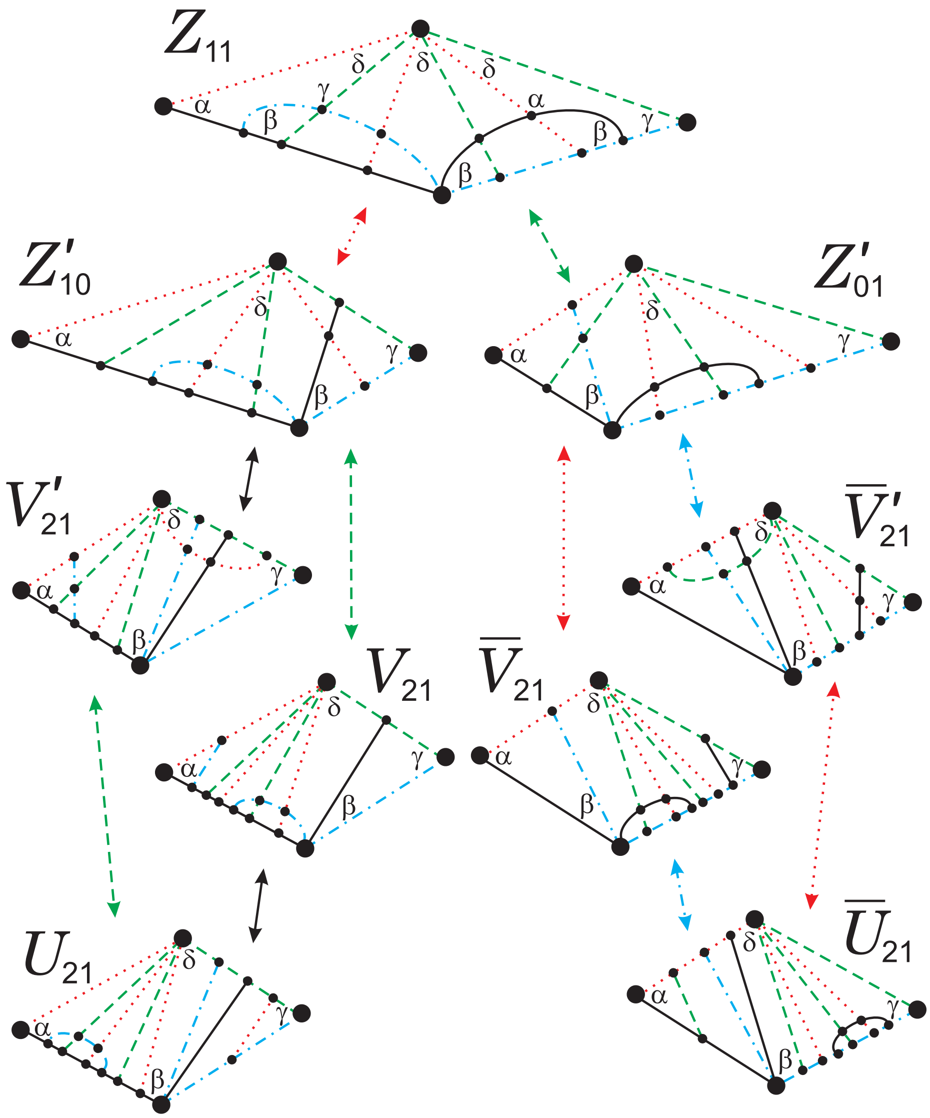

FIGURE 38. The chains of quadrilaterals associated with Z_{11}.

contracted to a point) when configuration P_1 is deformed to a configuration with a triple intersection corresponding to the quadrilateral in Fig. 36b. Accordingly, no disk can be attached to the side of order 7 of the quadrilateral R_{11}.

Example 5.17. The chains of quadrilaterals associated with the quadrilateral Z'_{00} are shown in Fig. 37. All quadrilaterals in Fig. 37 have the angles α, $1+\beta$, γ, $1+\delta$. The net Z'_{00} of the quadrilateral in the center of Fig. 37 has a quadrilateral face with the angles $(\alpha, 1-\beta, \gamma, 1-\delta)$

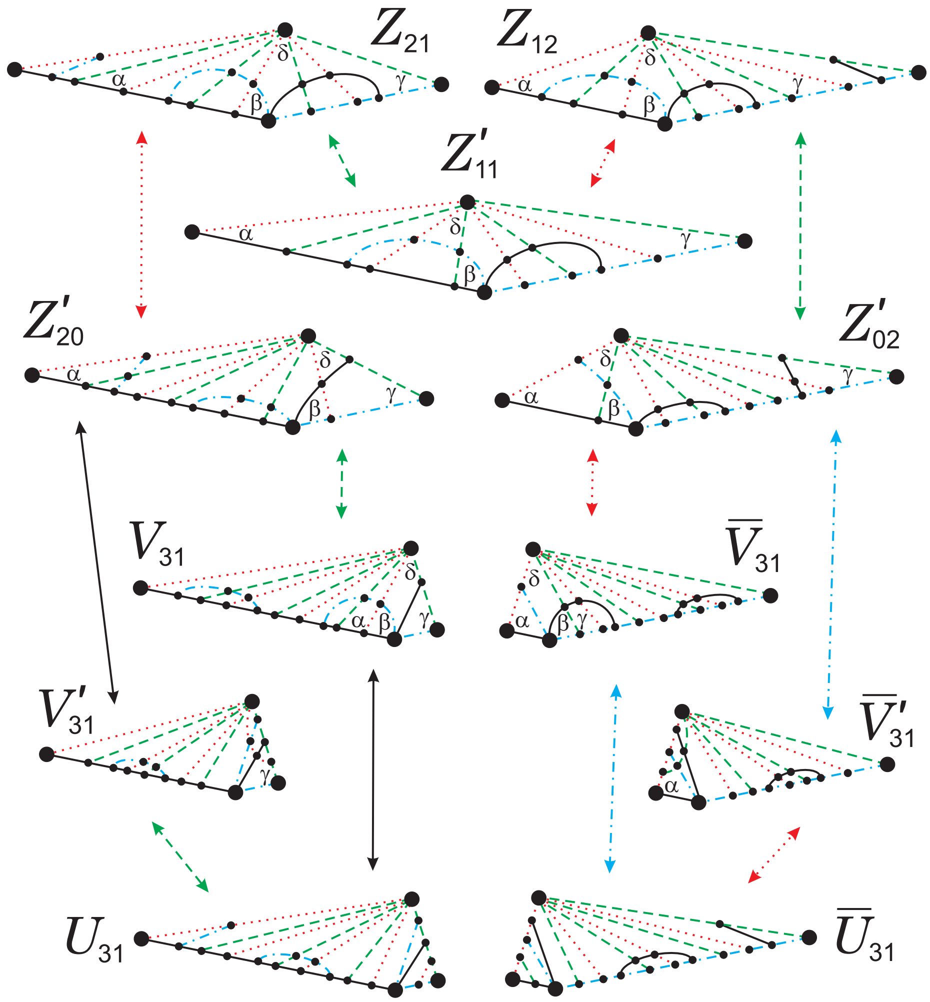

FIGURE 39. The chains of quadrilaterals associated with Z'_{11}.

satisfying

$$0 < \alpha - \beta + \gamma - \delta < 2\min(\alpha, 1-\beta, \gamma, 1-\delta) \tag{9}$$

If $\alpha+\delta > \beta+\gamma$ then the quadrilateral Z'_{00} can be deformed through a triple intersection to the quadrilateral Z_{10}. If $\alpha+\delta < \beta+\gamma$ then the quadrilateral Z'_{00} can be deformed through a triple intersection to the quadrilateral $\bar{Z}_{01}$. Note that these two deformations are not compatible.

If $\alpha+\beta < \gamma+\delta$ then the quadrilateral Z'_{00} can be deformed through a triple intersection to the quadrilateral Z_{01}. If $\alpha+\beta > \gamma+\delta$ then the quadrilateral Z'_{00} can be deformed through a triple intersection

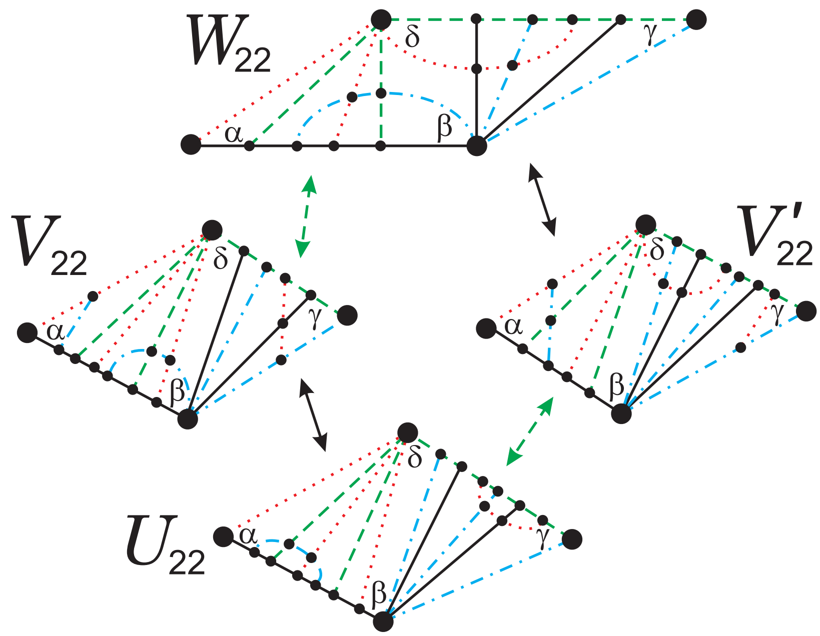

FIGURE 40. The chains of quadrilaterals associated with W_{22}.

to the quadrilateral $\bar{Z}_{10}$. Note that these two deformations are not compatible.

If instead of (9) we have

$$0 < \beta - \alpha + \delta - \gamma < 2\min(1-\alpha, \beta, 1-\gamma, \delta) \tag{10}$$

then each of the quadrilaterals Z_{10} and $\bar{Z}_{01}$ can be deformed through a triple intersection to the quadrilateral U_{11}, and each of the quadrilaterals Z_{01} and $\bar{Z}_{10}$ can be deformed through a triple intersection to the quadrilateral $\bar{U}_{11}$. Note that these deformations are not compatible with the deformations from Z_{10}, $\bar{Z}_{01}$, Z_{01} and $\bar{Z}_{10}$ to Z'_{00}.

Thus a chain of length 2 containing the quadrilateral Z'_{00} and either quadrilaterals Z_{10} and Z_{01} or the quadrilaterals $\bar{Z}_{01}$ and Z_{10} exists when inequalities (9) are satisfied and $\alpha+\delta \neq \beta+\gamma$, $\alpha+\beta \neq \gamma+\delta$. Two chains of length 1, one of them containing the quadrilateral U_{11} and one of the quadrilaterals Z_{10} and $\bar{Z}_{01}$, and another one containing the quadrilateral $\bar{U}_{11}$ and one of the quadrilaterals Z_{01} and $\bar{Z}_{10}$, exist when inequalities (10) are satisfied and $\alpha+\delta \neq \beta+\gamma$, $\alpha+\beta \neq \gamma+\delta$.

Example 5.18. Fig. 38 shows the chains of quadrilaterals associated with the quadrilateral Z_{11}. All quadrilaterals in Fig. 38 have the angles α, $1+\beta$, γ, $2+\delta$. The net of Z_{11} has a quadrilateral face with the angles $(\alpha, 1-\beta, \gamma, \delta)$ satisfying

$$1 < \alpha - \beta + \gamma + \delta > 1 + 2\min(\alpha, 1-\beta, \gamma, \delta). \tag{11}$$

If $\alpha+\beta+\delta < 1+\gamma$, the quadrilateral Z_{11} can be deformed to the quadrilateral Z'_{10}. If $\beta+\gamma+\delta < 1+\alpha$, the quadrilateral Z_{11} can be deformed to the quadrilateral Z'_{01}. If $\alpha+\beta+\gamma < 1+\delta$, the quadrilateral Z'_{10} can be deformed to the quadrilateral V_{21}, and the quadrilateral Z'_{01} can be deformed to the quadrilateral $\bar{V}_{21}$. If $\alpha+\beta+\gamma > 1+\delta$, the quadrilateral Z'_{10} can be deformed to the quadrilateral V'_{21}, and the quadrilateral Z'_{01} can be deformed to the quadrilateral $\bar{V}'_{21}$. Note that conditions on the angles of V'_{21} and $\bar{V}'_{21}$ are opposite to those on the angles of V_{21} and $\bar{V}_{21}$, thus only one of the two deformations is possible for Z'_{10} and Z'_{01}.

If $\alpha+\beta+\delta > 1+\gamma$, the quadrilaterals V_{21} and V'_{21} can be deformed to the quadrilateral U_{21}. If $\beta+\gamma+\delta > 1+\alpha$, the quadrilaterals $\bar{V}_{21}$ and $\bar{V}'_{21}$ can be deformed to the quadrilateral $\bar{U}_{21}$. Note that the condition on the angles of U_{21} and $\bar{U}_{21}$ are opposite to those on the angles of Z'_{10} and Z'_{01}, respectively.

Summing up, if in addition to (11) the inequalities

$$\alpha+\beta+\delta > 1+\gamma, \quad \beta+\gamma+\delta < 1+\alpha, \quad \alpha+\beta+\gamma < 1+\delta \tag{12}$$

are satisfied, then there is a chain $\{V_{21}, Z'_{10}, Z_{11}, Z'_{01}, \bar{V}_{21}\}$ of length 4. If the last inequality in (12) is replaced by the opposite inequality, V_{21} is replaced by V'_{21} and $\bar{V}_{21}$ is replaced by $\bar{V}'_{21}$, still having a chain of length 4. If the first inequality in (12) is replaced by the opposite inequality, Z'_{10} is replaced by U_{21}, and we get a chain $\{Z_{11}, Z'_{01}, \bar{V}_{21}\}$ of length 2 and a chain $\{V_{21}, U_{21}\}$ of length 1. If both the first and the last inequalities are replaced by their opposite inequalities, we get a chain $\{Z_{11}, Z'_{01}, \bar{V}'_{21}\}$ of length 2 and a chain $\{V'_{21}, U_{21}\}$ of length 1.

The chains described in this example remain the same if we add pseudo-diagonals and attach disks to the sides of our quadrilaterals, except if a disk D is attached to the side of order 5 of Z'_{10} (resp., Z'_{01}) the deformation to V_{21} (resp., to $\bar{V}_{21}$) becomes impossible even when (12) is satisfied, as the quadrilateral Z'_{10} (resp., Z'_{01}) degenerates at the triple intersection. Accordingly, no disks can be attached to the "long" sides of order 7 of V_{21}, $\bar{V}_{21}$, U_{21}, and $\bar{U}_{21}$.

Example 5.19. The chains of quadrilaterals associated with the quadrilateral Z'_{11} are shown in Fig. 39. The angles of all quadrilaterals in Fig. 39 are α, $1+\beta$, γ, $3+\delta$. The net Z'_{11} has a quadrilateral face with the angles $(1-\alpha, 1-\beta, 1-\gamma, 1-\delta)$ satisfying

$$2 > \alpha+\beta+\gamma+\delta > 2\max(\alpha, \beta, \gamma, \delta). \tag{13}$$

If $\alpha+\beta<\gamma+\delta$ then the quadrilateral Z'_{11} can be deformed through a triple intersection to the quadrilateral Z_{21}. If $\beta+\gamma<\alpha+\delta$ then the quadrilateral Z'_{11} can be deformed through a triple intersection to the quadrilateral Z_{12}. Each of the other two deformations of the four-circle configuration to a triple intersection results in degeneration of the quadrilateral Z'_{11}.

The net Z_{21} has a quadrilateral face with the angles $(1-\alpha,1-\beta,\gamma,\delta)$. If $\alpha+\gamma>\beta+\delta$ then the quadrilateral Z_{21} can be deformed through a triple intersection to the quadrilateral Z'_{20}. The net Z_{12} has a quadrilateral face with the angles $(\alpha,1-\beta,1-\gamma,\delta)$. If $\alpha+\gamma>\beta+\delta$ then the quadrilateral Z_{12} can be deformed through a triple intersection to the quadrilateral Z'_{02}.

The net Z'_{20} has a quadrilateral face with the angles $(\alpha,1-\beta,\gamma,1-\delta)$. If $\alpha+\delta>\beta+\gamma$ then the quadrilateral Z'_{20} can be deformed through a triple intersection to the quadrilateral V_{31}. If $\alpha+\delta<\beta+\gamma$ then the quadrilateral Z'_{20} can be deformed through a triple intersection to the quadrilateral V'_{31}. Note that these two deformations are not compatible.

The net Z'_{02} has a quadrilateral face with the angles $(\alpha,1-\beta,\gamma,1-\delta)$. If $\alpha+\beta<\gamma+\delta$ then the quadrilateral Z'_{02} can be deformed through a triple intersection to the quadrilateral $\bar V_{31}$. If $\alpha+\beta>\gamma+\delta$ then the quadrilateral Z'_{02} can be deformed through a triple intersection to the quadrilateral $\bar V'_{31}$. Note that these two deformations are not compatible.

The nets V_{31}, V'_{31}, $\bar V_{3,1}$ and $\bar V'_{31}$ have quadrilateral faces with the angles $(\alpha,1-\beta,1-\gamma,\delta)$, $(1-\alpha,\beta,\gamma,1-\delta)$, $(1-\alpha,1-\beta,\gamma,\delta)$ and $(\alpha,\beta,1-\gamma,1-\delta)$, respectively. Note that V_{31} and V'_{31} do not exist with the same values of the angles $\alpha,\beta,\gamma,\delta$. Similarly, $\bar V_{31}$ and $\bar V'_{31}$ do not exist with the same values of the angles $\alpha,\beta,\gamma,\delta$.

Each of the nets U_{31} and $\bar U_{31}$ has a quadrilateral face with the angles $(1-\alpha,\beta,1-\gamma,\delta)$ satisfying

$$0<\beta-\alpha+\delta-\gamma<2\min(1-\alpha,\beta,1-\gamma,\delta). \tag{14}$$

If $\alpha+\gamma<\beta+\delta$ then each of the quadrilaterals V_{31} and V'_{31} can be deformed through a triple intersection to the quadrilateral U_{31}, and each of the quadrilaterals $\bar V_{31}$ and $\bar V'_{31}$ can be deformed through a triple intersection to the quadrilateral $\bar U_{31}$. Note that these deformations are not compatible with the deformations between Z_{21} and Z'_{20}, and between Z_{12} and Z'_{02}.

Summing up, if in addition to (13) the inequalities

$$\alpha+\beta<\gamma+\delta,\quad \alpha+\delta>\beta+\gamma,\quad \alpha+\gamma>\beta+\delta \tag{15}$$

are satisfied, then there is a chain $\{V_{31}, Z'_{20}, Z_{21}, Z'_{11}, Z_{12}, Z'_{02}, \bar{V}_{31}\}$ of length 6. Note that, according to Remark 5.7, the last inequality in (13) can be removed when the inequalities (15) are satisfied.

If the first inequality in (13) is violated then Z'_{11} is removed from the chain, and we have two chains of length 2 each. If either the first or the second inequality in (15) is violated then either Z_{21} is removed and $\bar{V}_{31}$ is replaced by $\bar{V}'_{31}$, or Z_{12} is removed and V_{31} is replaced by V'_{31}. In both cases we have a chain of length 3 and a chain of length 1. If the third inequality in (15) is violated then Z'_{20} and Z'_{02} are removed, U_{31} and $\bar{U}_{31}$ are added, and we have a chain of length 2 and two chains of length 1.

If, in addition to (13), only the first inequality in (15) is satisfied, then there are three chains $\{Z'_{11}, Z_{21}\}$, $\{V'_{31}, U_{31}\}$ and $\{\bar{V}_{31}, \bar{U}_{31}\}$ of length 1.

If, in addition to (13), only the second inequality in (15) is satisfied, then there are three chains $\{Z'_{11}, Z_{12}\}$, $\{V_{31}, U_{31}\}$ and $\{\bar{V}'_{31}, \bar{U}_{31}\}$ of length 1.

If, in addition to (13), only the third inequality in (15) is satisfied, then there is a chain of length 0 consisting of a single net Z'_{11}, and two chains $\{Z'_{20}, V'_{31}\}$ and $\{Z'_{02}, \bar{V}'_{31}\}$ of length 1.

Example 5.20. Fig. 40 shows the chains of quadrilaterals associated with the quadrilateral W_{22} with the angles α, $2+\beta$, γ, $2+\delta$. The net of W_{22} has a quadrilateral face with the angles $(1-\alpha, 1-\beta, 1-\gamma, 1-\delta)$ satisfying

$$2 > \alpha+\beta+\gamma+\delta > 2\max(\alpha,\beta,\gamma,\delta). \tag{16}$$

If $\alpha+\beta<\gamma+\delta$, the quadrilateral W_{22} can be deformed through a triple intersection to a quadrilateral V_{22}. If $\alpha+\beta>\gamma+\delta$, the quadrilateral W_{22} can be deformed through a triple intersection to a quadrilateral V'_{22}. Note that these two transformations are incompatible. If $\alpha+\beta+\gamma+\delta>2$, each of the quadrilaterals V_{22} and V'_{22} can be deformed through a triple intersection to a quadrilateral U_{22}. Note that these transformations are incompatible with the transformations between V_{22} and W_{22} or between V'_{22} and W_{22}. In any case, we have one chain of length 1.

Proposition 5.21. *The chains containing generic quadrilaterals X_{kl} and X'_{pq} with even $n=k+l=p+q+1$ and fixed angles α, β, γ, $n+\delta$ may be of the following kind:*
(i) *The chain $X_{0,n}$, $X'_{0,n-1}$, $X_{1,n-1}, \dots, X_{n,0}$, of length $2n$, if*

$$\alpha+\beta+\gamma+\delta>2,\ \alpha+\delta<\beta+\gamma,\ \alpha+\gamma<\beta+\delta,\ \alpha+\beta>\gamma+\delta. \tag{17}$$

(ii) *If all inequalities except the first in (17) are satisfied, there are* $n/2$ *chains of length 2 obtained from the chain in (i) by removing all entries* $X_{2m,n-2m}$ *for* $m=0,\dots,n/2$.

(iii) *If all inequalities except the second in (17) are satisfied, there are* $n/2$ *chains of length 2 and one chain of length 0 obtained from the chain in (i) by removing all entries* $X'_{2m,n-2m-1}$ *for* $m=0,\dots,n/2-1$.

(iv) *If all inequalities except the third in (17) are satisfied, there are* $n/2+1$ *chains of length 1 obtained from the chain in (i) by removing all entries* $X_{2m+1,n-2m-1}$ *for* $m=0,\dots,n/2-1$.

(v) *If all inequalities except the fourth in (17) are satisfied, there are* $n/2$ *chains of length 2 and one chain of length 0 obtained from the chain in (i) by removing all entries* $X'_{2m+1,n-2m-2}$ *for* $m=0,\dots,n/2-1$.

(vi) *If only the first and second inequalities in (17) are satisfied, there are* $n/2$ *chains of length* 1 *and one chain of length 0 obtained from the chain in (i) by removing all entries* $X_{2m+1,n-2m-1}$ *and* $X'_{2m+1,n-2m-2}$ *for* $m=0,\dots,n/2-1$.

(vii) *If only the first and third inequalities in (17) are satisfied, there are* $n+1$ *chains of length* 0 *obtained from the chain in (i) by removing all entries* $X'_{m,n-m-1}$ *for* $m=0,\dots,n-1$.

(viii) *If only the first and fourth inequalities in (17) are satisfied, there are* $n/2$ *chains of length* 1 *and one chain of length 0 obtained from the chain in (i) by removing all entries* $X'_{2m,n-2m-1}$ *and* $X_{2m+1,n-2m-1}$ *for* $m=0,\dots,n/2-1$.

(ix) *If only the second and third inequalities in (17) are satisfied, there are* $n/2$ *chains of length* 1 *obtained from the chain in (i) by removing all entries* $X_{2m,n-2m}$ *for* $m=0,\dots,n/2$ *and* $X'_{2m+1,n-2m-2}$*, for* $m=0,\dots,n/2-1$.

(x) *If only the second and fourth inequalities in (17) are satisfied, there are* n *chains of length* 0 *obtained from the chain in (i) by removing all entries* $X_{m,n-m}$ *for* $m=0,\dots,n$.

(xi) *If only the third and fourth inequalities in (17) are satisfied, there are* $n/2$ *chains of length* 1 *obtained from the chain in (i) by removing all entries* $X_{2m,n-2m}$ *for* $m=0,\dots,n/2$ *and* $X'_{2m,n-2m-1}$ *for* $m=0,\dots,n/2-1$.

(xii) *If only the first inequality in (17) is satisfied, there are* $n/2+1$ *chains of length 0, each of them consisting of a single quadrilateral* $X_{2m,n-2m}$ *for* $m=0,\dots n$.

(xiii) *If only one inequality in (17), either second, third or fourth, is satisfied, there are* $n/2$ *chains of length 0, each of them consisting of a single quadrilateral.*

Proposition 5.22. *The chains containing generic quadrilaterals X_{kl} and X'_{pq} with odd $n=k+l=p+q+1\geq 3$ and fixed angles α, β, γ, $n+\delta$ may be of the following kind:*

(i) *The chain $X_{0,n}$, $X'_{0,n-1}$, $X_{1,n-1},\dots,X_{n,0}$, of length $2n$, if*

(18)

$$\alpha+\beta+\delta>1+\gamma,\ \alpha+\gamma+\delta<1+\beta,\ \beta+\gamma+\delta>1+\alpha,\ \alpha+\beta+\gamma>1+\delta.$$

(ii) *If all inequalities except the first in (18) are satisfied, there are $(n-1)/2$ chains of length 2 and one chain of length 1 obtained from the chain in (i) by removing all entries $X_{2m,n-2m}$ for $m=0,\dots,(n-1)/2$.*

(iii) *If all inequalities except the second in (18) are satisfied, there are $(n-1)/2$ chains of length 2 and two chains of length 0 obtained from the chain in (i) by removing all entries $X'_{2m,n-2m-1}$ for $m=0,\dots,(n-1)/2$.*

(iv) *If all inequalities except the third in (18) are satisfied, there are $(n-1)/2$ chains of length 2 and one chain of length 1 obtained from the chain in (i) by removing all entries $X_{2m+1,n-2m-1}$ for $m=0,\dots,(n-1)/2$.*

(v) *If all inequalities except the fourth in (18) are satisfied, there are $(n+1)/2$ chains of length 2 obtained from the chain in (i) by removing all entries $X'_{2m+1,n-2m-2}$ for $m=0,\dots,n/2-1$.*

(vi) *If only the first and second inequalities in (18) are satisfied, there are $(n+1)/2$ chains of length 1 obtained from the chain in (i) by removing all entries $X_{2m+1,n-2m}$ for $m=0,\dots,(n-1)/2$ and $X'_{2m+1,n-2m-2}$ for $m=0,\dots,(n-3)/2$.*

(vii) *If only the first and third inequalities in (18) are satisfied, there are $n+1$ chains of length 0 obtained from the chain in (i) by removing all entries $X'_{m,n-m-1}$ for $m=0,\dots,n-1$.*

(viii) *If only the first and fourth inequalities in (18) are satisfied, there are $(n-1)/2$ chains of length 1 and one chain of length 0 obtained from the chain in (i) by removing all entries $X_{2m+1,n-2m-1}$ for $m=0,\dots,(n-1)/2$ and $X'_{2m,n-2m-1}$ for $m=0,\dots,(n-1)/2$.*

(ix) *If only the second and third inequalities in (18) are satisfied, there are $(n+1)/2$ chains of length 1 obtained from the chain in (i) by removing all entries $X_{2m,n-2m}$ for $m=0,\dots,(n-1)/2$ and $X'_{2m+1,n-2m-2}$ for $m=0,\dots,(n-3)/2$.*

(x) *If only the second and fourth inequalities in (18) are satisfied, there are n chains of length 0 obtained from the chain in (i) by removing all entries $X_{m,n-m}$ for $m=0,\dots,n$.*

(xi) *If only the third and fourth inequalities in (18) are satisfied, there are $(n-1)/2$ chains of length 1 and one chain of length 0 obtained from the chain in (i) by removing all entries $X_{2m,n-2m}$ and $X'_{2m+1,n-2m-2}$*

for $m = 0, \dots, (n-1)/2$.
(xii) *If only one inequality, first, second or third in (18) is satisfied, there are* $(n+1)/2$ *chains of length 0, each of them consisting of a single quadrilateral.*
(xiii) *If only fourth inequality in (18) is satisfied, there are* $(n-1)/2$ *chains of length 0, each of them consisting of a single quadrilateral.*

5.2. **Lower bounds on the number of generic spherical quadrilaterals with given angles.** Given a generic spherical quadrilateral Q_0, we want to understand how many generic spherical quadrilaterals with the same angles and the same modulus as Q_0 may exist. The chains of quadrilaterals provide a lower bound for that number.

For each quadrilateral Q with the same angles and modulus as Q_0, consider the chain $\mathcal{C}$ of quadrilaterals containing Q. If the quadrilaterals at both ends of $\mathcal{C}$ conformally degenerate, the modulus of these quadrilaterals converges either to 0 at both ends, or to ∞ at both ends, or else to 0 at one end and to ∞ at another. We claim that the first two options are realized when $\mathcal{C}$ has odd length (contains an even number of nets) while the third possibility is realized when $\mathcal{C}$ has even length.

Let Q be a spherical quadrilaterals with the corners $a_0, \dots, a_3$ and the sides $[a_{j-1}, a_j]$ mapped to the circle C_j of a generic four-circle configuration $\mathcal{P}$. According to [16], Lemma 13.1 (see also [12], Lemma A4), when $\mathcal{P}$ degenerates to a four-circle configuration with a triple intersection, Q conformally degenerates with the modulus tending to 0 when intrinsic distance between its sides mapped to C_1 and C_3 tends to 0 while intrinsic distance between its other two sides does not. Accordingly, Q conformally degenerates with the modulus tending to ∞ when intrinsic distance between its sides mapped to C_2 and C_4 tends to 0 while intrinsic distance between its other two sides of does not.

When the configuration $\mathcal{P}$ degenerates to a four-circle configuration with a triple intersection that does not include a circle C_j, the quadrilateral Q conformally degenerates only when an arc γ of its net Γ is contracted to a point. This happens when γ is mapped to the complement of C_j, if the following three conditions are satisfied:

- γ has order 1 (either it is an interior arc without interior vertices or a boundary arc without lateral vertices),
- two ends of γ are on the opposite sides of Q, mapped to C_k and C_ℓ where k and ℓ have opposite parity of j,
- none of the two ends of γ is mapped to C_j.

Since the values j and j' for the two ends of I_Γ always have opposite parities, the arcs γ and γ' contracted to points in the limits at the two ends of any chain $\mathcal{C}$ have the ends on the same opposite sides of Q if $\mathcal{C}$ has odd length and on different opposite sides of Q if $\mathcal{C}$ has even length.

If a chain $\mathcal{C}$ has even length (contains an odd number of nets) and the quadrilaterals at both ends of $\mathcal{C}$ conformally degenerate, then the limit of the values of modulus at one end of $\mathcal{C}$ is 0, and the limit at its other end is ∞. This implies that there exists at least one quadrilateral with a net in in $\mathcal{C}$ with any given modulus $0 < K < \infty$. Thus the number of chains $\mathcal{C}$ such that length of $\mathcal{C}$ is even and quadrilaterals at both ends of $\mathcal{C}$ conformally degenerate is a lower bound for the number of quadrilaterals with the given angles and modulus.

If a chain $\mathcal{C}$ has odd length (contains an even number of nets) and the quadrilaterals at both ends of $\mathcal{C}$ conformally degenerate, then the limits of the values of modulus at its ends are either both 0 or both ∞. In the first (resp., second) case, there exist two quadrilaterals with the nets in $\mathcal{C}$ with small enough (resp., large enough) modulus.

Finally, if the quadrilaterals at only one end of a chain $\mathcal{C}$ of any length conformally degenerate, with the limit of the values of modulus at that end is 0 (resp., ∞) then there is exactly one quadrilateral with the net in $\mathcal{C}$ with small enough (resp., large enough) modulus..

Thus chains of generic quadrilaterals allow one to count the number of quadrilaterals with the given angles and either small enough or large enough modulus.

Example 5.23. Quadrilaterals with the angles $(\alpha, \beta, 1 + \gamma, 1 + \delta)$ satisfying (7) in Example 5.16 belong to a single chain $\mathcal{C}$ of length 1, consisting of the quadrilaterals with the nets R_{11} and S_{11} when $\alpha + \beta + \gamma + \delta < 2$. At both ends of that chain, the modulus tends to 0, thus the lower bound for the number of quadrilaterals with the given modulus is 0. Since $\mathcal{C}$ is the only chain of quadrilaterals with such angles, there are exactly two quadrilaterals with such angles for small enough values of the modulus, and no quadrilaterals with such angles for large enough values of the modulus.

When $\alpha + \beta + \gamma + \delta > 2$, there may be either one or two chains of length 0, with the nets either R_{11} or $P_0 \cup D_{15}$, or both. This implies that there may be either at least one or at least two spherical quadrilaterals with such angles for any value of the modulus, depending on the angles.

Example 5.24. Quadrilaterals with the angles $(\alpha, \beta, \gamma, n + \delta)$, where n is even, are considered in Proposition 5.21 (see also Example 5.14 for $n = 2$). If the inequalities (17) are satisfied (item (i) of Proposition 5.21) then there is a single chain of quadrilaterals with such angles of even length $2n$, such that the modulus tends to 0 at the $X_{0,n}$ end of the chain and to ∞ at the $X_{n,0}$ end. This gives the lower bound of one quadrilateral for each value of the modulus. If the first inequality in (17) is $\alpha+\beta+\gamma+\delta<2$ instead of $\alpha+\beta+\gamma+\delta>2$ (item (ii) of Proposition 5.21), then there are $n/2$ chains of length 2, and the lower bound becomes$n/2$. The maximal lower bound $n+1$ appears when the second and fourth inequalities in (17) are reversed (item (vii) of Proposition 5.21).

References

[1] L. Ahlfors, Conformal invariants. Topics in geometric function theory, Reprint of the 1973 original, AMS Chelsea Publishing, Providence, RI, 2010.

[2] D. Bartolucci, F. de Marchis, A. Malchiodi, Supercritical conformal metrics on surfaces with conical singularities. Int. Math. Res. Not., 24 (2011) 5625-–5643.

[3] C.-C. Chen and C.-S. Lin, Mean field equation of Liouville type with singular data: topological degree, Comm. Pure Appl. Math., 68 (2015) 887—947.

[4] Z. Chen and C.-S. Lin, Sharp Nonexistence Results for Curvature Equations with Four Singular Sources on Rectangular Tori. Amer. J. Math., 142 (2020) 1269–1300.

[5] A. Eremenko, Metrics of positive curvature with conic singularities on the sphere, Proc. Amer. Math. Soc., 132 (2004) 3349–3355.

[6] A. Eremenko, Metrics of constant positive curvature with four conic singularities on the sphere, Proc. Amer. Math. Soc., 148 (2020) 3957–3965.

[7] A. Eremenko, Co-axial monodromy, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 20 (2020), 619–634.

[8] A. Eremenko, Metrics of constant positive curvature with conic singularities. A survey, arXiv:2103.13364 (2021).

[9] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, Ann. Math., 155 (2002) 105–129.

[10] A. Eremenko and A. Gabrielov, On metrics of curvature 1 with four conic singularities on tori and on the sphere. Ill. J. Math., 59 (2015) 925-947.

[11] A. Eremenko and A. Gabrielov, Spherical rectangles. Arnold Math. Journal, 2 (2016) 463–486.

[12] A. Eremenko and A. Gabrielov, Circular pentagons and real solutions of Painlevé VI equations, Comm. Math. Phys., 355, 1 (2017) 51–95.

[13] A. Eremenko and A. Gabrielov, Schwarz–Klein triangles, arXiv:cw2006.16874 (2020), J. Math. Phys. An. Geom., 16 (2020) 263–282.

[14] A. Eremenko, A. Gabrielov, G. Mondello and D. Panov, Moduli spaces for Lamé functions and Abelian differentials of the second kind, Comm. Contemp. Math., 24 (2022) 1–68.

[15] A. Eremenko, A. Gabrielov and V. Tarasov, Metrics with conic singularities and spherical polygons, Illinois J. Math., 58 (2014) 739–755.

[16] A. Eremenko, A. Gabrielov and V. Tarasov, Metrics with four conic singularities and spherical quadrilaterals, Conformal geometry and Dynamics, 20 (2016) 128–175.

[17] A. Eremenko, A. Gabrielov and V. Tarasov, Spherical quadrilaterals with three non-integer angles, Journal of Math. Phys. Analysis and Geometry, 12 (2016) 134–167.

[18] A. Eremenko, G. Mondello and D. Panov, Moduli of spherical tori with one conical point, arXiv:2008.02772 (2020).

[19] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada, CMC-1 trinoids in hyperbolic 3-space and metrics of constant curvature one with conical singularities on the 2-sphere, Proc. Japan Acad., 87 (2011) 144–149.

[20] W. Ihlenburg, Über die geometrischen Eigenschaften der Kreisbopgenvierecke, Nova Acta Leopoldina, 92 (1909) 1–79+5 pages of tables.

[21] F. Klein, Mathematical seminar at Göttingen, winter semester 1905/6 under the direction of Professors Klein, Hilbert and Minkowski, talks by F. Klein, notes by O. Toeplitz,
www.claymath.org/sites/default/files/klein1math.sem__ws1905-06.pdf

[22] F. Klein, Forlesungen über die hypergeometrische Funktion, reprint of the 1933 original, Springer Verlag, Berlin-NY, 1981.

[23] Felix Klein, Vorlesungen über die hypergeometrische Funktion, Springer-Verlag, Berlin-New York, 1981.

[24] F. Luo and G. Tian, Liouville equation and spherical convex polytopes, Proc. Amer. Math. Soc., 116 (1992) 1119—1129.

[25] C.-S. Lin and C.-L. Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. Math., 172 (2010) 911–954.

[26] R. Mazzeo and X. Zhu, Conical metrics on Riemann surfaces, II: spherical metrics. arXiv:1906.09720 (2019)

[27] G. Mondello and D. Panov, Spherical metrics with conical singularities on a 2-sphere: angle constraints, Int. Math. Res. Not. IMRN (2016), no. 16, 4937–4995.

[28] G. Mondello and D. Panov, Spherical surfaces with conical points: systole inequality and moduli spaces with many connected components, Geom. Funct. Analysis, 29 (2019) 1110—1193.

[29] F. Schilling, Ueber die Theorie der symmetrischen s-Funktion mit einem einfachen Nebenpunkte, Math. Ann., 51, 481-522.

[30] M. Troyanov, Metrics of constant curvaturer on a sphere with two conical singularities, Lect. Notes Math., 1410, Springer, Berlin, 1989, 296–306.

Department of Mathematics, Purdue Univ., West Lafayette, IN 47907-2067 USA gabrielov@purdue.edu