Canad. Math. Bull. Vol. **46** (3), 2003 pp. 457–472

Strongly Perforated K_0-Groups of Simple C^*-Algebras

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Abstract. In the sequel we construct simple, unital, separable, stable, amenable C^*-algebras for which the ordered K_0-group is strongly perforated and group isomorphic to Z. The particular order structures to be constructed will be described in detail below, and all known results of this type will be generalised.

1 Statement of the Main Result

Theorem 1.1 *Suppose that for $i \in \{1, \dots, N\}$, q_i and m_i are relatively prime positive integers with q_i prime. Let L be a positive integer coprime with each q_i and m_i. Define*

$$S \equiv \frac{1}{L}\Big(\bigcap_{i=1}^{N}\langle q_i, m_i\rangle\Big) \cap Z,$$

where $\langle q_i, m_i\rangle$ denotes the subsemigroup of the positive integers consisting of non-negative integral linear combinations of q_i and m_i.

It follows that there exists a simple, separable, amenable, unital C^*-algebra with ordered K_0-group order isomorphic to the integers with positive cone S.

It is not known whether the subsemigroups of the positive integers constructed as above exhaust all of the subsemigroups of the positive integers that generate Z, but they do include subsemigroups of the form $\langle m, l\rangle$, where m and l are any two coprime positive integers, amongst others.

2 Background and Essential Results

We begin by reviewing the definition of the generalised mapping torus. Unless otherwise noted, all results from this section can be found in [E-V]. Let C, D be C^*-algebras and let ϕ_0, ϕ_1 be $*$-homomorphisms from C to D. Then the generalised mapping torus of C and D with respect to ϕ_0 and ϕ_1 is

$$A := \{(c, d) \mid d \in C([0, 1]; D), c \in C, d(0) = \phi_0(c), d(1) = \phi_1(c)\} \tag{1}$$

We will denote A by $A(C, D, \phi_0, \phi_1)$ where appropriate for clarity. We now list (without proof) some theorems which will be used in the sequel.

Received by the editors November 1, 2001.
AMS subject classification: 46, 19.

Theorem 2.1 *The index map $b_*\colon K_*C \to K_{1-*}SD = K_*D$ in the six-term periodic exact sequence for the extension*

$$0 \to SD \to A \to C \to 0$$

is the difference

$$K_*\phi_1 - K_*\phi_0\colon K_*C \to K_*D$$

Thus, the six-term exact sequence may be written as the short exact sequence

$$0 \to \text{Coker } b_{1-*} \to K_*A \to \text{Ker } b_* \to 0$$

In particular, if b_{1-i} is surjective, then K_iA is isomorphic to its image, Ker b_i, in K_iC.

Suppose that cancellation holds for D. It follows that if b_1 is surjective, so that $K_0A \subseteq K_0C$, then

$$(K_0A)^+ = (K_0C)^+ \cap K_0A.$$

The preceding conclusion also holds if cancellation is only known to hold for each pair of projections in $D \otimes K$ obtained as the images under the maps ϕ_0 and ϕ_1 of a single projection in $C \otimes K$.

Theorem 2.2 *Let A_1 and A_2 be building block algebras as described above,*

$$A_i = A(C, D, \phi_0^i, \phi_1^i), \quad i = 1, 2.$$

Let there be given four maps between the fibres,

$$\gamma\colon C_1 \to C_2,$$
$$\delta, \delta'\colon D_1 \to D_2, \quad \textit{and},$$
$$\epsilon\colon C_1 \to D_2,$$

such that δ, δ' and ϵ have mutually orthogonal images, and

$$\delta\phi_0^1 + \delta'\phi_1^1 + \epsilon = \phi_0^2\gamma,$$
$$\delta\phi_1^1 + \delta'\phi_0^1 + \epsilon = \phi_1^2\gamma.$$

Then there exists a unique map

$$\theta\colon A_1 \to A_2,$$

respecting the canonical ideals, giving rise to the map $\gamma\colon C_1 \to C_2$ between the quotients (or fibres at infinity), and such that for any $0 < s < 1$, if e_s denotes evaluation at s, and e_∞ the evaluation at infinity,

$$e_s\theta = \delta e_s + \delta' e_{1-s} + \epsilon e_\infty.$$

Theorem 2.3 *Let A_1 and A_2 be building block algebras as in Theorem 2. Let $\theta\colon A_1 \to A_2$ be a homomorphism constructed as in Theorem 2.2, from maps $\gamma\colon C_1 \to C_2$, δ, $\delta'\colon D_1 \to D_2$, and $\epsilon\colon C_1 \to D_2$.*

Let there be given a map $\beta\colon D_1 \to C_2$ such that the composed map $\beta\phi_1^1$ is a direct summand of the map γ, and such that the composed maps $\phi_0^2\beta$ and $\phi_1^2\beta$ are direct summands of the maps δ' and δ, respectively. Suppose that the decomposition of γ as the orthogonal sum of $\beta\phi_1^1$ and another map is such that the image of the second map is orthogonal to the image of β. (Note that this requirement is automatically satisfied if C_1, D_1, and the map $\beta\phi_1^1$ are unital.)

It follows that, for any $0 < t < \frac{1}{2}$, the map $\theta\colon A_1 \to A_2$ is homotopic to a map $\theta_t\colon A_1 \to A_2$ differing from it only as follows: the map $e_\infty\theta_t$ has the direct summand βe_t instead of one of the direct summands $\beta\phi_0^1 e_\infty$ and $\beta\phi_1^1 e_\infty$ of $e_\infty\theta$, and for each $0 < s < 1$ the map $e_s\theta_t$ has either the direct summand $\phi_0^2\beta e_t$ instead of the direct summand $\phi_0^2\beta e_s$ of $e_s\theta$, or the direct summand $\phi_1^2\beta e_t$ instead of the direct summand $\phi_1^2\beta e_s$ of $e_s\theta$, or both.

Furthermore, let $\alpha\colon D_1 \to C_2$ be any map homotopic to β within the hereditary sub-C^*-algebra of C_2 generated by the image of β. Then the map θ_t is homotopic to a map $\theta_t'\colon A_1 \to A_2$ differing from θ_t only in the direct summands mentioned, and such that $e_\infty\theta_t'$ has the direct summand αe_t instead of βe_t, and for each $0 < s < 1$, $e_s\theta_t'$ has either $\phi_0^2\alpha e_t$ instead of $\phi_0^2\beta e_t$, or $\phi_1^2\alpha e_t$ instead of $\phi_1^2\beta e_t$.

Theorem 2.4 *Let*

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

be a sequence of separable building block C^*-algebras,

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2, \ldots$$

with each map $\theta_i\colon A_i \to A_{i+1}$ obtained by the construction of Theorem 2.2 (and thus respecting the canonical ideals). For each $i = 1, 2, \ldots$ let $\beta_i\colon D_i \to C_{i+1}$ be a map verifying the hypotheses of Theorem 2.3.

Suppose that for every $i = 1, 2, \ldots$, the intersection of the kernels of the boundary maps ϕ_0^i and ϕ_1^i from C_i to D_i is zero.

Suppose that, for each i, the image of each of ϕ_0^{i+1} and ϕ_1^{i+1} generates D_{i+1} as a closed two-sided ideal, and that this is in fact true for the restriction of ϕ_0^{i+1} and ϕ_1^{i+1} to the smallest direct summand of C_{i+1} containing the image of β_i. Suppose that the closed two-sided ideal of C_{i+1} generated by the image of β_i is a direct summand.

Suppose that, for each i, the maps $\delta_i' - \phi_0^i\beta_i$ and $\delta_i - \phi_1^i\beta_i$ from D_i to D_{i+1} are injective.

Suppose that, for each i, the map $\gamma_i - \beta_i\phi_1^i$ takes each non-zero direct summand of C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal.

Suppose that, for each i, the map $\beta_i\colon D_i \to C_{i+1}$ can be deformed—inside the hereditary sub-C^*-algebra generated by its image—to a map $\alpha_i\colon D_i \to C_{i+1}$ with the following property: There is a direct summand of α_i, say $\bar{\alpha}_i$, such that $\bar{\alpha}_i$ is non-zero on an arbitrary given element x_i of D_i, and has image a simple sub-C^*-algebra of C_{i+1}, the closed two-sided ideal generated by which contains the image of β_i.

Choose a dense sequence (t_n) *in the open interval* $(0, \frac{1}{2})$*, such that* $t_{2n} = t_{2n-1}$*,* $n = 1, 2, \ldots$.

Choose a sequence of elements $x_3 \in D_3, x_5 \in D_5, x_7 \in D_7, \ldots$ *(necessarily non-zero) with the following property: For some countable basis for the topology of the spectrum of each of* $D_1, D_2, \ldots$*, and for some choice of non-zero element of the closed two-sided ideal associated to each of these (non-empty) open sets, under successive application of the maps* $\delta_i - \phi_1^{i+1}\beta_i$ *each one of these elements is taken into* x_j *for all* j *in some set* $S \subseteq \{3, 5, 7, \ldots\}$ *such that* $\{t_j, j \in S\}$ *is dense in* $(0, \frac{1}{2})$*. Choose* α_j *as above such that* $\bar{\alpha}_j(x_j) \neq 0$ *for some direct summand* $\bar{\alpha}_j$ *of* α_j *for each* $j \in \{3, 5, 7, \ldots\}$*. For each* $j \in \{4, 6, 8, \ldots\}$ *choose* α_j *with respect to the non-zero element* $(\delta'_{j-1} - \phi_0^j\beta_{j-1})(x_{j-1})$ *of* D_j*. (If* $j = 1$ *or* 2*, choose* $\alpha_j = \beta_j$*.)*

It follows that, if θ'_i *denotes the deformation of* θ_i *constructed in Theorem 4, with respect to the point* $t_i \in (0, \frac{1}{2})$ *and the maps* α_i *and* β_i *(and a fixed homotopy of* β_i *to* α_i*), then the inductive limit of the sequence*

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_2} \cdots$$

is simple.

3 The Main Result

In this section we will apply the theorems of Section 2 to the problem of constructing simple, stable, separable, amenable C^*-algebras having specific ordered K_0-groups. The algebras to be constructed will all be stably finite, thus allowing us to refer unambiguously to the ordered (as opposed to pre-ordered) K_0-group [B].

Consider the subsemigroup S of the positive integers given by

$$S = \frac{1}{L}\Big(\bigcap_{i=1}^{N}\langle q_i, m_i\rangle\Big) \cap Z$$

where m_i and q_i are coprime positive integers for each i, q_i is prime, L is any positive integer coprime to each q_i and m_i, Z is the integers, $\langle q_i, m_i\rangle$ is the additive subsemigroup of the positive integers generated by q_i and m_i, and $\frac{1}{L}(\bigcap_{i=1}^{N}\langle q_i, m_i\rangle)$ is the set of rational numbers with denominator L and numerator an element of the set $\bigcap_{i=1}^{N}\langle q_i, m_i\rangle$. Examples of subsemigroups of the positive integers which can be constructed in this manner include $\langle k, l\rangle$, where k and l are any coprime positive integers.

Let us construct a sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

with $A_j = (C_j, D_j, \phi_0^j, \phi_1^j)$ as in Section 2, and with θ_j constructed as in Theorem 2.2 from maps

$$\gamma_j \colon C_i \to C_{j+1}, \quad \delta_j, \delta'_j \colon D_j \to D_{j+1}.$$

In order to deform the θ_j to obtain a simple limit, we wish to have a map

$$\beta_j : D_j \to C_{j+1}$$

with the properties specified in Theorem 2.4.

We begin by specifying the algebras C_j to be used in the construction of the building blocks. For each $i \in \{1, \ldots, N+1\}$ let $X_{i,1}$ be a compact metrizable space, and let $X_{i,j}$ be the Cartesian product of n_{j-1} copies of $X_{i,j-1}$, with the n_j to be specified. For each $j \in \{1, 2, \ldots\}$ let Y_j be the disjoint union of the $X_{i,j}$, $i \in \{1, \ldots, N+1\}$. For each j let

$$C_j = p_j \big(C(Y_j) \otimes K \big) p_j$$

where p_j is a projection in $C(Y_j) \otimes K$. In the sequel we will specify p_1 and set $p_j = \gamma_{j-1}(p_{j-1})$. Let $p_{i,j}$ be the restriction of p_j to the component $X_{i,j}$ of Y_j. Setting $C_{i,j} = p_{i,j}\big(C(X_{i,j}) \otimes K\big)\, p_{i,j}$ we can write $C_j = \bigoplus_{i=1}^{N+1} C_{i,j}$. K is the C^*-algebra of compact operators on an infinite-dimensional separable Hilbert space.

Let $D_j = \bigoplus_{i=1}^{N+1} (C_{i,j} \otimes M_{(N+1)k_j \dim(p_{i,j})})$, here k_j is a non-zero positive integer to be specified. Let $(\dim(p_j))$ be the ordered $N+1$-tuple $\big(\dim(p_{1,j}), \ldots, \dim(p_{N+1,j})\big)$. In the sequel we will choose p_j so that $\dim(p_{i,j}) = \dim(p_{k,j}), \forall i, k \in \{1, \ldots, N+1\}$, and will denote this quantity by $\dim(p_j)$. D_j can then be written as $C_j \otimes M_{(N+1)k_j \dim(p_j)}$.

For each $i \in \{1, \ldots, N+1\}$ we will specify two maps $\phi_j^{0,i}$ and $\phi_j^{1,i}$ from C_j to $C_j \otimes M_{k_j \dim(p_j)}$, and set $\phi_j^t = \bigoplus_{i=1}^{N+1} \phi_j^{t,i}$, $t = 0, 1$.

Let $\mu_{i,j}$ and $\nu_{i,j}$ be maps from C_j to $C_j \otimes M_{\dim(p_j)}$ as follows:

$$\mu_{i,j}(a) = p_j \otimes a(x_{i,j}) \cdot 1_{\dim(p_j)}$$

(where $x_{i,j}$ is a point in $X_{i,j}$ to be specified and $1_{\dim(p_j)}$ is the unit of the $C_j \otimes M_{\dim(p_j)}$) and

$$\nu_{i,j}(a) = a \otimes 1_{\dim(p_j)}.$$

Let $\phi_j^{t,i}$ be the direct sum of l_j^t and $k_j - l_j^t$ copies of $\mu_{i,j}$ and $\nu_{i,j}$, respectively, where the l_j^t are non-negative integers such that $l_j^0 \neq l_j^1$. We will also require that $l_j^1 - l_j^0$ be coprime with each of the q_i. Then $\phi_j^{t,i}$ is a map from C_j to $C_j \otimes M_{k_j \dim(p_j)}$, as desired. In this manner ϕ_j^t is specified only up to the order of its direct summands, but it is only necessary to specify ϕ_j^t up to unitary equivalence (*i.e.*, up to composition with an inner automorphism). In the sequel we shall, in fact, modify the ϕ_j^t by inner automorphisms at each stage.

Note that C_j and D_j are both unital. The maps ϕ_j^t are unital since $\mu_{i,j}(1) = p_j \otimes 1_{\dim(p_j)}$ and $\nu_{i,j}(1) = \nu_{i,j}(p_j) = p_j \otimes 1_{\dim(p_j)}$. They are also injective as $a \neq b \Rightarrow \nu_{i,j}(a) \neq \nu_{i,j}(b)$.

By Theorem 2.1, for each $e \in K_0(C_j)$,

$$\begin{aligned} b_0(e) &= (l_j^1 - l_j^0)\Big(\sum_{i=1}^{N+1}\big(K_0(\mu_{i,j}) - K_0(\nu_{i,j})\big)\Big)(e) \\ &= (l_j^1 - l_j^0)\Big(\sum_{i=1}^{N+1}(\dim(e_i)\cdot K_0(p_j) - \dim(p_j)\cdot e)\Big) \\ &= (l_j^1 - l_j^0)\bigg(\Big(\sum_{i=1}^{N+1}\dim(e_i)\Big)\cdot K_0(p_j) - (N+1)\dim(p_j)\cdot e\bigg) \end{aligned}$$

where $\dim(e_i)$ denotes the dimension of e over $X_{i,j}$. Since $l_j^1 - l_j^0$ is a non-zero quantity which can be chosen (as will be shown later) to be coprime to each q_i, we conclude (since the torsion coefficients of $K_0(C_{i,j})$ are all q_i [R-V]) that $b_0(e) = 0$ implies

$$\bigg(\Big(\sum_{i=1}^{N+1}\dim(e_i)\Big)\cdot K_0(p_j) - (N+1)\dim(p_j)\cdot e\bigg) = 0.$$

If both $N+1$ and $\dim(p_j)$ are chosen to be coprime to each q_i (the former by adding copies of the connected component $X_{N,j}$ to Y_j as necessary, and the latter as will be shown below), then e is necessarily an element of the maximal free cyclic subgroup of $K_0(C_j)$ containing $K_0(p_j)$.

Given a subsemigroup of the positive integers S, where

$$S = \frac{1}{L}\Big(\bigcap_{i=1}^{N}\langle q_i, m_i\rangle\Big) \cap Z,$$

choose the spaces $X_{i,1}$ as follows: Let $X_{i,1}$ be the Cartesian product of $(q_i - 1)m_i$ copies of D_{q_i} for $i \in \{1, \ldots, N\}$, where D_{q_i} is the quotient of the closed unit disc in C by the equivalence relation that identifies elements of T having like q_i-th powers. Let $X_{N+1,1}$ be the Cartesian product of $L+1$ copies of S^2. Note that $K^1(X_{i,j}) = 0\ \forall i \in \{1, \ldots, N+1\}, \forall j \in N$, so that $K_1(C_j) = 0$. It follows that b_1 is surjective. Applying Theorem 2.1 we see that $K_0(A_j)$ is isomorphic as a group to its image, $\mathrm{Ker}\, b_0$, in $K_0(C_j)$—which is isomorphic as a group to Z.

In order for $K_0(A_j)$ to be isomorphic as an ordered group to its image in $K_0(C_j)$, with the relative order, it is sufficient (by Theorem 2.1) that for any projection q in $C_j \otimes K$ such that the images of q under $\phi_j^0 \otimes \mathrm{id}$ and $\phi_j^1 \otimes \mathrm{id}$ have the same K_0 class, these images be in fact equivalent. For any such q, the image of $K_0(q)$ under $b_0 = K_0(\phi_j^0) - K_0(\phi_j^1)$ is zero—in other words, $K_0(q)$ belongs to $Ker b_0$. By construction, $K_0(q)$ belongs to the largest subgroup of $K_0(C_j)$ containing $K_0(p_j)$ and isomorphic to Z. The choice of k_j below will ensure that the dimension of both $\phi_j^1(q)$ and $\phi_j^0(q)$ is at least half of the largest dimension of any $X_{i,j}$ over each connected component of Y_j. By Theorem 8.1.5 of [H], $\phi_j^1(q)$ and $\phi_j^0(q)$ are thus equivalent (as they have the same K_0 class).

Let us now specify the projection $p_1 \in C_1$. Let ξ_{q_i} be a complex line bundle over D_{q_i} with euler class a generator of $H^2(D_{q_i}) = Z/q_iZ$. Such bundles are known to exist [R-V]. Let $\omega_{q_i} = \xi_{q_i}^{\otimes(q_i-1)}$. Since q_i and m_i are coprime for each $i \in \{1, \ldots, N\}$, there exist integers a_i and b_i such that $a_iq_i + b_im_i = 1$. Set $g_{i,1} = a_i[\theta_{q_i}] + b_i[\omega_{q_i}^{\times m_i}]$ in $K^0\big(D_{q_i}^{\times(q_i-1)m_i}\big) = K_0\big(C(D_{q_i}^{\times(q_i-1)m_i}\big)$ ($[\,\cdot\,]$ denotes the stable isomorphism class of a vector bundle, and θ_d is the trivial vector bundle of fibre dimension d). Let ξ denote the Hopf line bundle over S^2, and put $g_{N+1,1} = [\xi^{\times L+1}] - [\theta_1]$. Finally, let $g_1 = (\bigoplus_{i=1}^N L \cdot g_{i,1}) \oplus g_{N+1,1}$. Let p_1 be a projection whose K_0 class is a multiple of g_1, and whose dimension is both coprime to each q_i and larger than half the largest dimension found amongst the $X_{i,1}$.

It follows from [R-V] that the ordered group $\big\langle \langle g_{i,1}\rangle, \langle g_{i,1}\rangle \cap K_0^+\big(C(X_{i,1})\big) \big\rangle$ is isomorphic to $\big\langle Z, \langle q_i, m_i\rangle\big\rangle$ for each $i \in \{1, \ldots, N\}$. It is shown in [V] that $\big\langle \langle g_{N+1,1}\rangle, \langle g_{N+1,1}\rangle \cap K_0^+\big(C(X_{N+1,1})\big) \big\rangle$ is isomorphic to $\langle Z, \{0, 2, 3, 4, \ldots\}\rangle$. We will now compute the order structure on $\langle g_1\rangle$ in $K_0\big(C(Y_1)\big)$. $K_0\big(C(Y_1)\big)$ is the direct sum of the $K_0\big(C(X_{i,1})\big)$ equipped with the direct sum order (an element x of $K_0\big(C(Y_1)\big)$ is positive if and only if the restriction of x to each of the direct summands $K_0\big(C(X_{i,1})\big)$ is positive). Thus a multiple $n \cdot g_1$ of g_1 is positive if and only if $nL \cdot g_{i,1} \in \langle q_i, m_i\rangle \cdot g_{i,1}$ for each $i \in \{1, \ldots, N\}$ and $n > 1$. Since we are only interested in perforated order structures, the element g_1 itself will never be positive. Thus if $n \cdot g_1$ is to be positive, n must be at least two. This fact renders moot the requirement that n be larger than one. Returning to the conditions involving $g_{1,1}, \ldots, g_{N,1}$, we may drop the $g_{i,1}$'s altogether, resulting in the condition

$$nL \in \langle q_i, m_i\rangle, \quad i \in \{1, \ldots, N\}$$

which is equivalent to the condition

$$nL \in \bigcap_{i=1}^{N} \langle q_i, m_i\rangle$$

Dividing both sides of the above equation by L and intersecting the right hand side with the integers (indicating that n must be an integer) we have

$$n \in \frac{1}{L}\Big(\bigcap_{i=1}^{N} \langle q_i, m_i\rangle\Big) \cap Z$$

as desired.

We now wish to specify the maps $\gamma_j \colon C_j \to C_{j+1}$ for each $j \in N$. First we recall that for a connected, compact Hausdorff space X we have $C(X^{\times n}) = C(X)^{\otimes n}$. Consider the maps

$$\gamma'_{i,j} := (\mathrm{id} \otimes 1 \otimes \cdots \otimes 1) \oplus (1 \otimes \mathrm{id} \otimes 1 \otimes \cdots \otimes 1) \oplus \cdots \oplus (1 \otimes \cdots \otimes 1 \otimes \mathrm{id})$$

from $C(X_{i,j})$ to $M_{n_j}\big(C(X_{i,j+1})\big) = M_{n_j}\big(C(X_{i,j}) \otimes \cdots \otimes C(X_{i,j})\big)$, where 1 denotes the unit of $C(X_{i,j})$, id denotes the identity function from $C(X_{i,j})$ to $C(X_{i,j})$, and $i \in \{1, \ldots, N+1\}$.

Consider also the maps

$$\beta'_{i,j} := 1 \cdot e_{x_{i,j}}$$

from $C(Y_j)$ to $C(Y_{j+1})$ where $e_{x_{i,j}}$ denotes evaluation at the point $x_{i,j} \in X_{i,j}$, and 1 denotes the unit of $C(Y_{j+1})$. Let us specify $x_{i,j}$ as the point in $X_{i,j}$ with all co-ordinates equal to a fixed point $x_{i,1} \in X_{i,1}$.

Let

$$\gamma'_j = \bigoplus_{i=1}^{N+1} \gamma'_{i,j}$$

where the direct sum is to be understood as a direct sum over the connected components of Y_j, resulting in a map from $C(Y_j)$ to $M_{n_j}\big(C(Y_{j+1})\big)$.

Let us define γ_j inductively to be the map from C_j to $C(Y_{j+1}) \otimes M_{N+2}(K)$ consisting of the direct sum of $N+2$ maps. For the first map, take the restriction to $C_j \subseteq C(Y_j) \otimes K$ of the tensor product of γ'_j with the identity map from K to K. The remaining $N+1$ maps are obtained as follows: for each $i \in \{1, \ldots, N+1\}$, compose the map ϕ^1_j with the direct sum of η_j copies of the tensor product of $\beta'_{i,j}$ with the identity from K to K (restricted to $D_j \subseteq C(Y_j) \otimes K$), where η_j is to be specified. The induction consists in first considering the case $i = 1$ (as p_1 has already been chosen)), then setting then setting $p_2 = \gamma_1(p_1)$, so that C_2 is specified as the cut-down of $C(Y_2) \otimes M_{N+2}(K)$, and continuing in this way.

With $\beta_j \colon D_j \to C_{j+1}$ taken to be the restriction to $D_j \subseteq C(Y_j) \otimes M_{N+1}(K)$ of $\bigoplus_{i=1}^{N+1} \beta'_{i,j} \otimes \mathrm{id}$ we have, by construction, that $\beta_j \phi^1_j$ is a direct summand of γ_j—and, furthermore, the second direct summand and β_j map into orthogonal blocks (and hence orthogonal subalgebras)—as desired.

We will now need to verify that $p_j := \gamma_{j-1} \cdots \gamma_1(p_1)$ has the following property: the set of all rational multiple of $K_0(p_j)$ in the ordered group $K_0C_j = K^0Y_j$ should be isomorphic as a sub ordered group to Z with positive cone

$$\frac{1}{L}\Big(\bigcap_{i=1}^{N} \langle q_i, m_i \rangle\Big) \cap Z.$$

This property has been established in the case $j = 1$. It remains to show that the map γ_j induces an order isomorphism from the rational multiples of $K_0(p_j)$ to the rational multiples of $K_0(p_{j+1})$.

We will first show that γ_j gives a group isomorphism between the groups in general. To establish this fact we require that $g_2 := \gamma_1(g_1)$ generate a maximal free cyclic subgroup of K_0C_2, $g_3 := \gamma_2(g_2)$ generate a maximal free cyclic subgroup of K_0C_3, and so on. This amounts to showing (in the case of g_2) that g_2 is not a positive integral multiple of any other element in $K_0C_2 = K^0Y_2$. Since Y_2 is a disjoint union of connected components, we may consider the restriction of $g_{i,2}$ of g_2 to each component $X_{i,2}$ of Y_2. If g_2 is a positive integral multiple of some other element of K^0Y_2, say $g_2 = l \cdot h$, then (denoting by h_i the restriction of h to $X_{i,2}$) we have that $g_{i,2} = l \cdot h_i$ for each $i \in \{1, \ldots, N\}$. Thus in order to show that g_2 is not a positive integral multiple of some $h \in K^0Y_2$, it is enough to establish this fact for one of the $g_{i,2}$.

Let $g_{i,j+1}$ denote the restriction to $X_{i,j+1}$ of $\gamma_j(g_j)$.

Consider $g_{N+1,2}$, recalling that $X_{N+1,2}$ is a product of spheres. We reproduce here the proof found in [E-V] which establishes the desired maximality condition for $g_{N+1,2}$. Note that $g_{N+1,1}$ generates a maximal free cyclic subgroup of $K^0(X_{N+1,1})$ (since $g_{N+1,1}$ is of the form $L \oplus 1 \oplus a_3 \oplus \cdots \oplus a_{2^{L+1}} \in Z^{(2^{L+1})} = K^0(S^{2\times L+1})$. Also note that $g_{N+1,1}$ is independent of $K_0(1_{X_{N+1,1}})$ in $K^0X_{N+1,1}$ (*i.e.* the free cyclic subgroups generated by these K_0 classes have zero intersection). Since $K^0X_{N+1,1}$ is torsion free and $K^1X_{N+1,1} = 0$ we have (by the Künneth theorem) that $K^0X_{N+1,2}$ is isomorphic as a group to the tensor product of n_1 copies of $K^0X_{N+1,1}$. Note that the map id $\otimes$ dim $\otimes \cdots \otimes$ dim, where id denotes the identity map on $K^0X_{N+1,1}$ and dim: $K^0X_{N+1,1} \to Z$ the dimension function, takes $K^0X_{N+1,2} = K^0X_{N+1,1} \otimes \cdots \otimes K^0X_{N+1,1}$ onto $K^0X_{N+1,1}$ and takes $g_{N+1,2}$ onto $g_{N+1,1}$ plus a multiple of $K_0(1_{X_{N+1,1}})$. If $g_{N+1,2}$ is a multiple of some other element of $K^0X_{N+1,2}$, say $g_{N+1,2} = k \cdot g$, then it follows that $g_{N+1,1}$ plus a multiple of $K_0(1_{X_{N+1,1}})$ is k times the image of g. Then, modulo the subgroup of $K^0X_{N+1,1}$ generated by $K_0(1_{X_{N+1,1}})$, $g_{N+1,1}$ is k times some element (the image of g). But the subgroup of $K^0X_{N+1,1}$ generated by $g_{N+1,1}$ has zero intersection with the subgroup generated by $K_0(1_{X_{N+1,1}})$, and so its image modulo $K_0(1_{X_{N+1,1}})$ is still isomorphic to Z, and has the image of $g_{N+1,1}$ as its generator. This shows that $k = \pm 1$, as desired.

We have now shown that $g_{N+1,2}$ has the same properties as $g_{N+1,1}$ used above (namely, that $g_{N+1,2}$ generates a maximal subgroup of rank one which has zero intersection with the subgroup generated by $K_0(1_{X_{N+1,2}})$). We may thus deduce as above that $\gamma_2(g_{N+1,2})$ generates a maximal subgroup of $K^0X_{N+1,3}$ of rank one, *i.e.*, γ_2 gives a group isomorphism between the subgroups under consideration (namely, Ker b_0 restricted to $X_{N+1,2}$ and $X_{N+1,3}$, respectively). Clearly, we may proceed in this way to establish that γ_j gives a group isomorphism for every j between Ker b_0 at the j-th and $(j+1)$-st stages, restricted to $X_{N+1,j}$ and $X_{N+1,j+1}$, respectively.

Let us now show that, for each j, if n_j is chosen sufficiently large, then γ_j restricted to Ker b_0 is an order isomorphism between the subgroups Ker $b_0 = Zg_j$ and Ker $b_0 = Zg_{j+1}$ of K^0Y_j and K^0Y_{j+1} with the relative order, where $g_j = \gamma_{j-1}\cdots\gamma_1(g_1)$. To this end it will serve us to recall the details of [R-V] concerning the proof of the fact that $(Z \cdot g_{i,1})^+ = \langle q_i, m_i\rangle$ for $i \in \{1,\ldots,N\}$.

For $i \neq N+1$, $g_{i,1} = a_i[\theta_{q_i}] + b_i[\omega_{q_i}^{\times m_i}]$, where ω_{q_i} is a non-trivial line bundle with the property that $\bigoplus_{l=1}^{q_i} \omega_{q_i} \simeq \theta_{q_i}$. Thus

$$\begin{aligned}
q_i \cdot g_{i,1} &= a_i q_i[\theta_{q_i}] + b_i q_i[\omega_{q_i}^{\times m_i}] \\
&= a_i q_i[\theta_{q_i}] + b_i\Big[\bigoplus_{l=1}^{q_i} \omega_{q_i}^{\times m_i}\Big] \\
&= a_i q_i[\theta_{q_i}] + b_i[\theta_{q_i m_i}] \\
&= a_i q_i[\theta_{q_i}] + b_i m_i[\theta_{q_i}] \\
&= (a_i q_i + b_i m_i)[\theta_{q_i}] \\
&= [\theta_{q_i}]
\end{aligned}$$

and

$$\begin{aligned} m_i \cdot g_{i,1} &= a_i m_i[\theta_{q_i}] + b_i m_i[\omega_{q_i}^{\times m_i}] \\ &= a_i[\theta_{q_i m_i}] + b_i m_i[\omega_{q_i}^{\times m_i}] \\ &= a_i\Big[\bigoplus_{l=1}^{q_i} \omega_{q_i}^{\times m_i}\Big] + b_i m_i[\omega_{q_i}^{\times m_i}] \\ &= a_i q_i[\omega_{q_i}^{\times m_i}] + b_i m_i[\omega_{q_i}^{\times m_i}] \\ &= (a_i q_i + b_i m_i)[\omega_{q_i}^{\times m_i}] \\ &= [\omega_{q_i}^{\times m_i}] \end{aligned}$$

since a_i and b_i were chosen so that $a_i q_i + b_i m_i = 1$. This shows that both $q_i \cdot g_{i,1}$ and $m_i \cdot g_{i,1}$ are positive element of $K^0(X_{i,1})$. The subsemigroup of the positive integers $S_{i,1}$ with the property that $s \cdot g_{i,1} \in K^0(X_{i,1})^+$ if and only if $s \in S_{i,1}$ thus contains the subsemigroup $\langle q_i, m_i\rangle$ of the positive integers.

Lemma 3.1 *If S is a subsemigroup of the positive integers containing the coprime integers k and l, and if S does not contain the integer $kl - k - l$, then $S = \langle k, l\rangle$ (the subsemigroup of the positive integers generated by k and l).*

The above lemma (whose proof can be found in [R-V]) has the following consequence: in order to show that $\big\langle \langle g_{i,1}\rangle, \langle g_{i,1}\rangle \cap K^0(X_{i,1})^+ \big\rangle$ is isomorphic as an ordered group to $\big\langle Z, \langle q_i, m_i\rangle \big\rangle$, it suffices to establish the non-positivity of $\big((q_i-1)m_i - q_i\big) \cdot g_{i,1}$ ($i \neq N+1$). Using the expressions for $q_i \cdot g_{i,1}$ and $m_i \cdot g_{i,1}$ above, we have that $\big((q_i-1)m_i - q_i\big) \cdot g_{i,1} = (q_i - 1)[\omega_{q_i}^{\times m_i}] - [\theta_{q_i}]$.

Consider a difference of stable isomorphism classes of vector bundles $[\xi] - [\theta_l]$ over a connected space X ($l \neq 0$), and suppose that this difference is in fact equal to $[\eta]$ for some vector bundle η over X. Then, by definition, $\xi \oplus \theta_r \equiv \eta \oplus \theta_{r+l}$ for some natural number r. Taking the Chern class of both sides of the preceding equation yields $c(\xi) = c(\eta)$, where $c(\cdot)$ denotes the Chern class of a vector bundle. The dim(ξ)-th Chern class, (or Euler class, if ξ is a sum of line bundles) of ξ must be zero in this case, as the n-th Chern class of any vector bundle of dimension less than n is zero [H]. Thus choosing ξ to be a vector bundle with non-zero Euler class ensures that the difference $[\xi] - [\theta_l]$ with $l \neq 0$ is not positive in $K^0(X)$.

In [R-V] it is shown that the Euler class of the vector bundle $\bigoplus_{l=1}^{q_i-1} \omega_{q_i}^{\times m_i}$ (with corresponding stable isomorphism class $(q_i - 1)[\omega_{q_i}^{\times m_i}]$) is non-zero. In fact, their proof establishes that the Euler class of the vector bundle $\bigoplus_{l=1}^{q_i-1} \omega_{q_i}^{\times m_i n}$ over $X_{i,1}^{\times n}$ is non-zero for any natural number n. Thus $(q_i - 1)[\omega_{q_i}^{\times m_i}] - [\theta_{q_i}]$ is non-positive in $K^0(X_{i,1})$, and

$$\big\langle \langle g_{i,1}\rangle, \langle g_{i,1}\rangle \cap K^0(X_{i,1})^+ \big\rangle \equiv \big\langle Z, \langle q_i, m_i\rangle \big\rangle, \quad i \in \{1, \ldots, N\}$$

as desired. The fact that

$$\big\langle \langle g_{N+1,1}\rangle, \langle g_{N+1,1}\rangle \cap K^0(X_{N+1,1})^+ \big\rangle \equiv \langle Z, \{0, 2, 3, 4, \ldots\}\rangle$$

is established in [V].

Returning now to the matter of verifying that γ_j (with an appropriate choice of n_j) restricted to Ker b_0 is an order isomorphism as described above, note that for a complex vector bundle π over $X_{i,1}$, $i \in \{1, \ldots, N+1\}$ we have that $K_0(\gamma_{j-1} \cdots \gamma_1)([\pi]) = [\pi^{\times n_1 \cdots n_{j-1}}] + [\theta_l]$, some $l \in N$. Since all induced maps on K_0 are positive, we have that

$$\{g_j N\}^+ \supseteq g_j \Big\{ \frac{1}{L} \Big(\bigcap_{i=1}^{N} \langle q_i, m_i \rangle \Big) \cap Z \Big\}$$

In order to show that the right and left hand sides of the above equation are in fact equal, we need only show that for each j and each $i \in \{1, \ldots, N\}$ the group $\langle g_{i,j} \rangle$ is isomorphic as an ordered group to $\langle g_{i,1} \rangle$ (whose order structure has already been established).

Since the map $\gamma_{j-1} \cdots \gamma_1$ is positive, we have that for any positive multiple lg_1 of g_1 (necessarily a positive multiple of $g_{i,1}$ for each i), the restriction of lg_j to $X_{i,j}$ (*i.e.*, $lg_{i,j}$) is also positive. Thus the positive multiples of $g_{i,j}$ considered as a subset of the integers contain the positive multiples of $g_{i,1}$. Now consider $\big((q_i - 1)m_i - q_i\big) g_{i,j} = (q_i - 1)[\omega_{q_i}^{\times m_i n_1 \cdots n_{j-1}}] - [\theta_{l_{i,j}}]$. If $l_{i,j}$, through judicious choice of the n_j, can be made positive, then the multiple of $g_{i,j}$ in question will be non-positive. This will establish the desired order isomorphism.

In order to prove the positivity of $l_{i,j}$ we will proceed by induction. Assume that $l_{i,k}$ is positive for all $k < j$ and all i. Now

$$\begin{aligned} \big((q_i - 1) - m_i\big) g_{i,j} &= \big((q_i - 1) - m_i\big) \gamma_{j-1}(g_{j-1})|_{X_{i,j}} \\ &= [\omega_{q_i}^{\times m_i n_1 \cdots n_{j-1}}] - [\theta_{l_{i,j}}] \end{aligned}$$

where

$$l_{i,j} = l_{i,j-1} n_{j-1} - (N+1)\eta_{j-1} k_{j-1} \dim(p_{j-1}) \dim\Big(\big((q_i - 1) - m_i\big) g_{i,j-1} \Big).$$

Recall that k_{j-1} and p_{j-1} have already been chosen; we may also suppose that η_{j-1} has already been chosen in the manner to be specified below, which does not depend on the choice of n_{j-1}. Thus $l_{i,j}$ is easily seen to be positive for n_{j-1} sufficiently large. Choose n_{j-1} to be large enough that $l_{i,j}$ is positive for each i, and such that it is coprime to each q_i, $i \in \{1, \ldots, N\}$. This choice establishes the desired order isomorphism between Ker b_0 at the $(j-1)$-st and j-th stages with the relative order.

Note that $\gamma_j - \beta_j \phi_j^1$ takes a full element of C_j into a full element of C_{j+1} and so takes C_j into a subalgebra of C_{j+1} not contained in any proper closed two-sided ideal (as required in the hypotheses of Theorem 2.4). (C_j is unital, and any non-zero projection of C_{j+1} generates it as a closed two sided ideal.)

Let us now construct maps δ_j and δ'_j from D_j to D_{j+1} with orthogonal images such that

$$\begin{aligned} \delta_j \phi_j^0 + \delta'_j \phi_j^1 &= \phi_{j+1}^0 \gamma_j, \\ \delta'_j \phi_j^0 + \delta_j \phi_j^1 &= \phi_{j+1}^1 \gamma_j, \end{aligned}$$

and $\phi^0_{j+1}\beta_j$ and $\phi^1_{j+1}\beta_j$ are direct summands of δ'_j and δ_j, respectively. To achieve this end we will modify ϕ^0_{j+1} and ϕ^1_{j+1} by inner automorphisms. As stated above, these modifications will not affect K_0.

Now notice that (up to the order of direct summands, with μ_j denoting the direct sum over i of the $\mu_{i,j}$) we have the following string of equalities:

$$\begin{aligned}
\mu_{j+1}\gamma_j &= \bigoplus_{i=1}^{N+1} \mu_{i,j+1}\gamma_j \\
&= \bigoplus_{i=1}^{N+1} p_{j+1} \otimes e_{x_{i,j+1}}\gamma_j \\
&= \bigoplus_{i=1}^{N+1} \gamma_j(p_j) \otimes e_{x_{i,j+1}}\gamma_j \\
&= \bigoplus_{i=1}^{N+1} \gamma_j(p_j) \otimes \left(n_j e_{x_{i,j}} \oplus \left(\bigoplus_{l=1}^{N+1} \eta_j k_j \dim(p_j) e_{x_{l,j}} \right) \right) \\
&= \bigoplus_{i=1}^{N+1} \gamma_j(p_j) \otimes \big(n_j + (N+1)\eta_j k_j \dim(p_j) \big) e_{x_{i,j}} \\
&= \bigoplus_{i=1}^{N+1} \mathrm{mult}(\gamma_j)\gamma_j(p_j \otimes e_{x_{i,j}}) \\
&= \mathrm{mult}(\gamma_j)\gamma_j\mu_j
\end{aligned}$$

Similarly (with ν_j being the direct sum over i of the $\nu_{i,j}$),

$$\begin{aligned}
\nu_{j+1}\gamma_j &= \bigoplus_{i=1}^{N+1} \gamma_j \otimes 1_{\dim(p_{j+1})} \\
&= \bigoplus_{i=1}^{N+1} \mathrm{mult}(\gamma_j)\gamma_j \otimes 1_{\dim(p_j)} \\
&= \bigoplus_{i=1}^{N+1} \mathrm{mult}(\gamma_j)\gamma_j\nu_{i,j} \\
&= \mathrm{mult}(\gamma_j)\gamma_j\nu_j
\end{aligned}$$

Note that $\mathrm{mult}(\gamma_j)$ is well defined, as the dimension of $p_{i,j}$ is independent of i.

Let us take δ_j and δ'_j to be the direct sum of r_j and s_j copies of γ_j, respectively, where r_j and s_j are to be specified. The condition, for $t = 0, 1$,

$$\delta_j\phi^t_j + \delta'_j\phi^{1-t}_j = \phi^t_{j+1}\gamma_j,$$

understood up to unitary equivalence (in particular, up to the order of direct summands) then becomes the condition

$$r_j\gamma_j\big(l_j^t\mu_j + (k_j - l_j^t)\nu_j\big) + s_j\gamma_j\big(l_j^{t-1}\mu j + (k_j - l_j^{t-1})\nu_j\big) = \big(l_{j+1}^t\mu_{j+1} + (k_{j+1} - l_{j+1}^t)\nu_{j+1}\big)\gamma_j,$$

also up to unitary equivalence. Since $K_0(\nu_j)$ is injective, it is independent of $K_0(\mu_j)$. The above equation is thus equivalent to the two equations

$$r_j l_j^t + s_j l_j^{1-t} = \mathrm{mult}(\gamma_j) l_{j+1}^t$$
$$(r_j + s_j)k_j = \mathrm{mult}(\gamma_j)k_{j+1}$$

Let us choose $r_j = (p - \lfloor \frac{p}{2} \rfloor)\,\mathrm{mult}(\gamma_j)$ and $s_j = \lfloor \frac{p}{2} \rfloor\,\mathrm{mult}(\gamma_j)$, so that

$$k_{j+1} = pk_j,$$

and

$$l_{j+1}^t = \left(p - \left\lfloor \frac{p}{2} \right\rfloor\right) l_j^t + \left\lfloor \frac{p}{2} \right\rfloor l_j^{1-t}.$$

The integer p should be a prime number coprime to each q_i having further the property that it is greater than the largest positive integer not contained in the subsemigroup of the positive integers given by

$$\Big(\bigcap_{i=1}^{N} \langle q_i, m_i\rangle\Big) \cap Z.$$

Take $k_1 = p$, $l_1^1 = (p - \lfloor \frac{p}{2} \rfloor)$, and $l_1^0 = \lfloor \frac{p}{2} \rfloor$. These choices yield $k_j = p^j$ and $l_j^1 - l_j^0 = 1$ for all j. Note that $l_j^1 - l_j^0$ is both non-zero and coprime to each q_i, as required above. In addition, k_j thus chosen is large enough to ensure that K_0A_j is isomorphic as an ordered group to its image in K_0C_j, with the relative order, also required above.

Next let us show that, up to unitary equivalence preserving the equations $\delta_j\phi_j^t + \delta_j'\phi_j^{1-t} = \phi_{j+1}^t\gamma_j$, $\phi_{j+1}^0\beta_j$ is a direct summand of $\delta_j' = \lfloor \frac{p}{2} \rfloor\,\mathrm{mult}(\gamma_j)$, and $\phi_{j+1}^1\beta_j$ is a direct summand of $\delta_j = (p - \lfloor \frac{p}{2} \rfloor)\,\mathrm{mult}(\gamma_j)\gamma_j$.

Note that $\phi_{j+1}^t\beta_j$ is the direct sum of l_{j+1}^t copies of $p_{j+1} \otimes \beta_j$ and $(k_{j+1} - l_{j+1}^t) \cdot \dim(p_{j+1})$ copies of β_j, whereas δ_j' and δ_j contain, respectively, $\eta_j\lfloor \frac{p}{2} \rfloor\,\mathrm{mult}\,\gamma_j$ and $\eta_j(p - \lfloor \frac{p}{2} \rfloor)\,\mathrm{mult}\,\gamma_j$ copies of β_j. By Theorem 8.1.2 of [H], a trivial projection of dimension at least $\dim(p_{j+1}) + \mathrm{maxdim}(Y_{j+1})$ (where $\mathrm{maxdim}(Y_{j+1}) = \max_{i=1}^{N+1} / \dim(X_{i,j+1}))$ over each component of Y_{j+1} contains a copy of p_{j+1}. Therefore $\dim(p_{j+1}) + \mathrm{maxdim}(Y_{j+1})$ copies of β_j contain a copy of $p_{j+1} \otimes \beta_j$. It follows that $k_{j+1}\big(2\dim(p_{j+1}) + \dim(X_{j+1})\big)$ copies of β_j contain a copy of $\phi_{j+1}^t\beta_j$ for $t = 0, 1$. Here a copy of a given map from D_j to D_{j+1} is taken to be a map obtained from the original by way of a partial isometry in D_{j+1} with initial projection the image of the unit.

Note that

$$\begin{aligned} k_{j+1}\big(2\dim(p_{j+1}) + \mathrm{maxdim}(Y_{j+1})\big) &= pk_j\big(2\,\mathrm{mult}(\gamma_j)\big)\dim(p_j) + n_j\,\mathrm{maxdim}(Y_j) \\ &\leq pk_j\big(2\dim(p_j) + \mathrm{maxdim}(Y_j)\big)\,\mathrm{mult}(\gamma_j). \end{aligned}$$

Since k_j, $\dim(p_j)$, and $\mathrm{maxdim}(Y_j)$ have already been specified and are independent of n_j put

$$\eta_j = pk_j\big(2\dim(p_j) + \mathrm{maxdim}(Y_j)\big).$$

With this η_j, $\eta_j\,\mathrm{mult}(\gamma_j)$ copies of β_j contain a copy of $\phi^t_{j+1}\beta_j$ for $t = 0, 1$. Thus δ'_j and δ_j contain copies of $\phi^0_{j+1}\beta_j$ and $\phi^1_{j+1}\beta_j$, respectively.

With this choice of η_j, let us show that for each $t = 0, 1$ there exists a unitary $u_t \in D_{j+1}$ commuting with the image of ϕ^t_{j+1}, *i.e.*, with

$$(\mathrm{Ad}\,u_t)\phi^t_{j+1}\gamma_j = \phi^t_{j+1}\gamma_j,$$

such that $(\mathrm{Ad}\,u_0)\phi^0_{j+1}\beta_j$ is a direct summand of δ'_j and $(\mathrm{Ad}\,u_1)\phi^1_{j+1}\beta_j$ is a direct summand of δ_j. In other words, for each $t = 0, 1$, we must show that the partial isometry constructed in the preceding paragraph, producing a copy of $\phi^t_{j+1}\beta_j$ inside δ'_j or δ_j may be chosen in such a way that it extends to a unitary element of D_{j+1}—which in addition commutes with the image of $\phi^t_{j+1}\gamma_j$.

Consider the case $t = 0$. The case $t = 1$ is, for all intents and purposes, the same. First we will show that the partial isometry in D_{j+1} transforming $\phi^0_{j+1}\beta_j$ into a direct summand of δ'_j may be chosen to lie in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Note that the unit of the image of $\phi^0_{j+1}\beta_j$—the initial projection of the partial isometry—lies in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Indeed, this projection is the image by $\phi^0_{j+1}\beta_j$ of the unit of D_j, which, by construction, is the image by ϕ^1_j of the unit of C_j. The property that $\beta_j\phi^1_j$ is a direct summand of γ_j implies in particular that the image by $\beta_j\phi^1_j$ of the unit of C_j commutes with the image of γ_j. The image by $\phi^0_{j+1}\beta_j\phi^1_j$ of the unit of C_j (*i.e.*, the unit of the image of $\phi^0_{j+1}\beta_j$) therefore commutes with the image of $\phi^0_{j+1}\gamma_j$, as claimed.

The final projection of the partial isometry also commutes with the image of $\phi^0_{j+1}\gamma_j$. Indeed, it is the unit of the image of a direct summand of δ'_j, and since D_j is unital it is the image of the unit of D_j by this direct summand. Since C_j and ϕ^0_j are unital, the projection in question is the image of the unit of C_j by a direct summand of $\delta'_j\phi^1_j$, which is in turn a direct summand of $\phi^0_{j+1}\gamma_j$. Thus the projection in question is the image of the unit of C_j by a direct summand of $\phi^0_{j+1}\gamma_j$, and commutes with the image of $\phi^0_{j+1}\gamma_j$.

Note that both direct summands of $\phi^0_{j+1}\gamma_j$ (namely $\phi^0_{j+1}\beta_j\phi^1_j$ and a copy of it) are direct sums of $N + 1$ maps, each of which factors through the evaluation of C_j at $x_{i,j}$ for some i, and are thus contained in the largest such direct summand of $\phi^0_{j+1}\gamma_j$, say π_j. This largest direct summand is seen to exist by inspection of the construction of $\phi^0_{j+1}\gamma_j$. Write $\pi_j = \bigoplus_{i=1}^{N+1}\pi_{i,j}$, where $\pi_{i,j}$ denotes the direct summand of π_j that factors through the evaluation of C_j at $x_{i,j}$. Since both of the projections under consideration (the images of the unit of C_j by two different copies of $\phi^0_{j+1}\beta_j\phi^1_j$) are less than $\pi_j(1)$, to show that they are unitarily equivalent in the commutant of the image of $\phi^0_{j+1}\gamma_j$ it is sufficient to show that they are unitarily equivalent in the commutant of the image of π_j in $\pi_j(1)D_{j+1}\pi_j(1)$. In fact, since any partial unitary defined only on the cut-down of D_{j+1} by $\pi_{i,j}(1)$ for some $i \in \{1, \ldots, N+1\}$ can be extended to a unitary on D_{j+1} equal to one inside the complement of $\pi_{i,j}(1)$, the problem of proving the unitary equivalence of the two projections in question is reduced to the problem of proving their unitary equivalence in the commutant of the image of $\pi_{i,j}$ in $\pi_{i,j}(1)D_{j+1}\pi_{i,j}(1)$. This image is isomorphic to $M_{\dim(p_j)}(C)$.

By construction, the two projections in question are Murray-von Neumann equivalent in D_{j+1}, and thus have the same class in $K^0(Y_{j+1})$. Note that the dimension of these projections is $(N+1)^2\big(\dim(p_j)\big)^2\dim(p_{j+1})k_jk_{j+1}$, and the dimension of $\pi_{i,j}(1)$ is $l^0_{j+1}k_{j+1}\dim(p_{j+1})\dim(p_j)\big(n_j+\eta_jk_j\dim(p_j)\big)$. Since the two projections in question commute with $\pi_{i,j}(C_j)$, to prove unitary equivalence in the commutant of $\pi_{i,j}(C_j)$ in $\pi_{i,j}(1)D_{j+1}\pi_{i,j}(1)$, it is sufficient to prove unitary equivalence of the product of these projections with a fixed minimal projection of $\pi_{i,j}(C_j)$, say e. Since $\dim(p_j)$ is coprime to q_i for each i, the products of the two projections with e will have the same class in $K^0(Y_{j+1})$.

To prove that these projections are unitarily equivalent inside $eD_{j+1}e$, it is sufficient to establish that both they and their complements (inside e) are Murray-von Neumann equivalent. Since the two projections and their complements have the same class in $K^0(Y_{j+1})$, we need only show that all four projections have dimension greater than $\frac{1}{2}\operatorname{maxdim}(Y_{j+1})$. Then by Theorem 8.1.5 of [H], the two pairs of projections will be Murray-von Neumann equivalent, as desired.

Dividing the dimensions of the two projections (images of the unit of C_j) and $\pi_j(1)$ by the order of the matrix algebra ($\dim(p_j)$), we find that the dimension of the first two projections is $\big((N+1)\dim(p_j)\big)^2k_jk_{j+1}\operatorname{mult}(\gamma_j)$ and the dimension of e is $l^0_{j+1}k_{j+1}\operatorname{mult}(\gamma_j)\dim(p_j)\big(n_j+\eta_jk_j\dim(p_j)\big)$. The dimension of the second pair of projections is thus $\operatorname{mult}(\gamma_j)l^0_{j+1}k_{j+1}\dim(p_j)\Big(n_j+\eta_jk_j\dim(p_j)-k_jk_{j+1}\big((N+1)\dim(p_j)\big)^2\Big)$. Recall that $\dim(p_1) > \operatorname{maxdim}(Y_1)$, $\dim(p_{j+1}) = \operatorname{mult}(\gamma_j)\dim(p_j)$, $\operatorname{maxdim}(Y_{j+1}) = n_j\operatorname{maxdim}(Y_j)$, and that $\operatorname{mult}(\gamma_j) \geq n_j$ (for all j). These facts imply that $\dim(p_{j+1}) \geq \frac{1}{2}\operatorname{maxdim}(Y_{j+1})$ (for all j). The fact that $k_{j+1}k_j$ is non-zero then implies the first inequality. The second inequality holds if

$$\begin{aligned} &l^0_{j+1}k_{j+1}\dim(p_j)\big(n_j+\eta_jk_j\dim(p_j)\big) - \big((N+1)\dim(p_j)\big)^2k_jk_{j+1} \\ &\quad = \big(l^0_{j+1}\eta_j-(N+1)^2\big)k_jk_{j+1}\dim(p_j)^2 + n_jl^0_{j+1}k_{j+1}\dim(p_j) \end{aligned}$$

is strictly bigger than $\dim(p_j)$. We may assume that p, and hence l^0_{j+1} have been chosen large enough to ensure the aforementioned inequality holds.

Thus the two projections in D_{j+1} under consideration are unitarily equivalent by a unitary in the commutant of the image of $\phi^0_{j+1}\gamma_j$. Replacing $\phi^0_{j+1}\gamma_j$ by its composition with the corresponding inner automorphism, we may assume that the two projections in question are in fact equal. In other words, $\phi^0_{j+1}\beta_j$ is unitarily equivalent to the cut-down of δ'_j by the projection $\phi^0_{j+1}\beta_j(1)$.

Consider the composition of the two maps above with ϕ^1_j ($\phi^0_{j+1}\beta_j\phi^1_j$ and the cut-down of $\delta'_j\phi^1_j$ by the projection $\phi^0_{j+1}\beta_j(1)$). Both of these maps can be viewed as the cut-down of $\phi^0_{j+1}\gamma_j$ by the same projection ($\beta_j\phi^1_j$ is the cut-down of γ_j by $\beta_j\phi^1_j(1)$, and $\phi^0_{j+1}\beta_j(1) = \phi^0_{j+1}\big(\beta_j\phi^1_j(1)\big)$), so they are in fact the same map.

Now any unitary inside the cut-down of D_{j+1} by $\phi^0_{j+1}\beta_j(1)$ taking $\phi_{j+1}\beta_j$ into the cut-down of δ'_j by this projection (such a unitary is known to exist) must commute with the image of $\phi^0_{j+1}\beta_j\phi^1_j$, and hence with the image of $\phi^0_{j+1}\gamma_j$. If we extend such a partial unitary to a unitary u_{j+1} in D_{j+1} equal to one inside the complement of $\phi^0_{j+1}\beta_j(1)$, then u_{j+1} will commute with the image of $\phi^0_{j+1}\gamma_j$ and transform $\phi_{j+1}\beta_j$ into the cut-down of δ'_j by this projection, as desired.

Inspection will show that $\delta'_j - \phi^0_j\beta_j$ and $\delta_j - \phi^1_j\beta_j$ are injective maps, as required.

Replacing ϕ^t_{j+1} with $(\mathrm{Ad}\, u_{j+1})\phi^t_{j+1}$ completes the inductive construction of the desired sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots ,$$

satisfying the hypotheses of Theorems 2, 3, and 5. The existence of α_j homotopic to β_j, non-zero on a specified element of D_j, defined by another direct sum of point evaluations (thus satisfying the requirements of Theorem 2.4 with $\bar{\alpha}_j = \alpha_j$) is clear.

By Theorem 2.4 there exists a sequence

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_2} \cdots$$

such that θ'_j agrees with θ_j on K_0 (by virtue of its being homotopic to θ_j). The limit of this sequence is simple, and has the desired order structure on K_0.

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