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A class of simple C*-algebras with stable rank one ☆

George A. Elliott [a], Toan M. Ho [b], Andrew S. Toms [b,*]

[a] *Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4*
[b] *Department of Mathematics and Statistics, York University, 4700 Keele St., Toronto, Ontario, Canada M3J 1P3*

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Abstract

We study the limits of inductive sequences (A_i, ϕ_i) where each A_i is a direct sum of full matrix algebras over compact metric spaces and each partial map of ϕ_i is diagonal. We give a new characterisation of simplicity for such algebras, and apply it to prove that the said algebras have stable rank one whenever they are simple and unital. Significantly, our results do not require any dimension growth assumption.

Keywords: AH algebras; Stable rank; Dimension growth

1. Introduction

Let X and Y be compact Hausdorff spaces. A $*$-homomorphism

$$\phi : \mathrm{M}_m\big(\mathrm{C}(X)\big) \to \mathrm{M}_{nm}\big(\mathrm{C}(Y)\big)$$

is called *diagonal* if there are n continuous maps $\lambda_i : Y \to X$ such that

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* Corresponding author. Fax: +1 416 736 5757.
E-mail addresses: elliott@math.toronto.edu (G.A. Elliott), toan@mathstat.yorku.ca (T.M. Ho), atoms@mathstat.yorku.ca (A.S. Toms).

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$$\phi(f)=\begin{pmatrix} f\circ\lambda_1 & 0 & \dots & 0\\ 0 & f\circ\lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & f\circ\lambda_n \end{pmatrix}.$$

The λ_i are called the *eigenvalue maps* or simply *eigenvalues* of ϕ. The multiset $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ is called the *eigenvalue pattern* of ϕ and is denoted by $\mathrm{ep}(\phi)$. This definition can be extended to $*$-homomorphisms

$$\phi:\bigoplus_{i=1}^{n}\mathrm{M}_{n_i}\big(\mathrm{C}(X_i)\big)\to\bigoplus_{j=1}^{m}\mathrm{M}_{m_j}\big(\mathrm{C}(Y_j)\big)$$

by requiring, roughly, that each partial map

$$\phi^{ij}:\mathrm{M}_{n_i}\big(\mathrm{C}(X_i)\big)\to\mathrm{M}_{m_j}\big(\mathrm{C}(Y_j)\big)$$

induced by ϕ be diagonal. (A precise definition can be found in Section 2.)

C^*-algebras obtained as limits of inductive systems (A_i,ϕ_i) where

$$A_i=\bigoplus_{j=1}^{n_i}\mathrm{M}_{n_{i,j}}\big(\mathrm{C}(X_{i,j})\big)$$

and each ϕ_i is diagonal form a rich class. They include AF algebras, simple unital $\mathrm{A}\mathbb{T}$ algebras (and hence the irrational rotation algebras) [6], Goodearl algebras [10], and some interesting examples of Villadsen and the third named author connected to Elliott's program to classify amenable C^*-algebras via K-theory [15,17]. The structure of these algebras is only well understood when they satisfy some additional conditions such as (very) slow dimension growth or the combination of real rank zero, stable rank one, and weak unperforation of the K_0-group—situations in which the strong form of Elliott's classification conjecture can be verified [3,5,7–9].

In this paper we give a new characterisation of simplicity for AH algebras with diagonal connecting maps. As a consequence we are able to prove that such algebras have stable rank one whenever they are unital and simple. The significance of our result derives from the fact that we make no assumptions on the dimension growth of the algebras; we obtain a general theorem on the structure of algebras heretofore considered "wild." As suggested by M. Rørdam in his recent ICM address, it is high time we became friends with such algebras, as opposed to treating them simply as a source of pathological examples.

2. Preliminaries

2.1. Basic notation

We use M_n to denote the set of $n\times n$ complex matrices. Given a closed subset E of a compact metric space (X,d) and $\delta>0$ we set

$$B_\delta(E)=\big\{x\in X\;\big|\;d(E,x)<\delta\big\},$$

and make the convention that $B_\delta(\emptyset)=\emptyset$.

2.2. AH systems with diagonal maps

Definition 2.1. We will say that a unital $*$-homomorphism

$$\phi : \bigoplus_{i=1}^{n} M_{n_i}\big(\mathrm{C}(X_i)\big) \to M_k\big(\mathrm{C}(Y)\big) \cong M_k \otimes \mathrm{C}(Y)$$

is diagonal if there exist natural numbers $k_1, \dots, k_n$ such that $\sum_i k_i = k$ and $n_i | k_i$, an embedding

$$\iota : \bigoplus_{i=1}^{n} M_{k_i} \hookrightarrow M_k,$$

and diagonal maps

$$\phi_i : M_{n_i}\big(\mathrm{C}(X_i)\big) \to M_{k_i} \otimes \mathrm{C}(Y)$$

such that

$$\phi = \bigoplus_{i=1}^{n} \phi_i.$$

(Notice that $k_i = 0$ is allowed.) We will say that a unital $*$-homomorphism

$$\phi : \bigoplus_{i=1}^{n} \mathrm{M}_{n_i}\big(\mathrm{C}(X_i)\big) \to \bigoplus_{j=1}^{m} \mathrm{M}_{m_j}\big(\mathrm{C}(Y_j)\big)$$

is diagonal if each restriction

$$\phi_j : \bigoplus_{i=1}^{n} \mathrm{M}_{n_i}\big(\mathrm{C}(X_i)\big) \to \mathrm{M}_{m_j}\big(\mathrm{C}(Y_j)\big)$$

is diagonal.

Let A be the limit of the inductive sequence (A_i, ϕ_i), where

$$A_i = \bigoplus_{t=1}^{k_i} \mathrm{M}_{n_{i,t}}\big(\mathrm{C}(X_{i,t})\big), \tag{1}$$

$X_{i,t}$ is a connected compact metric space, and $n_{i,t}$ and k_i are natural numbers. Define

$$A_{i,t} := \mathrm{M}_{n_{i,t}}\big(\mathrm{C}(X_{i,t})\big),$$
$$X_i := X_{i,1} \sqcup X_{i,2} \sqcup \cdots \sqcup X_{i,k_i},$$

and

$$\phi_{i,j} := \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let

$$\phi_{i,j}^{t,l} : \mathrm{M}_{n_{i,t}}\big(\mathrm{C}(X_{i,t})\big) \to \mathrm{M}_{n_{j,l}}\big(\mathrm{C}(X_{j,l})\big)$$

and

$$\phi_{i,j}^{l} : \bigoplus_{t=1}^{k_i} \mathrm{M}_{n_{i,t}}\big(\mathrm{C}(X_{i,t})\big) \to \mathrm{M}_{n_{j,l}}\big(\mathrm{C}(Y_{j,l})\big)$$

denote the appropriate restrictions of ϕ. If each ϕ_i is unital and diagonal, then we refer to (A_i, ϕ_i) as an *AH system with diagonal maps*. The limit algebra A will be called a *diagonal AH algebra*, and we will refer to (A_i, ϕ_i) as a *decomposition* of A.

Assume that A as above is diagonal. We will view $\phi_{i,j}^{t,l}$ as a diagonal map from $\mathrm{M}_{n_{i,t}}(\mathrm{C}(X_{i,t}))$ into the cut-down of $\mathrm{M}_{n_{j,l}}(\mathrm{C}(X_{j,l}))$ by $\phi_{i,j}^{t,l}(1)$. For fixed i and j, we will denote by ep_{ij} the multiset which is the union, counting multiplicity, of the eigenvalue patterns of each $\phi_{i,j}^{t,l}$; ep_{ij} is the *eigenvalue pattern* of $\phi_{i,j}$; an element of ep_{ij} is an *eigenvalue map* of $\phi_{i,j}$. For fixed i, j, and l, we will denote by ep_{ij}^{l} the multiset which is the union of the eigenvalue patters of each $\phi_{i,j}^{t,l}$.

Let us now show that the bonding maps ϕ_i may be assumed to be injective. Let (A_i, ϕ_i) be a decomposition for a diagonal AH algebra A as above. Fix $i \in \mathbb{N}$ and $1 \leqslant t \leqslant k_i$. For each $j > i$ and $1 \leqslant l \leqslant k_j$, Let $X_{i,t}^{j,l}$ denote the closed subset of $X_{i,t}$ which is the union of the images of the eigenvalue maps of $\phi_{i,j}^{t,l}$. Put $X_{i,t}^{j} = \bigcup_l X_{i,t}^{j,l}$, and $\tilde{X}_{i,t} = \bigcap_j X_{i,t}^{j}$. Since $X_{i,t}^{j} \supseteq X_{i,t}^{j+1}$, we have that $\tilde{X}_{i,t}$ is closed subset of $X_{i,t}$. Define

$$\tilde{A}_{i,t} = \mathrm{M}_{n_{i,t}}\big(\mathrm{C}(\tilde{X}_{i,t})\big)$$

and

$$\tilde{A}_i = \bigoplus_{t=1}^{k_i} \tilde{A}_{i,t}.$$

Define diagonal maps $\tilde{\phi}_{i,i+1}^{t,l} : \tilde{A}_{i,t} \to \tilde{A}_{i+1,l}$ by replacing the eigenvalue maps of $\phi_{i,i+1}^{t,l}$ with their restrictions to $\tilde{X}_{i+1,l}$. Define $\tilde{\phi}_i : \tilde{A}_i \to \tilde{A}_{i+1}$ in a manner analogous to the definition of ϕ_i. It follows that $(\tilde{A}_i, \tilde{\phi}_i)$ is a diagonal AH system with limit A, and $\tilde{\phi}_i$ is injective by construction. We assume from here on that all bonding maps in diagonal AH systems are injective.

One way to construct a simple diagonal AH algebra is to ensure that for each $i \in \mathbb{N}$ and x in a specified dense subset of X_i there is some $j \geqslant i$ such that for each $l \in \{1, \ldots, k_j\}$ the diagonal map $\phi_{i,j}^{t,l}$ contains the eigenvalue map $ev_x : X_{j,l} \to X_{i,t}$ given by $ev_x(y) = x$. The next definition gives and approximate version of this situation.

Definition 2.2. Say that a diagonal AH algebra A with decomposition (A_i, ϕ_i) has the property $\mathcal{P}$ if for any $i \in \mathbb{N}$, element f in A_i, $\epsilon > 0$, and $x \in X_i$ there exist $j \geqslant i$ and unitaries $u_l \in A_{j,l}$, $l \in \{1, \ldots, k_j\}$ such that

$$\left\| u_l \phi_{i,j}^{t,l}(f) u_l^* - \begin{pmatrix} f(x) & 0 \\ 0 & b_l \end{pmatrix} \right\| < \epsilon$$

for some appropriately sized b_l. (Note that $\operatorname{diag}(f(x_0), b_l) \in A_{j,l}$.)

We will prove in the sequel that A as in Definition 2.2 is simple if and only if it has property $\mathcal{P}$.

2.3. A characterisation of simplicity

Proposition 2.1 of [4] gives some necessary and sufficient conditions for the simplicity of an AH algebra. We will have occasion to apply these in the proof of our main result, and so restate the said proposition in the particular case of an AH system with diagonal injective maps.

Proposition 2.3. *Let (A_i, ϕ_i) be an AH system with diagonal injective maps, and set $A = \lim_{i\to\infty}(A_i, \phi_i)$. The following conditions are equivalent*:

(i) *A is simple*;
(ii) *For any positive integer i and any nonempty open subset U of X_i, there is a $j_0 \geqslant i$ such that for every $j \geqslant j_0$ and $l \in \{1, \dots, k_j\}$ we have*

$$\left(\mathrm{ep}_{ij}^l\right)^{-1}(U) = X_{j,l},$$

where $(\mathrm{ep}_{ij}^l)^{-1}(U)$ denotes the union of the sets $\lambda^{-1}(U)$, $\lambda \in \mathrm{ep}_{ij}^l$;
(iii) *For any nonzero element a in A_i, there is a $j_0 \geqslant i$ such that for every $j \geqslant j_0$, $\phi_{ij}(a)(x)$ is not zero, for every x in X_j.*

2.4. Paths between permutation matrices

Given any permutation $\pi \in S_n$, let $U[\pi]$ denote the permutation matrix in M_n corresponding to π, that is, $U[\pi]$ is obtained from the identity of M_n by moving the ith row to the $\pi(i)$th row, for $i \in \{1, 2, \dots, n\}$. Any two permutation matrices are homotopic inside the unitary group $\mathcal{U}(\mathrm{M}_n)$ of M_n, but we want to define some particular homotopies for use in the sequel. Let π and σ be elements of S_n, viewed as permutations of the canonical basis vectors $e_1, \dots, e_n$ of $\mathbb{C}^n$. Let $R = \{e_{W,1}, \dots, e_{W,\dim(W)}\}$ be the set of basis vectors upon which π and σ agree, and choose $\gamma \in S_n$ be such that

$$\gamma\sigma(v) = \gamma\pi(v) = v, \quad \forall v \in R.$$

Then $U[\gamma]U[\sigma]$ and $U[\gamma]U[\pi]$ fix $e_1, \dots, e_{|R|}$. Put $W = \operatorname{span}\{e_1, \dots, e_{|R|}\}$, and let V be the orthogonal complement of W. There are a canonical unital embedding of $\mathrm{M}_{\dim(W)} \oplus \mathrm{M}_{\dim(V)}$ into M_n and unitaries $u, v \in \mathcal{U}(\mathrm{M}_{\dim(V)})$ such that

$$U[\gamma]U[\pi] = \mathbf{1}_{\mathrm{M}_{\dim(W)}} \oplus v; \qquad U[\gamma]U[\sigma] = \mathbf{1}_{\mathrm{M}_{\dim(W)}} \oplus u.$$

Choose a homotopy $g(t)$ between u and v inside $\mathcal{U}(\mathrm{M}_{\dim(V)})$—$g(0) = v$ and $g(1) = u$—and put

$$u(t) = U[\gamma]^{-1}\left(\mathbf{1}_{\mathrm{M}_{\dim(V)}} \oplus g(t)\right).$$

Then $u(t)$ is a homotopy of unitaries between $U[\pi]$ and $U[\sigma]$ such that

$$u(t)(v) = U[\pi](v) = U[\sigma](v), \quad \forall v \in R,\ \forall t \in [0,1].$$

2.5. Applications of Urysohn's Lemma

Lemma 2.4. *Let $\sigma \in S_n$ be given. There is a homotopy $u : [0,1] \to \mathcal{U}(M_n)$ between $u(0) = 1_n$ and $u(1) = U[\sigma]$ which moreover has the following property: for any complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_i = \lambda_{\sigma(i)}$ for every $i = 1, 2, \ldots, n$, we have*

$$u(t)\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} u^*(t) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}, \quad \forall t \in [0,1].$$

Proof. Let us first consider the case that σ is a k-cycle. The hypotheses of the lemma guarantee that the desired conclusion holds already for $t \in \{0,1\}$. Choose the homotopy $u(t)$ as in Section 2.4 by using our given value of σ and setting π equal to the identity element of S_n. The hypothesis $\lambda_i = \lambda_{\sigma(i)}$, $i \in \{1, \ldots, n\}$, implies that there is a $\lambda \in \mathbb{C}$ such that for each $i \in \{1, \ldots, n\}$ which is not fixed by σ we have $\lambda_i = \lambda$. In other words, if one decomposes $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ into a direct sum of two diagonal matrices using the decomposition $\mathbb{C}^n = W \oplus V$—$V$ and W as in Section 2.4—then the direct summand corresponding to V is scalar $k \times k$ matrix. By construction, $u(t) = v(t) \oplus 1_{n-k}$, with $v(t) \in \mathcal{U}(M_k)$. It follows that $u(t)$ commutes with $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ for each $t \in (0,1)$.

Now suppose that σ is any permutation on n letters, and write σ as a product of disjoint cycles: $\sigma = \sigma_1\sigma_2\ldots\sigma_l$. For each $j \in \{1, \ldots, l\}$, let $u_i(j)$ denote the unitary path between $U[\mathrm{id}]$ and $U[\sigma_j]$, constructed as in Section 2.4. Now

$$u(t) := u_1(t)u_2(t)\cdots u_l(t)$$

is a path of unitaries with $u(0) = U[\mathrm{id}]$ and $u(1) = U[\sigma]$, and $u(t)$ commutes with $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ since each $u_j(t)$ does. □

Lemma 2.5. *Let σ be any permutation in S_n, and A, B disjoint nonempty closed subsets of a metric space X. Let $\lambda_1, \ldots, \lambda_n : X \to \mathbb{C}$ be continuous. Then, there exists a unitary $v \in M_n(\mathrm{C}(X))$ such that*

(i) *$v(x) = 1_n$, $\forall x \in A$,*
(ii) *$v(x) = U[\sigma]$, $\forall x \in B$, and*
(iii) *$v(x)$ commutes with $\mathrm{diag}(\lambda_1(x), \ldots, \lambda_n(x))$ whenever $\lambda_i(x) = \lambda_{\sigma(i)}(x)$ for each $i \in \{1, \ldots, n\}$.*

Proof. Find a unitary path $u(t)$ connecting $U[\mathrm{id}]$ to $U[\sigma]$ using Lemma 2.4, so that $u(t)$ commutes with $\mathrm{diag}(\lambda_1(x), \ldots, \lambda_n(x))$ for each $t \in (0,1)$ and each $x \in X$ for which $\lambda_i(x) = \lambda_{\sigma(i)}(x)$, $i \in \{1, \ldots, n\}$. By Urysohn's Lemma, there is a continuous map $f : X \to [0,1]$ which is equal to zero on A and equal to one on B. It is straightforward to check that $v(x) := u(f(x))$ satisfies the conclusion of the lemma. □

Lemma 2.6. *Let Y be a closed subset of a normal space X, and let $f : Y \to S^n$ be continuous. Then, there is a continuous map $\tilde{f}$ from X to the $n+1$ disk D^{n+1} which extends f.*

Proof. View D^{n+1} as the cube $[0,1]^{n+1}$, with S^n as its boundary. Then

$$f(y) = \big(f_1(y), \dots, f_{n+1}(y)\big),$$

where each $f_i : Y \to [0,1]$ is continuous. Extend each f_i to a continuous map $\tilde{f}_i : X \to [0,1]$, and put

$$\tilde{f}(y) = \big(\tilde{f}_1(y), \dots, \tilde{f}_{n+1}(y)\big). \qquad \square$$

3. The main theorem

Let A be a diagonal AH algebra with decomposition (A_i, ϕ_i), and assume that the ϕ_i are injective. In this section we will prove that if A is simple then it has the property $\mathcal{P}$ (cf. Section 2). (The converse also holds, but is easier by far.) Let us begin with an outline of our strategy, before plunging headlong into the proof.

Assume first that A is simple and $A_i = \mathrm{M}_{n_i}(\mathrm{C}(X_i))$, so that there are no partial maps to contend with. Let there be given a natural number i, an element f of A_i, a point $x_0 \in X_i$, and some $\epsilon > 0$. Put $U = B_\epsilon(x_0)$. By Proposition 2.3 there is a j_0 with $j_0 \geqslant i$ such that for any $j \geqslant j_0$,

$$X_j = \lambda_1^{-1}(U) \cup \lambda_2^{-1}(U) \cup \cdots \cup \lambda_n^{-1}(U),$$

where

$$\phi_{i,j}(f) = \begin{pmatrix} f \circ \lambda_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ \lambda_n \end{pmatrix}.$$

On each closed subset $\lambda_t^{-1}(\overline{U})$, the range of the eigenvalue map λ_t is within ϵ of x_0. To show that A has the property $\mathcal{P}$, we require a unitary u in A_j and an element $b_f \in \mathrm{M}_{n_j - n_i}(\mathrm{C}(X_j))$ such that

$$\left\| u\phi_{i,j}(f)u^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_f \end{pmatrix} \right\| < \epsilon.$$

We would like $u(y)$ to exchange the first and tth diagonal entries of $\phi_{i,j}(f)$ whenever $y \in \lambda_t^{-1}(\overline{U})$, but this operation is unlikely to be well-defined—the sets $\lambda_1^{-1}(\overline{U}), \dots, \lambda_n^{-1}(\overline{U})$ need not be mutually disjoint. The remainder of this section is devoted to overcoming this complication.

Given positive integers $m \leqslant n$, let us denote by $\mathrm{M}_m(\mathrm{C}(Y)) \oplus 1_{n-m}$ the set of all $n \times n$ matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

where $a \in \mathrm{M}_m(\mathrm{C}(Y))$ and $1_{n-m} \in \mathrm{M}_{n-m}$.

Theorem 3.1. *Let there be given a diagonal $*$-homomorphism $\phi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ with the eigenvalue pattern $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, a point x_0 in X, an element f of $\mathrm{C}(X)$, and a tolerance $\epsilon > 0$. Choose $\eta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < 2\eta$ (d is the metric on X). Suppose that $F_1, \ldots, F_m$ are nonempty closed subsets of Y ($m \leqslant n$) such that $d(\lambda_i(y), x_0) < \eta$ whenever $y \in F_i$.*

Then, there is a unitary u in $\mathrm{M}_m(\mathrm{C}(Y)) \oplus 1_{n-m}$ and an element $b \in \mathrm{M}_{n-m-1}(\mathrm{C}(Y))$ such that for each $y \in \bigcup_{i=1}^m F_i$ we have

$$\left\| u(y)\phi(f)(y)u^*(y) - \begin{pmatrix} f(x_0) & 0 & 0 & \ldots & 0 \\ 0 & b(y) & 0 & \ldots & 0 \\ 0 & 0 & \lambda_{m+1}(y) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n(y) \end{pmatrix} \right\| < 2\epsilon. \tag{2}$$

(Note that if $n - m - 1 = 0$, then there is no b in the matrix in (2).)

Proof. Let ρ denote the metric on Y. Choose $\delta > 0$ such that $d(\lambda_i(x), \lambda_i(y)) < \eta$ whenever $\rho(x, y) \leqslant \delta$, $i \in \{1, \ldots, n\}$. For each $1 \leqslant i \leqslant m$,

$$\left| f \circ \lambda_i(y) - f(x_0) \right| < \epsilon, \quad \text{for all } y \in \overline{B_\delta(F_i)}.$$

Set $\varepsilon_i(y) = f \circ \lambda_i(y) - f(x_0)$ for all y in $\overline{B_\delta(F_i)}$. Then, ε_i is a continuous map from $\overline{B_\delta(F_i)}$ to the disk of radius ϵ in the complex plane. By Lemma 2.6, ε_i can be extended to a continuous function from Y to the complex plane such that $\|\varepsilon_i\| \leqslant \epsilon$ (let us also denote this extension map by ε_i). For $m < i \leqslant n$, set $\varepsilon_i = 0$.

For each $i \in \{1, \ldots, n\}$, put $g_i = f \circ \lambda_i - \epsilon_i$, so that $g_i \in \mathrm{C}(X)$. Set

$$g = \mathrm{diag}(g_1, g_2, \ldots, g_n).$$

Then, for each $i \in \{1, \ldots, m\}$ and $y \in \overline{B_\delta(F_i)}$, we have

$$g_i(y) = f(x_0);$$

if $i \in \{m+1, \ldots, n\}$, then $g_i = f \circ \lambda_i$. For any unitary $u \in \mathrm{M}_n(\mathrm{C}(Y))$ we have

$$\left\| u\phi(f)u^* - ugu^* \right\| = \left\| \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) \right\| < 2\epsilon.$$

We have therefore reduced our problem to proving the following claim.

Claim. *There is a unitary u in $\mathrm{M}_m(\mathrm{C}(Y)) \oplus 1_{n-m}$ such that*

$$ugu^* = \begin{pmatrix} f(x_0) & 0 & 0 & \dots & 0 \\ 0 & b(x) & 0 & \dots & 0 \\ 0 & 0 & g_{m+1}(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_n(x) \end{pmatrix}, \quad \forall x \in \bigcup_{i=1}^{m} F_i,$$

where $b \in \mathrm{M}_{n-m-1}(\mathrm{C}(Y))$.

Proof. We will assume that for some $1 \leqslant k < m$ there is a unitary $u_k \in \mathrm{M}_k(\mathrm{C}(Y)) \oplus 1_{n-k}$ such that

$$u_k g u_k^* = \begin{pmatrix} f(x_0) & 0 & 0 & \dots & 0 \\ 0 & b(x) & 0 & \dots & 0 \\ 0 & 0 & g_{k+1}(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_n(x) \end{pmatrix}, \quad \forall x \in \bigcup_{i=1}^{k} F_i, \tag{3}$$

and then prove that the same statement holds with k replaced by $k+1$. Since (3) clearly holds when $k = 1$—just take u_1 to be the identity matrix of $\mathrm{M}_n(\mathrm{C}(Y))$—this recursive argument will prove our claim. □

Assume that (3) holds for some $k < m$. Put $B = F_{k+1}$ and $A = Y \setminus B_\delta(B)$. Apply Lemma 2.5 with these choices of A and B and with $\sigma = (1\ k+1)$ to obtain a unitary $v \in \mathrm{M}_n(\mathrm{C}(Y))$. We then have that $v = 1_n$ on A and $v = U[(1\ k+1)]$ on B. Inspecting the construction of v, we find that it has the following form:

$$v(y) = U\big[(2\ k+1)\big] \begin{pmatrix} v'(y) & 0 \\ 0 & 1 \end{pmatrix} U\big[(2\ k+1)\big], \quad \forall y \in Y, \tag{4}$$

where $v'(y)$ is a unitary matrix in $\mathrm{M}_2(\mathrm{C}(Y))$ equal to 1_2 on A and equal to $U[(12)]$ on B. Define $u_{k+1} := vu_k$.

Let us show that u_{k+1} satisfies the requirements of the claim. It is clear that u_{k+1} is an element of $\mathrm{M}_{k+1}(\mathrm{C}(Y)) \oplus 1_{n-k-1}$. First suppose that $y \in B$, so that $g_{k+1}(y) = f(x_0)$. Since $u_{k+1} = vu_k$ we have

$$u_{k+1}(y)g(y)u_{k+1}^*(y) = U\big[(1\ k+1)\big] \begin{pmatrix} c(y) & 0 & \dots & 0 \\ 0 & f(x_0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n(y) \end{pmatrix} U\big[(1\ k+1)\big]$$

$$= \begin{pmatrix} f(x_0) & 0 & 0 & \dots & 0 \\ 0 & b(y) & 0 & \dots & 0 \\ 0 & 0 & g_{k+2}(y) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_n(y) \end{pmatrix}$$

for some $c(y), b(y) \in \mathrm{M}_k$.

Now suppose that $y \in \bigcup_{i=1}^m F_i \setminus B_\delta(B) \subseteq Y \setminus B_\delta(B)$. In this case $v(y) = 1_n$ and $u_{k+1}(y) = u_k(y)$ and there is nothing to prove.

Finally, suppose that $y \in (B_\delta(B) \setminus B) \cap (\bigcup_{i=1}^m F_i)$. As in the case $y \in B$, we have $g_{k+1}(y) = f(x_0)$. From (3) and this last fact we have

$$U\big[(2\,k+1)\big]u_k(y)g(y)u_k^*(y)U\big[(2\,k+1)\big] = \begin{pmatrix} f(x_0) & 0 & 0 & 0 & \dots & 0 \\ 0 & f(x_0) & 0 & 0 & \dots & 0 \\ 0 & 0 & d(y) & 0 & \dots & 0 \\ 0 & 0 & 0 & g_{k+2}(y) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & g_n(y) \end{pmatrix}$$

for some $d(y) \in M_{k-1}$. Since the upper left 2×2 corner of the matrix above is scalar, the entire matrix commutes with $v'(y) \oplus 1_{n-2}$. It follows that $u_{k+1}(y)g(y)u_{k+1}^*(y)$ is equal to

$$U\big[(2\,k+1)\big] \begin{pmatrix} f(x_0) & 0 & 0 & 0 & \dots & 0 \\ 0 & f(x_0) & 0 & 0 & \dots & 0 \\ 0 & 0 & b_1(y) & 0 & \dots & 0 \\ 0 & 0 & 0 & g_{k+2}(y) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & g_n(y) \end{pmatrix} U\big[(2\,k+1)\big].$$

Computing this product yields a matrix of the form

$$\begin{pmatrix} f(x_0) & 0 & 0 & \dots & 0 \\ 0 & b(y) & 0 & \dots & 0 \\ 0 & 0 & g_{k+2}(y) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & g_n(y) \end{pmatrix},$$

as required. □

For any C*-algebra B there is an isomorphism

$$\pi : B \otimes \mathrm{M}_n \to \mathrm{M}_n(B)$$

given by

$$\pi\big(b \otimes (a_{ij})\big) = \begin{pmatrix} ba_{11} & \cdots & ba_{1n} \\ \vdots & \ddots & \vdots \\ ba_{n1} & \cdots & ba_{nn} \end{pmatrix}. \tag{5}$$

Proposition 3.2. *Suppose that $\phi : \mathrm{M}_m(\mathrm{C}(X)) \to \mathrm{M}_{nm}(\mathrm{C}(Y))$ is a diagonal $*$-homomorphism with $\mathrm{ep}(\phi) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $\tilde{\phi} : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ be the diagonal $*$-homomorphism given by*

$$\tilde{\phi}(f) = \begin{pmatrix} f \circ \lambda_1 & 0 & \ldots & 0 \\ 0 & f \circ \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f \circ \lambda_n \end{pmatrix}.$$

Then, $\tilde{\phi} \otimes \mathbf{id}_{\mathrm{M}_m} : \mathrm{C}(X) \otimes \mathrm{M}_m \to \mathrm{M}_n(\mathrm{C}(Y)) \otimes \mathrm{M}_m$ is unitarily equivalent to ϕ.

Proof. On the one hand we have

$$\tilde{\phi} \otimes \mathbf{id}_{\mathrm{M}_m}\big(f \otimes (c_{ij})\big) = \begin{pmatrix} f \circ \lambda_1 & 0 & \ldots & 0 \\ 0 & f \circ \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f \circ \lambda_n \end{pmatrix} \otimes (c_{ij}),$$

while on the other we have

$$\phi\big(f \otimes (c_{ij})\big) = \begin{pmatrix} (f \circ \lambda_1) \otimes (c_{ij}) & 0 & \ldots & 0 \\ 0 & (f \circ \lambda_2) \otimes (c_{ij}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (f \circ \lambda_n) \otimes (c_{ij}) \end{pmatrix}.$$

With the identifications $\mathrm{C}(X) \otimes \mathrm{M}_n \cong \mathrm{M}_n(\mathrm{C}(X))$ and $\mathrm{M}_n(\mathrm{C}(Y)) \otimes \mathrm{M}_m \cong \mathrm{M}_{nm}(\mathrm{C}(Y))$ given by (5) in mind, one sees that

$$\mathrm{Ad}\big(U[\pi]\big) \circ (\tilde{\phi} \otimes \mathbf{id}_{\mathrm{M}_m}) = \phi,$$

where π is the permutation in S_{nm} and given by $\pi(kn + i) = (i - 1)m + k + 1$ for $k = 0, 1, 2, \ldots, m - 1$ and $i = 1, 2, \ldots, n$. $\square$

Corollary 3.3. *Let $\phi : \mathrm{M}_m(\mathrm{C}(X)) \to \mathrm{M}_{nm}(\mathrm{C}(Y))$ be a diagonal $*$-homomorphism with eigenvalue pattern $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, and let $\epsilon > 0$ be given. Let f be any element of $\mathrm{M}_m(\mathrm{C}(X))$ and choose $\eta > 0$ such that*

$$\big\| f(x) - f(y) \big\| < \epsilon \quad \textit{whenever } d(x, y) < 2\eta$$

(d is the metric on X). Let U be the open ball centered at $x_0 \in X$ with radius η, and suppose that

$$Y = \lambda_1^{-1}(U) \cup \lambda_2^{-1}(U) \cup \cdots \cup \lambda_n^{-1}(U).$$

Then there is a unitary u in $\mathrm{M}_{nm}(\mathrm{C}(Y))$ and element $b \in \mathrm{M}_{nm-m}(\mathrm{C}(Y))$ such that

$$\left\| u\phi(f)u^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b \end{pmatrix} \right\| < \epsilon.$$

Proof. By Proposition 3.2, we can assume that $m = 1$. Let F_i be the closure of $\lambda_i^{-1}(U)$ for each i. Now, apply Theorem 3.1. Since $Y = \bigcup_{i=1}^{n} F_i$, we are done. □

Now, we are ready to prove the main theorem of this section.

Theorem 3.4. *Let $A = \varinjlim(A_i, \phi_i)$ be a unital diagonal AH algebra. Then, A is simple if and only if A has the property $\mathcal{P}$ of Definition* 2.2.

Proof. Suppose that A has property $\mathcal{P}$. Let $f \in A_i$ be nonzero, so that there is a point x_0 in X_i such that $f(x_0) \neq 0$. By the definition of property $\mathcal{P}$, there are an integer $j > i$ and unitaries $u_l \in A_{j,l}$, $l \in \{1, \ldots, k_j\}$ such that

$$\left\| u_l \phi_{i,j}^{t,l}(f) u_l^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_l \end{pmatrix} \right\| < \epsilon$$

for some appropriately sized b_l. We may assume that $\epsilon < \| f(x_0) \|$, so that $\phi_{ij}(f)$ is nowhere zero. This implies that the ideal of A_j generated by $\phi_{ij}(f)$ is all of A_j, and that the ideal of A generated by the image of f is all of A. Since f was arbitrary, A is simple.

Now assume that A is simple, and let $f \in A_{i,t}$ be nonzero. Recall that the ϕ_i may be taken to be injective. By Proposition 2.3 there exists, for each $x_0 \in X_{i,t}$ and $\epsilon > 0$, a $j_0 > i$ with the following property: for every $j \geqslant j_0$ and every $l \in \{1, \ldots, k_j\}$ we have

$$\mathrm{ep}^{-1}\big(\phi_{i,j}^{t,l}\big)\big(B_\delta(x_0)\big) = X_{j,l},$$

where δ is some positive number such that

$$d\big(f(x), f(y)\big) < \epsilon \quad \text{whenever } d(x, y) < 2\delta.$$

As pointed out in Section 2, the map $\phi_{i,j}^{t,l}$ may be viewed as a diagonal map from $A_{i,t}$ into the cut-down of $A_{j,l}$ by the projection $\phi_{i,j}^{t,l}(1)$; any unitary u in this corner of $A_{j,l}$ gives rise to a unitary $\tilde{u}$ in $A_{j,l}$ by setting $\tilde{u} = u \oplus (1_{A_{j,l}} - \phi_{i,j}^{t,l}(1)$. Combining this observation with Corollary 3.3 we see that there exists, for each $l \in \{1, \ldots, k_j\}$, a unitary $u_l \in A_{j,l}$ such that

$$\left\| u_l \phi_{i,j}^{t,l}(f) u_l^* - \begin{pmatrix} f(x_0) & 0 \\ 0 & b_l \end{pmatrix} \right\| < \epsilon.$$

Thus, A has property $\mathcal{P}$, as desired. □

4. Stable rank

Theorem 4.1. *Let $A = \varinjlim(A_i, \phi_i)$ be a simple unital diagonal AH algebra. Then, A has stable rank one.*

Before proving Theorem 4.1, we situate it relative to other results on the stable rank of general approximately homogeneous (AH) algebras. Recall that an AH algebra is an inductive limit C*-algebra $A = \lim_{i\to\infty}(A_i, \phi_i)$, where

$$A_i = \bigoplus_{l=1}^{n_i} p_{i,l}\big(\mathrm{C}(X_{i,l}) \otimes \mathcal{K}\big)p_{i,l} \tag{6}$$

for compact connected Hausdorff spaces $X_{i,l}$, projections $p_{i,l} \in \mathrm{C}(X_{i,l}) \otimes \mathcal{K}$, and natural numbers n_i. If A is separable, then one may assume that the $X_{i,l}$ are finite CW-complexes [1,11]. The inductive system (A_i, ϕ_i) is referred to as a *decomposition* for A. All AH algebras in this paper are assumed to be separable.

If an AH algebra A admits a decomposition as in (6) for which

$$\max_{1\leqslant l\leqslant n_i}\left\{\frac{\dim(X_{i,1})}{\mathrm{rank}(p_{i,1})}, \ldots, \frac{\dim(X_{i,n_i})}{\mathrm{rank}(p_{i,n_i})}\right\} \overset{i\to\infty}{\longrightarrow} 0,$$

then we say that A has *slow dimension growth*. Theorem 1 of [2] states that every simple unital AH algebra with slow dimension growth has stable rank one. Villadsen in [17] constructed simple diagonal AH algebras which do not have slow dimension growth, but which do have stable rank one; the converse of [2, Theorem 1] does not hold. There are in fact a wealth of simple diagonal AH algebras without slow dimension growth which exhibit all sorts of interesting behaviour (cf. [14–16]), whence Theorem 4.1 is widely applicable.

Simple AH algebras may have stable rank strictly greater than one, and there is reason to believe that Theorem 4.1 is quite close to being best possible. One might be able to generalise our result to the setting of AH algebras where the projections $\phi_{i,j}(p_{i,l})$ appearing in (6) can be decomposed into a direct sum of a trivial projection θ_j and a second projection q_j such that $\tau(q_j) \to 0$ as $j \to \infty$ for any trace τ. Otherwise, one finds oneself in a situation very similar to the construction of [18], where the stable rank is always strictly greater than one.

Let us now prepare for the proof of Theorem 4.1.

Lemma 4.2. *Let $a \in \mathrm{M}_m(\mathrm{C}(X))$ be block diagonal, i.e.,*

$$a = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & a_1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \ldots & a_n \end{pmatrix},$$

where $a_i \in \mathrm{M}_{k_i}(\mathrm{C}(X))$ for natural numbers $k_1, \ldots, k_n$. If the size of the matrix 0 in the upper left-hand corner of a is strictly greater than $\max_{1\leqslant i\leqslant n} k_i$, then a can be approximated arbitrarily closely by invertible elements in $\mathrm{M}_m(\mathrm{C}(X))$.

Proof. Let $\epsilon > 0$ be given, and let k denote the size of the matrix 0 in the upper left-hand corner of a. Since $k > k_i$, $i \in \{1, \dots, n\}$, there is a permutation matrix U such that

$$Ua = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

is nilpotent. As was proved in [13], every nilpotent element in a unital C*-algebra can be approximated arbitrarily closely by invertible elements. We may thus find an invertible element $b \in \mathrm{M}_m(\mathrm{C}(X))$ such that $\|Ua - b\| < \epsilon$, and

$$\|a - U^{-1}b\| = \|U^{-1}Ua - U^{-1}b\| \leqslant \|U^{-1}\| \cdot \|Ua - b\| < \epsilon.$$

The lemma now follows from the fact that $U^{-1}b$ is invertible. □

It easy to prove that $a \in \mathrm{M}_n(\mathrm{C}(X))$ is invertible if and only if $a(x)$ is invertible for each $x \in X$. The proof of the next lemma is also straightforward.

Lemma 4.3. *Let p, q be orthogonal projections in a C*-algebra A, and let $\epsilon > 0$ be given. If elements a and b in A can be approximated to within ϵ by invertible elements in pAp and qAq, respectively, then $a + b$ can be approximated to within ϵ by an invertible element in $(p+q)A(p+q)$.*

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since every element in A can be approximated arbitrarily closely by elements in $\bigcup_{i=1}^{\infty} A_i$, it will suffice to prove for any $\epsilon > 0$ and any $a \in A_i$, there is an invertible element in A whose distance to a is less than ϵ. (Note that we are using the injectivity of the ϕ_i to identify A_i with its image in A.)

By Lemma 4.3, we may assume that $A_i = \mathrm{M}_{n_i}(\mathrm{C}(X_i))$. We also assume that a is not invertible. By the comment preceding Lemma 4.3, there is a point $x_0 \in X_i$ such that $\det(a(x_0)) = 0$. There are permutation matrices $u, v \in \mathrm{M}_{n_i}$ and a matrix $c \in \mathrm{M}_{n_i - 1}$ such that

$$ua(x_0)v = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Let b denote the element uav. Following the lines of the proof of Lemma 4.2, it will suffice to prove that b can be approximated to within ϵ by an invertible element of A.

For each $j > i$, $\phi_{i,j}(b)$ is a tuple of k_j elements. If each coordinate of $\phi_{i,j}(b)$ can be approximated to within ϵ by an invertible element in the corner of A generated by the unit of $A_{j,l}$, then $\phi_{i,j}(b)$ can be approximated to within ϵ by an invertible element of A. We may therefore assume that $A_j = \mathrm{M}_{n_j}(\mathrm{C}(Y_j))$, and concern ourselves with proving that $\phi_{i,j}(b)$ is approximated to within ϵ by an invertible element in A.

By Theorem 3.4, there exist an integer $j > i$, a unitary $w \in A_j$ and an element b' such that

$$\left\| w\phi_{ij}(b)w^* - \begin{pmatrix} b(x_0) & 0 \\ 0 & b' \end{pmatrix} \right\| < \epsilon/2.$$

We have

$$\begin{pmatrix} b(x_0) & 0 \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b'' \end{pmatrix},$$

where $b'' = \mathrm{diag}(c, b')$. Put $d = w\phi_{i,j}(b)w^*$, and note that it will suffice to prove that $\phi_{j,m}(d)$ is approximated to within $\epsilon/2$ by an invertible element in A_m for some $m > j$.

Since A is simple, there is an integer $m > j$ large enough so that, for each $t \in \{1, \dots, k_m\}$, either the number of the eigenvalue maps of $\phi_{j,m}^{1,t}$ counted with multiplicity is strictly larger than the size of the matrix b'', or the image of $\phi_{j,m}^{1,t}$ is finite-dimensional. In the latter case, $\phi_{j,m}^{1,t}(d)$ is approximated to within ϵ by an invertible element in the image of $\phi_{j,m}^{1,t}$ since finite-dimensional C*-algebras have stable rank one, so we may assume that the number of eigenvalue maps of $\phi_{j,m}^{1,t}$, counted with multiplicity, is strictly larger than the size of the matrix b''. Then, $\phi_{j,m}^{1,t}(d)$ is unitarily equivalent to

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & b_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_l \end{pmatrix}$$

inside $A_{m,t}$, where b_k is the composition of b'' and the kth eigenvalue map of $\phi_{j,m}^{1,t}$, and the size of the matrix 0 at the upper left-hand corner of the above matrix is strictly bigger than the size of the matrix b_k for every k. By Lemma 4.2, the matrix above can be approximated to within $\epsilon/2$ by an invertible in $A_{m,t}$, as required. □

Corollary 4.4. *Let A be a simple unital diagonal AH algebra. If A has real rank zero and weakly unperforated* K_0*-group, then A is tracially AF.*

Proof. By Theorem 4.1, A has stable rank one. The corollary then follows from a result of Lin [12, Theorem 2.1]. □

Corollary 4.4 applies, for instance, to simple unital diagonal AH algebras for which the spaces X_i in some diagonal decomposition for A are all contractible. This contractibility hypothesis may seem strong, but it does not substantially restrict the complexity of A; if one wants to classify all such A via K-theory and traces, then the additional assumption of very slow dimension growth and the full force of [8] and [9] are required; the collection of all such cannot be classified by topological K-theory and traces alone [15].

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References

[1] B. Blackadar, Matricial and ultramatricial topology, in: Operator Algebras, Mathematical Physics, and Low-Dimensional Topology, Istanbul, 1991, in: Res. Notes Math., vol. 5, pp. 11–38.
[2] B. Blackadar, M. Dadarlat, M. Rordam, The real rank of inductive limit C*-algebras, Math. Scand. 69 (1991) 211–216.
[3] M. Dadarlat, Reduction to dimension three of local spectra of real rank zero C*-algebras, J. Reine Angew. Math. 460 (1995) 189–212.
[4] M. Dadarlat, G. Nagy, A. Nemethi, C. Pasnicu, Reduction of topological stable rank in inductive limits of C*-algebras, Pacific J. Math. 153 (1992) 267–276.
[5] G.A. Elliott, On the classification of C*-algebras of real rank zero, J. Reine Angew. Math. 443 (1993) 179–219.
[6] G.A. Elliott, D.E. Evans, The structure of irrational rotation C*-algebras, Ann. of Math. (2) 138 (1993) 477–501.
[7] G.A. Elliott, G. Gong, On the classification of C*-algebras of real rank zero. II, Ann. of Math. (2) 144 (3) (1996) 497–610.
[8] G.A. Elliott, G. Gong, L. Li, On the classification of simple inductive limit C*-algebras, II. The isomorphism theorem, Invent. Math., in press.
[9] G. Gong, On the classification of simple inductive limit C*-algebras I. The reduction theorem, Doc. Math. 7 (2002) 255–461.
[10] K.R. Goodearl, Notes on a class of simple C*-algebras with real rank zero, Publ. Mat. 36 (2A) (1992) 637–654.
[11] K.R. Goodearl, Riesz decomposition in inductive limit C*-algebras, Rocky Mountain J. Math. 24 (1994) 1405–1430.
[12] H. Lin, Simple AH algebras of real rank zero, Proc. Amer. Math. Soc. 131 (12) (2003) 3813–3819.
[13] M. Rordam, On the structure of simple C*-algebras tensored with a UHF-algebra, J. Funct. Anal. 100 (1991) 1–17.
[14] A. Toms, Flat dimension growth for C*-algebras, J. Funct. Anal. 238 (2006) 678–708.
[15] A. Toms, On the classification problem for nuclear C*-algebras, Ann. of Math. (2), in press.
[16] A. Toms, An infinite family of non-isomorphic C*-algebras with identical K-theory, preprint, arXiv: math.OA/0609214, 2006.
[17] J. Villadsen, Simple C*-algebras with perforation, J. Funct. Anal. 154 (1998) 110–116.
[18] J. Villadsen, On the stable rank of simple C*-algebras, J. Amer. Math. Soc. 12 (1999) 1091–1102.