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CANCELLATION DOES NOT IMPLY STABLE RANK ONE

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Abstract

A unital C*-algebra A is said to have *cancellation of projections* if the semigroup $D(A)$ of Murray–von Neumann equivalence classes of projections in matrices over A is cancellative. It has long been known that stable rank one implies cancellation for any A, and some partial converses have been established. In this paper it is proved that cancellation does not imply stable rank one for simple, stably finite C*-algebras.

1. *Introduction*

Rieffel introduced the notion of stable rank for C*-algebras in his 1983 paper [**4**]: a unital C*-algebra A is said to have *stable rank* n ($\mathrm{sr}(A) = n$) if n is the least natural number such that the set

$$\mathrm{Lg}_n(A) \stackrel{\mathrm{def}}{=} \left\{(a_1, \ldots, a_n) \in A^n \mid \exists b_i \in A,\ 1 \leqslant i \leqslant n : \sum_{i=1}^{n} b_i a_i = 1\right\}$$

is dense in A^n. If no such n exists, then one says that the stable rank of A is *infinite*. In the case of a commutative C*-algebra, the stable rank is proportional to the covering dimension of the spectrum; stable rank may be viewed as a kind of non-commutative dimension.

Given a unital C*-algebra A, let $D(A)$ be the Abelian semigroup obtained by endowing the set of Murray–von Neumann equivalence classes of projections in matrix algebras over A with the addition operation coming from direct sums. The algebra A is said to have *cancellation of projections* if $x + y = x + z$ implies that $y = z$ for any $x, y, z \in D(A)$. Shortly after the appearance of Rieffel's paper, Blackadar showed that stable rank one implies cancellation of projections [**1**]. He also established a partial converse: if a C*-algebra of real rank zero has cancellation of projections, then it has stable rank one. The relationship between cancellation and stable rank for general simple, stably finite C*-algebras, however, remained unclear. The lack of examples of simple, stably finite C*-algebras with non-minimal stable rank was a serious obstacle. Villadsen provided the first such examples in [**7**], but determining whether his examples had cancellation of projections was all but impossible, due to their extremely complicated K-theory.

Recently, the author has been able to apply Villadsen's techniques to construct simple, stably finite C*-algebras with non-minimal stable rank and cyclic K_0-groups. These algebras constitute the first simple, nuclear and stably finite counterexamples to Elliott's classification conjecture for nuclear C*-algebras [**2**, **6**]. In this paper we study one such algebra in order to prove our main result.

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Theorem. *There is a simple, separable, nuclear, and stably finite C*-algebra with non-minimal stable rank, which nevertheless has cancellation of projections.*

Thus, Blackadar's partial converse cannot be extended to cover general simple, stably finite C*-algebras.

2. *The proof of the main result*

We proceed by a close analysis of the structure of the simple, separable, and stably finite C*-algebra B_2 of [**6**], which has non-minimal stable rank. We prove that B_2 nevertheless has cancellation of projections.

Let C and D be C*-algebras, and let ϕ_0 and ϕ_1 be $*$-homomorphisms from C to D. The generalised mapping torus of C and D with respect to ϕ_0 and ϕ_1 is

$$A := \{(c, d) \mid d \in C([0,1]; D),\ c \in C,\ d(0) = \phi_0(c),\ d(1) = \phi_1(c)\}.$$

We denote A by $A(C, D, \phi_0, \phi_1)$ for clarity when necessary. Let $\mathcal{U}(A)$ denote the unitary group of a unital C*-algebra A.

The algebra B_2 of [**6**] is constructed as the limit of an inductive sequence (A_i, θ_i) of generalised mapping tori $A_i = A(C_i, D_i, \phi_i^0, \phi_i^1)$ and unital $*$-homomorphisms $\theta_i : A_i \longrightarrow A_{i+1}$ where, for each $i \in \mathbb{N}$,

$$C_i \stackrel{\text{def}}{=} p_i(C(X_i) \otimes \mathcal{K})p_i$$

and

$$D_i \stackrel{\text{def}}{=} M_{k_i} \otimes C_i$$

for some connected compact Hausdorff space X_i, projection $p_i \in C(X_i) \otimes \mathcal{K}$ and natural number k_i. The maps ϕ_i^0 and ϕ_i^1 are unital. The spaces X_i, $i \in \mathbb{N}$, have the property that

$$\dim(p_i) = \frac{\dim(X_i)}{2},$$

and the maps ϕ_i^0 and ϕ_i^1 are chosen to ensure that

$$\big(K_0A_i, K_0A_i^+, [1_{A_i}]\big) = (\mathbb{Z}, \mathbb{Z}^+, 1),$$

where $1_{A_i} \in A_i$ is the unit; A_i is projectionless except for zero and 1_{A_i}.

To prove our theorem, it will suffice to prove that A_i has cancellation of projections for every $i \in \mathbb{N}$. Let $p, q \in M_n(A_i)$ be projections having the same K_0-class. We must show that p and q are Murray–von Neumann equivalent. Since $K_0(A_i) = \mathbb{Z}[1_{A_i}]$, we may assume that p is a multiple of the unit of A_i, say $p = l1_{A_i}$. Now $M_n(A_i)$ can be viewed as an algebra of functions from $[0,1] \times X_i$ into matrices. Given $f \in M_n(A_i)$, we let $f(t)$, $t \in [0,1]$, denote the restriction of f to $\{t\} \times X_i \subseteq [0,1] \times X_i$. Both $f(0)$ and $f(1)$ are images of a single element in $M_n(C_i)$, which we denote by $f(\infty)$. If two vector bundles over a compact, connected CW-complex X of covering dimension m with the same K^0-class have fibre dimension at least $m/2$, then the bundles are isomorphic (cf. [**3**, Theorem 1.5, Chapter 8]). In the language of C*-algebras, the projections in $M_k \otimes C(X)$, for some $k \in \mathbb{N}$, corresponding to these vector bundles are Murray–von Neumann equivalent. Since $p(\infty)$ and $q(\infty)$ can be viewed as vector bundles over X_i having the same K^0-class, and since they must both have fibre dimension at least $\dim(X_i)/2$ by the construction of A_i, they are Murray–von Neumann equivalent, as are their images under ϕ_i^0 and ϕ_i^1. Note that if one considers $M_n(A_i)$ as a unital sub-C*-algebra of $C_i \otimes \mathrm{M}_{nk_i} \otimes \mathrm{C}([0,1])$, then fibre dimension considerations show q and p to be Murray–von Neumann equivalent inside $C_i \otimes M_{nk_i} \otimes C([0,1])$. This does not, however, prove that q and p are Murray–von Neumann equivalent inside $M_n(A_i)$.

We may assume without loss of generality that $l1_{A_i}$ and q are constant over some small interval $[1/2-\varepsilon, 1/2+\varepsilon]$ in the interval factor of the spectrum of $M_n(A_i)$, since small perturbations do not disturb the Murray–von Neumann equivalence class. Consider $l1_{A_i}$ and q as vector bundles over $[0,1] \times X_i$. Define

$$q_0 := q|_{[0,1/2-\varepsilon]\times X_i}, \qquad q_1 := q|_{[1/2+\varepsilon,1]\times X_i}$$

and

$$1_{A_i,0} := 1_{A_i}|_{[0,1/2-\varepsilon]\times X_i}, \qquad 1_{A_i,1} := 1_{A_i}|_{[1/2+\varepsilon,1]\times X_i}.$$

The following statement appears as [**3**, Chapter 3, Corollary 4.4].

LEMMA. *Let γ be a vector bundle over $X \times [0,1]$, X paracompact, and ω a vector bundle over X such that $\gamma|_{X\times\{0\}} \cong \omega$. Then γ is isomorphic to the induced bundle $\pi^*(\omega)$, where $\pi : X \times [0,1] \longrightarrow X \times \{0\}$ is given by $\pi(x,t) = (x,0)$.*

Define maps

$$\pi_0 : \left[0, \tfrac{1}{2} - \varepsilon\right] \times X_i \longrightarrow \{0\} \times X_i, \qquad \pi_1 : \left[\tfrac{1}{2} + \varepsilon, 1\right] \times X_i \longrightarrow \{1\} \times X_i,$$

by

$$\pi_0(t,x) = (0,x), \qquad \pi_1(t,x) = (1,x).$$

We have $l1_{A_i}(j) \cong q(j)$ for $j \in \{0,1\}$. Moreover, $l1_{A_i,j} \cong \pi_j^*(l1_{A_i}(j))$ by construction. We may thus apply our lemma, with $\gamma = q_j$, $\omega = l1_{A_i}(j)$, and $\pi = \pi_j$, to conclude that $l1_{A_i,j} \cong q_j$. In other words, there is a continuous path of partial isometries $v(t)$, $t \in [0,1/2-\varepsilon] \cup [1/2+\varepsilon,1]$, such that $v(t)^*v(t) = l1_{A_i}(t)$, $v(t)v(t)^* = q(t)$, and, for each $j \in \{0,1\}$, the partial isometry $v(j)$ is the image under $\phi_i^j \otimes \mathrm{id}_{M_n}$ of a single partial isometry $v \in M_n(C_i)$ such that $v^*v = l1_{C_i}$ and $vv^* = q(\infty)$. This last property ensures that if we can find a continuous extension of $v(t)$ to a partial isometry defined on $[0,1]$, then our proof is complete: $v(t)$ will lie in $M_n(A_i)$.

From [**5**] we have the formula

$$\mathrm{sr}(p(C(X)\otimes\mathcal{K})p) = \left\lceil \frac{\lfloor \dim(X)/2 \rfloor}{\mathrm{rank}(p)} \right\rceil + 1,$$

where $\mathcal{K}$ denotes the compact operators on a separable Hilbert space, X is a compact connected Hausdorff space, and p is a projection in $C(X) \otimes \mathcal{K}$. Straightforward calculation then shows that $\mathrm{sr}(C_i) = 2$, for all $i \in \mathbb{N}$. For a unital C*-algebra A, let $\mathcal{U}(A)$ denote the unitary group of A, and let $\mathcal{U}(A)_0$ denote the connected component of $\mathcal{U}(A)$ containing the identity. Now, [**4**, Theorem 10.12] states that one has an isomorphism

$$\frac{\mathcal{U}(M_r(A))}{\mathcal{U}(M_r(A))_0} \longrightarrow K_1(A)$$

whenever $r \geqslant \mathrm{sr}(A)+2$. In the construction of A_i, the parameter k_i in the definition $D_i := M_{k_i}(C_i)$ is chosen to be much larger than $\mathrm{sr}(C_i)$. Furthermore, one has (again, by construction) that $K_1(C_i) = 0$, for all $i \in \mathbb{N}$. Thus, $\mathcal{U}(M_l(D_i))$ is connected for every $l \in \mathbb{N}$.

We may view $u := v(1/2+\varepsilon)^* v(1/2-\varepsilon)$ as a unitary element in $M_l(D_i)$. By the discussion above, there is a path of unitary elements $u(t)$, $t \in [1/2-\varepsilon, 1/2+\varepsilon]$, inside $M_l(D_i)$ such that $u(1/2+\varepsilon) = l1_{A_i}$ and $u(1/2-\varepsilon) = u$.

For $t \in [1/2-\varepsilon, 1/2+\varepsilon]$, define $\tilde{v}(t) = v(1/2+\varepsilon)u(t)$. Clearly, $\tilde{v}(t)$ is a partial isometry in $M_n(D_i)$ for each t in its domain. One has

$$\tilde{v}\big(\tfrac{1}{2}+\varepsilon\big) = v\big(\tfrac{1}{2}+\varepsilon\big)$$

and

$$\tilde{v}\big(\tfrac{1}{2}-\varepsilon\big) = v\big(\tfrac{1}{2}+\varepsilon\big)v\big(\tfrac{1}{2}+\varepsilon\big)^* v\big(\tfrac{1}{2}-\varepsilon\big) = q\big(\tfrac{1}{2}-\varepsilon\big)v\big(\tfrac{1}{2}-\varepsilon\big) = v\big(\tfrac{1}{2}-\varepsilon\big).$$

Then

$$v(t) := \begin{cases} v(t), & t \in \big[0, \tfrac{1}{2}-\varepsilon\big] \cup \big[\tfrac{1}{2}+\varepsilon, 1\big], \\ \tilde{v}(t), & t \in \big(\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon\big) \end{cases}$$

defines a partial isometry in $M_n(A_i)$ such that $v(t)^* v(t) = l1_{A_i}(t)$ and $v(t)v(t)^* = q(t)$, for all $t \in [0,1]$. Thus q and $l1_{A_i}$ are Murray–von Neumann equivalent, as desired. □

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References

1. B. Blackadar, *K-theory for operator algebras*, 2nd edn, MSRI Publications 5 (Cambridge University Press, 1998).
2. G. A. Elliott, 'The classification problem for amenable C*-algebras', *Proc. ICM '94*, Zurich, Switzerland (ed. S. D. Chatterji, Birkhauser, Basel, Switzerland, 1995) 922–932.
3. D. Husemoller, *Fibre bundles* (Mc-Graw–Hill, New York, 1966).
4. M. A. Rieffel, 'Dimension and stable rank in the K-theory of C*-algebras', *Proc. London Math. Soc.* (3) 46 (1983) 301–333.
5. V. Nistor, 'Stable rank for a certain class of type I C*-algebras', *J. Operator Theory* 17 (1987) 365–373.
6. A. S. Toms, 'On the independence of K-theory and stable rank for simple C*-algebras', *J. Reine Angew. Math.* 578 (2005) 185–199.
7. J. Villadsen, 'On the stable rank of simple C*-algebras', *J. Amer. Math. Soc.* 12 (1999) 1091–1102.

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