TOPOLOGICAL GALOIS THEORY

Problems of solvability and non-solvability of algebraic, transcendental and differential equations in explicit form, including Liouville theory, classical and differential Galois theory, and Picard-Vessiot theory, will be addressed from the geometry and analysis point of view. The course will be based on the book "Topological Galois Theory" by A. G. Khovanskii (2008, in Russian). Most of its contents can be found in Khovanskii's review paper "On solvability and unsolvability of equations in explicit form," Russian Math. Surveys, v. 59, pp. 661-736.

The students are not expected to have prior knowledge of any of the above mentioned theories.

1. Overview

- Algebraic equations (Abel, Galois).
- Quadratures (Liouville, Ritt).
- Differential equations (Picard-Vessiot, Kolchin).
- Topological obstructions to solvability (Arnol'd, Khovanskii).

2. Solvability problem setup

- Classes of functions defined by generators and admissible operations.
- Multivalued functions.
- Functions representable by radicals.
- Elementary functions.
- Functions representable by quadratures.
- Generalized elementary, etc., functions.
- Functions representable by k-radicals, etc.

3. Liouville theory: preliminaries

- When an indefinite integral of an elementary function is an elementary function?
- When solutions of a linear differential equation can be represented by (generalized) quadratures?
- A "simple" equation either has a "simple" solution or cannot be solved explicitly.
- Algebraization: replacing composition $y = \exp(f)$ and $z = \log(f)$ by differential equations $y' = f'y$ and $z' = f'/f$.
- All elementary functions are generated by exp and log.
- Differential fields, elementary and Liouville extensions, generalized and k-extensions.
- Differential function fields and their extensions.

4. Liouville theory: indefinite integrals

- An indefinite integral of a function $f \in K$ belongs to a generalized elementary extension of K if and only if

$$f = A_0' + \sum_{i=1}^{n} \lambda_i A_i'/A_i, \quad y = A_0 + \sum_{i=1}^{n} \lambda_i \log A_i \qquad (*)$$

where $A_i \in K$ for $i = 0, \ldots, n$.

$\Rightarrow$ An indefinite integral y of a generalized elementary function f is a generalized elementary function if and only if $(*)$ holds with A_i rational in f and its derivatives.

$\Rightarrow$ An indefinite integral y of an algebraic function f is a generalized elementary function if and only if $(*)$ holds with A_i algebraic, single-valued on the Riemann surface of f.

$\Rightarrow$ An indefinite integral I of fe^g where $f \not\equiv 0$ and $g \not\equiv \text{const}$ are rational functions is a generalized elementary function if and only if $a' + ag' = f$ for some rational function a. Then $I = ae^g + C$.

5. Liouville theory: linear differential equations

- A second order equation $y'' + py' + q = 0$ with p and q representable by generalized quadratures can be solved by generalized quadratures if and only if it has a solution $y_1 = \exp\left(\int f(t)dt\right)$ where

f is algebraic over the differential field K generated by p and q.
- Higher order equations (Liouville, Mordukhai-Boltovskii, Picard-Vessiot). An equation $y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0$ with p_i representable by generalized quadratures can be solved by generalized quadratures if and only if

1) It has a solution $y_1 = \exp\left(\int f(t)dt\right)$ where f belongs to an algebraic extension K_1 of the differential field K generated by p_i.
2) The differential equation for $z = y' - (y_1'/y_1)y$ of the order (n-1) with coefficients in K_1, obtained by reducing the order of the original equation, is solvable by generalized quadratures over K_1.

6. Galois theory: Solvability of algebraic equations by radicals

- Let V be an algebra over a field K and G a finite commutative group of automorphisms of V. Assume that $\mathrm{char}(K) = 0$ and K contains n-th roots of 1. Then each $x \in V$ is a sum of n-th roots of elements of the invariant algebra V_0.
- For V as above, let G be a finite solvable group of automorphisms of V. Then each $x \in V$ can be obtained from the elements of V_0 by extracting roots and summation.
- Permutation group $S(n)$ and solvability of equations of degree ≤ 4.
- Lagrange interpolation polynomials, eigenvectors of commuting matrices, and formulas for solutions of equations of degree ≤ 4.
- Galois group of an algebraic equation. Galois extension of a field and its Galois group.
- Main theorem of Galois theory: Let P be a Galois extension of a field K with the Galois group G. The Galois correspondence between subgroups H of G and intermediate fields F, $K \subseteq F \subseteq P$, is one-to-one. An intermediate field F is a Galois extension of K, with the Galois group G/H, if and only if the group H corresponding to F is a normal subgroup of P.
- Galois extension of the field of coefficients and the change of the Galois group of an algebraic equation.
- An algebraic equation over a field K is solvable by radicals if and only if its Galois group is solvable.
- Abel's theorem: The general algebraic equation of degree ≥ 5 is not solvable by radicals.
- k-solvable groups and reduction of the degree of an algebraic equation.

7. Picard-Vessiot theory: Solvability of linear differential equations by quadratures

- Division with remainder for linear differential operators.
- Reduction of order of a linear differential equation.
- Wronskians and the analog of Viète formula for linear differential equations.
- Galois group of a linear differential equation. Picard-Vessiot extension of a differential field.
- Main theorem of Picard-Vessiot theory: Let P be a Picard-Vessiot extension of a differential field K, with the Galois group G. The Galois correspondence between Zariski closed subgroups H of G and intermediate differential fields F, $K \subseteq F \subseteq P$, is one-to-one. An intermediate differential field F is a Picard-Vessiot extension of K, with the Galois group G/H, if and only if the group H corresponding to F is a normal subgroup of P.
- Picard-Vessiot extension of the field of coefficients and the change of the Galois group of a linear differential equation.
- Solvable, k-solvable, almost solvable algebraic groups and solvability of linear differential equations by quadratures, k-quadratures, generalized quadratures, resp.
- Kolchin's theory: Quasi-compact and anticompact groups. Special triangular and diagonal groups. Solvability of linear differential equations by integrals, integrals and radicals, integrals and algebraic functions, exponentials of integrals, exponentials of integrals and algebraic functions.
- Picard-Vessiot extensions with triangular Galois groups.

8. Univariate topological Galois theory

- Fields of meromorphic functions on algebraic curves and ramified coverings of the Riemann surfaces.
- Galois group of an algebraic equation as its monodromy group.
- Topological obstructions to representability of functions by radicals: An algebraic function is representable by radicals (k-radicals) if and only if its monodromy group is solvable (k-solvable).

$\Rightarrow$ (Arnol'd) An algebraic function not representable by radicals is also not representable by radicals and any entire functions.

$\Rightarrow$ (Arnol'd) If a meromorphic function g is topologically equivalent to an elliptic function f then g is an elliptic function (possibly with different periods than those of f). In particular, g is not elementary.

- Dense sets of ramification points and monodromy groups with a continuum of elements. Functions analytic outside a "forbidden" set A and their A-monodromy groups. Closed monodromy groups.
- Functions with at most countable set of singular points ($\mathcal{S}$-functions). Closedness under differentiation, integration, composition, substitution into a meromorphic function, solution of algebraic and linear differential equations.
- Topological obstructions to representability of functions by quadratures, generalized quadratures, k-quadratures.

9. Topological Picard-Vessiot theory

- Monodromy of a linear differential equation and its Galois group.
- Fuchsian equations. Frobenius theorem: Single-valued solutions of a Fuchsian equation are rational functions.

$\Rightarrow$ Zariski closure of the monodromy group of a Fuchsian equation coincides with its Galois group.

- A Fuchsian equation is solvable by quadratures (k-quadratures, generalized quadratures) if and only if its monodromy group is solvable (k-solvable, almost solvable).
- Monodromy group of a system of linear differential equations and its Galois group.
- Regular singular points of a system of linear differential equations. Regular systems of linear differential equations.
- For a regular system of linear differential equations, the differential field generated by its solutions is a Picard-Vessiot extension of the field of rational functions. Its Galois group coincides with the Zariski closure of its monodromy group.
- Each component of each solution of a regular system of linear differential equations can be expressed by quadratures (k-quadratures, generalized quadratures) if and only if its monodromy group is solvable (k-solvable, almost solvable).

10. Multivariate topological Galois theory

- Will be covered if time permits.