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# Flat dimension growth for $C^*$-algebras ☆

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**Abstract**

Simple and nuclear $C^*$-algebras which fail to absorb the Jiang–Su algebra tensorially have settled many open questions in the theory of nuclear $C^*$-algebras, but have been little studied in their own right. This is due partly to a dearth of invariants sensitive to differences between such algebras. We present two new real-valued invariants to fill this void: the dimension–rank ratio (for unital AH algebras), and the radius of comparison (for unital and stably finite algebras). We establish their basic properties, show that they have natural connections to ordered K-theory, and prove that the range of the dimension–rank ratio is exhausted by simple algebras (this last result shows the class of simple, nuclear and non-$\mathcal{Z}$-stable $C^*$-algebras to be uncountable). In passing, we establish a theory of moderate dimension growth for AH algebras, the existence of which was first supposed by Blackadar. The minimal instances of both invariants are shown to coincide with the condition of being tracially AF among simple unital AH algebras of real rank zero and stable rank one, whence they may be thought of as generalised measures of dimension growth. We argue that the radius of comparison may be thought of as an abstract version of the dimension–rank ratio.


*Keywords:* Nuclear $C^*$-algebras; Non-commutative dimension

## 1. Introduction

The Jiang–Su algebra $\mathcal{Z}$ is by now well known in the study of nuclear $C^*$-algebras. All evidence indicates that the property of being $\mathcal{Z}$-stable—a $C^*$-algebra $A$ is said to be $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$—is connected naturally to Elliott's program to classify separable and nuclear

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$C^*$-algebras [7]; examples due to Rørdam and the author show that the largest class of simple, separable, unital and nuclear $C^*$-algebras which may be classified up to $*$-isomorphism by the Elliott invariant consists of those algebras which are, in addition, $\mathcal{Z}$-stable [25,27,28]. It has been surprising to find that almost all of our stock-in-trade simple, separable, and nuclear $C^*$-algebras are $\mathcal{Z}$-stable [14,31].

Little is known about non-$\mathcal{Z}$-stable $C^*$-algebras in general, save that they seem able to exhibit arbitrarily strange behaviour. Specific examples of such algebras have, over the past several years, been used to settle many open questions in the theory of separable and nuclear $C^*$-algebras—see [23,25,27–29,32,33]—but no attempt has been made to study their structure systematically. In this paper—a sequel to [30] in design, though technically independent of it—we study these algebras through the introduction of invariants which distill purely non-$\mathcal{Z}$-stable information: they are insensitive to differences between $\mathcal{Z}$-stable algebras, while detecting differences between non-$\mathcal{Z}$-stable $C^*$-algebras not readily manifest in known invariants.

In the early sections of the sequel we concentrate on approximately homogeneous (AH) $C^*$-algebras, as these provide the most tractable examples of simple and non-$\mathcal{Z}$-stable $C^*$-algebras. Recall that a *homogeneous* $C^*$-algebra has the form

$$p\big(\mathrm{C}(X)\otimes\mathcal{K}\big)p,$$

where $X$ is a compact Hausdorff space, $\mathcal{K}$ is the algebra of compact operators on a separable and infinite-dimensional Hilbert space $\mathcal{H}$, and $p\in\mathrm{C}(X)\otimes\mathcal{K}$ is a projection of constant rank. A *semi-homogeneous* $C^*$-algebra is a finite direct sum of homogeneous $C^*$-algebras.

**Definition 1.1.** (See Blackadar [1].) An approximately homogeneous (AH) $C^*$-algebra is an inductive limit

$$A=\lim_{i\to\infty}(A_i,\phi_i),$$

where each $A_i$ is semi-homogeneous.

Let

$$A\cong\lim_{i\to\infty}(A_i,\phi_i) \tag{1}$$

be an unital (i.e., both $A_i$ and $\phi_i: A_i\to A_{i+1}$ are unital for every $i\in\mathbb{N}$) AH algebra, where

$$A_i:=\bigoplus_{l=1}^{m_i}p_{i,l}\big(\mathrm{C}(X_{i,l})\otimes\mathcal{K}\big)p_{i,l} \tag{2}$$

for compact Hausdorff spaces $X_{i,l}$, projections $p_{i,l}\in\mathrm{C}(X_{i,l})\otimes\mathcal{K}$, and natural numbers $m_i$. Put

$$\phi_{ij}=\phi_{j-1}\circ\phi_{j-2}\circ\cdots\circ\phi_i,$$

and write $\phi_{i\infty}: A_i\to A$ for the canonical map. We refer to this collection of objects and maps as a *decomposition* for $A$. If the $\phi_i$ are injective, then we will refer to this collection as an *injective decomposition*.

**Definition 1.2.** Let $A$ be an unital AH algebra. Say that $A$ has flat dimension growth if it admits a decomposition for which

$$\limsup_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\operatorname{rank}(p_{i,l})} \right\} < \infty. \tag{3}$$

A simple unital AH algebra $A$ admitting a decomposition for which (3) is zero is said to have *slow dimension growth* [4]. (There are definitions of slow dimension growth for non-simple algebras in [10,18], but we will not require them here. Suffice it to say that these definitions coincide with Definition 1.2 for simple algebras.) If (3) is finite for some decomposition of $A$, then we may, by passing to a subsequence, replace the lim sup by a limit; proofs in the sequel with exploit this.

The beginnings of Definition 1.2 are contained in Blackadar's 1991 survey article "Matricial and ultramatricial topology" [1]. At the time, all known simple unital AH algebras had slow dimension growth, but Blackadar mused nonetheless about the possible existence of a theory of "AH algebras with moderate dimension growth" (synonymous with our flat dimension growth). His hoped-for theory was made plausible when Villadsen provided the first examples of simple unital AH algebras without slow dimension growth in 1996 [32]. In the sequel we prove that there does indeed exist a theory of flat dimension growth for AH algebras, and that the natural way to study this theory is through an invariant we call the *dimension–rank ratio*. This invariant for unital AH algebras takes values in the nonnegative reals and recovers, roughly, the minimum possible value of the limit in (3). It turns out to be of the variety we seek: it is insensitive to differences between $\mathcal{Z}$-stable algebras (provided that they are simple and of real rank zero); it detects subtle differences between non-$\mathcal{Z}$-stable algebras. It is also naturally connected to ordered K-theory. These connections lead us to define an invariant for general unital and stably finite $C^*$-algebras—the *radius of comparison*—which measures of the failure of comparison in the Cuntz semigroup. This, we argue, is the appropriate abstraction of the dimension–rank ratio. Both invariants can be viewed as measuring the ratio of the matricial size of a $C^*$-algebra to its topological dimension (as constituted by Kirchberg and Winter's decomposition rank—see [16]), despite the fact that both quantities are frequently infinite for non-$\mathcal{Z}$-stable algebras.

Our paper is organised as follows. In Section 2 we give the precise definition of the dimension–rank ratio, give a formula for it in the case of semi-homogeneous $C^*$-algebras, and examine its behaviour with respect to common constructions; in Section 3 we draw connections between the dimension–rank ratio and ordered K-theory; Section 4 shows that, among simple algebras of real rank zero, the minimal instance of the dimension–rank ratio coincides with the condition of being tracially AF; the range of the dimension–rank ratio is shown to be exhausted by simple algebras in Section 5; Section 6 introduces the radius of comparison, defined for any unital and stably finite $C^*$-algebra, and establishes analogues of some of our earlier results on the dimension–rank ratio.

## 2. The dimension–rank ratio of an AH algebra

**Definition 2.1.** Let $A$ be an unital AH algebra. Define the dimension–rank ratio of $A$ (write $\operatorname{drr}(A)$) to be the infimum of the set of strictly positive reals $c$ such that $A$ has a decomposition satisfying

$$\limsup_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\operatorname{rank}(p_{i,l})} \right\} = c,$$

whenever this set is not empty, and $\infty$ otherwise.

By compressing the inductive sequence decomposition for $A$ if necessary, one can replace the lim sup of Definition 2.1 with a limit. It is sketched in [1] and proved in [10] that the spaces $X_{i,l}$ in an injective decomposition for an unital AH algebra $A$ can always be replaced by CW-complexes $\tilde{X}_{i,l}$ of the same dimension. From here on we will assume, unless otherwise noted, that the $X_{i,l}$s are CW-complexes. It is also true that if one has a decomposition for $A$ as in Definition 2.1 which is not injective, then it can be replaced with an injective decomposition for which the limit in Definition 2.1 is no larger (cf. [8]). Thus, we may assume that the decomposition of the definition is injective whenever this is convenient.

Our first proposition collects some basic properties of the dimension–rank ratio.

**Proposition 2.2.** *Let $A$, $B$ be unital AH algebras, and $I$ an ideal of $A$. Then*:

(i) $\mathrm{drr}(A/I) \leqslant \mathrm{drr}(A)$;
(ii) $\mathrm{drr}(A \oplus B) = \max\{\mathrm{drr}(A), \mathrm{drr}(B)\}$;
(iii) $\mathrm{drr}(A \otimes \mathrm{M}_k) \leqslant (1/k)\,\mathrm{drr}(A)$;
(iv) *if $A$ and $B$ are simple and of finite dimension–rank ratio, then $A \otimes B$ has slow dimension growth and* $\mathrm{drr}(A \otimes B) = 0$.

**Proof.** For (i), let $\epsilon > 0$ be given, and fix an injective decomposition for $A$ such that for every $i \in \mathbb{N}$ one has

$$\max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} = \mathrm{drr}(A) + \epsilon.$$

Let $I$ be an ideal of $A$, and let $\phi_{i\infty} : A_i \to A$ be the canonical map. Then, $I_i := \phi_{i\infty}^{-1}(I)$ is an ideal of $A_i$ for every $i \in \mathbb{N}$. Define $\psi_i : A_i/I_i \to A_{i+1}/I_{i+1}$ by

$$\psi_i(a + I_i) = \phi_i(a) + I_{i+1}.$$

One can then check that $A/I = \lim_{i\to\infty}(A_i/I_i, \psi_i)$. It is well known that

$$A_i/I_i = \bigoplus_{l=1}^{m_i} p_{i,l}\big(\mathrm{C}(Y_{i,l}) \otimes \mathcal{K}\big) p_{i,l}$$

for closed subspaces $Y_{i,l} \subseteq X_{i,l}$, $i \in \mathbb{N}$, $1 \leqslant l \leqslant m_i$. Since $\dim(Y_{i,l}) \leqslant \dim(X_{i,l})$, we have

$$\lim_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(Y_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} \leqslant \mathrm{drr}(A) + \epsilon.$$

Since $\epsilon$ was arbitrary, we conclude that $\mathrm{drr}(A/I) \leqslant \mathrm{drr}(A)$.

For (ii), we clearly have $\mathrm{drr}(A \oplus B) \leqslant \max\{\mathrm{drr}(A), \mathrm{drr}(B)\}$. $A$ and $B$ are ideals of $A \oplus B$, so we may use (i) to obtain the reverse inequality.

(iii) is straightforward.

For (iv), fix decompositions $A = \lim_{i\to\infty}(A_i, \phi_i)$ and $B = \lim_{j\to\infty}(B_j, \psi_j)$, where

$$A_i = \bigoplus_{l=1}^{m_i} p_{i,l}\big(\mathrm{C}(X_{i,l}) \otimes \mathcal{K}\big) p_{i,l}, \quad \text{and} \quad B_j = \bigoplus_{s=1}^{n_i} q_{j,s}\big(\mathrm{C}(Y_{j,s}) \otimes \mathcal{K}\big) q_{j,s}.$$

Assume, as we may, that

$$\lim_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\operatorname{rank}(\mathrm{p_{i,l}})} \right\} = \operatorname{drr}(A) + \epsilon_1$$

and

$$\lim_{j\to\infty} \max_{1\leqslant s\leqslant n_i} \left\{ \frac{\dim(Y_{j,s})}{\operatorname{rank}(\mathrm{q_{j,s}})} \right\} = \operatorname{drr}(B) + \epsilon_2,$$

for some $\epsilon_1, \epsilon_2 > 0$. $A \otimes B$ is the limit of the inductive system $(A_i \otimes B_i, \phi_i \otimes \psi_i)$, and

$$A_i \otimes B_i = \bigoplus_{l,s} (p_{i,l} \otimes q_{j,s}) \big(\mathrm{C}(X_{i,l} \times Y_{j,s}) \otimes \mathcal{K}\big) (p_{i,l} \otimes q_{j,s}).$$

We have the inequalities

$$\operatorname{drr}(A \otimes B) \leqslant \limsup_{i\to\infty} \max_{l,s} \left\{ \frac{\dim(X_{i,l}) + \dim(Y_{i,s})}{\operatorname{rank}(p_{i,l}) \operatorname{rank}(q_{i,s})} \right\} \tag{4}$$

$$\leqslant \limsup_{i\to\infty} \max_{l,s} \left\{ \frac{\dim(X_{i,l})}{\operatorname{rank}(p_{i,l}) \operatorname{rank}(q_{i,s})} + \frac{\dim(Y_{i,s})}{\operatorname{rank}(p_{i,l}) \operatorname{rank}(q_{j,s})} \right\} \tag{5}$$

$$\leqslant \limsup_{i\to\infty} \max_{l,s} \left\{ \frac{\operatorname{drr}(A) + \epsilon_1}{\operatorname{rank}(q_{i,s})} + \frac{\operatorname{drr}(B) + \epsilon_2}{\operatorname{rank}(p_{i,l})} \right\}. \tag{6}$$

Since $A$ and $B$ are simple, we have

$$\operatorname{rank}(q_{i,s}), \operatorname{rank}(p_{i,l}) \xrightarrow{i\to\infty} \infty.$$

It follows that the right-hand side of (6) is equal to zero, whence $A \otimes B$ has slow dimension growth and $\operatorname{drr}(A \otimes B) = 0$. □

We suspect that equality holds in (iii) above, but it is unclear how a proof might proceed; a decomposition for $A \otimes \mathrm{M}_k$ need not respect the tensor product structure, and so does not give rise to an obvious decomposition of $A$. An inductive limit of AH algebras is only approximated locally by semi-homogeneous algebras, and the latter condition is strictly weaker than approximate homogeneity [6]. Thus, it does not make sense to investigate the behaviour of the dimension–rank ratio for inductive limits of AH algebras.

Our first theorem shows that the dimension–rank ratio behaves as one would like for semi-homogeneous $C^*$-algebras. Note that the spectrum of $B$ in the proposition below need not be a CW-complex, and need not be of finite covering dimension.

**Theorem 2.3.** *Let $B = \bigoplus_{j=1}^{n} B_j$ be a direct sum of homogeneous $C^*$-algebras*

$$B_j = p_j \big(\mathrm{C}(X_j) \otimes \mathcal{K}\big) p_j.$$

*Then,*

$$\mathrm{drr}(B) = \max_{1 \leqslant j \leqslant n} \left\{ \frac{\dim(X_j)}{\mathrm{rank}(p_j)} \right\}.$$

**Proof.** By part (i) of Proposition 2.2, it will be enough to establish the theorem for $n = 1$ and $X_1$ connected.

Suppose first that $B = p(\mathrm{C}(X) \otimes \mathcal{K})p$ for some connected compact Hausdorff space $X$ of finite covering dimension. Clearly,

$$\mathrm{drr}(B) \leqslant \frac{\dim(X)}{\mathrm{rank}(p)},$$

since we may write $B = \lim_{i\to\infty}(B, \mathrm{id}_B)$. Let $B = \lim_{i\to\infty}(A_i, \phi_i)$ be an injective decomposition for $B$, where the $A_i$ and $\phi_i$ are as in (1) and (2). Let $\mathrm{dr}(\cdot)$ denote the decomposition rank of a nuclear $C^*$-algebra. In [34] it is proved that

$$\mathrm{dr}\big(p(\mathrm{C}(X) \otimes \mathcal{K})p\big) = \dim(X)$$

whenever $X$ is a compact. Section 3 of [16] shows that

$$\mathrm{dr}(C \oplus D) = \max\big\{\mathrm{dr}(C), \mathrm{dr}(D)\big\}$$

for any nuclear $C$ and $D$. It follows that $\mathrm{dr}(B) = \dim(X)$, and that

$$\mathrm{dr}(A_i) = \max_{1 \leqslant l \leqslant m_i} \big\{\dim(X_{i,l})\big\}.$$

If $\mathrm{dr}(A_i) \leqslant n$ for every $i \in \mathbb{N}$, then $\mathrm{dr}(B) \leqslant n$, again by [16, Section 3]. By dropping terms from the inductive sequence for $B$, we may assume that $\mathrm{dr}(A_i) = \mathrm{dr}(B)$ for every $i \in \mathbb{N}$. In other words there exists, for each $i \in \mathbb{N}$, an $1 \leqslant l_i \leqslant m_i$ such that

$$\dim(X_{i,l_i}) = \dim(X).$$

If

$$\mathrm{rank}(p_{i,l}) > \max_{1 \leqslant j \leqslant n} \big\{\mathrm{rank}(p_j)\big\},$$

then the canonical map from $p_{i,l}(\mathrm{C}(X_{i,l}) \otimes \mathcal{K})p_{i,l}$ to $B$ must be zero, contradicting the injectivity of the decomposition. Thus,

$$\mathrm{rank}(p_{i,l}) \leqslant \max_{1 \leqslant j \leqslant n} \big\{\mathrm{rank}(p_j)\big\}, \quad \forall i \in \mathbb{N},\ \forall 1 \leqslant l \leqslant m_i.$$

Suppose that

$$\max_{1 \leqslant l \leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} < \frac{\dim(X)}{\mathrm{rank}(p)}.$$

Then,

$$\frac{\dim(X_{i,l_i})}{\operatorname{rank}(p_{i,l_i})} < \frac{\dim(X)}{\operatorname{rank}(p)},$$

which, since $\dim(X_{i,l_i}) = \dim(X)$, implies that

$$\frac{1}{\operatorname{rank}(p_{i,l_i})} < \frac{1}{\operatorname{rank}(p)}.$$

But this implies that $\operatorname{rank}(p_{i,l_i}) > \operatorname{rank}(p)$, a contradiction. It follows that

$$\operatorname{drr}\bigl(p\bigl(\mathrm{C}(X) \otimes \mathcal{K}\bigr)p\bigr) = \frac{\dim(X)}{\operatorname{rank}(p)}.$$

If $X$ is infinite-dimensional, then the decomposition rank argument from the second paragraph of the proof allows us to assume that for each $i \in \mathbb{N}$, there is $1 \leqslant l_i \leqslant m_i$ such that

$$\dim(X_{i,l_i}) \geqslant i,$$

and that the partial map from $p_{i,l_i}(\mathrm{C}(X_{i,l_i}) \otimes \mathcal{K})p_{i,l_i}$ is not zero. On the other hand, rank considerations show that there is $M > 0$ such that $\operatorname{rank}(p_{i,j}) < M$ whenever the partial map from $p_{i,j}(\mathrm{C}(X_{i,j}) \otimes \mathcal{K})p_{i,j}$ is not zero. Thus,

$$\lim_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\operatorname{rank}(p_{i,l})} \right\} = \infty$$

for every decomposition, and $\operatorname{drr}(B) = \infty$, as desired. □

**Corollary 2.4.** *Let* $A = \lim_{i\to\infty}(A_i, \phi_i)$ *be an unital AH algebra, where each* $A_i$ *is semi-homogeneous. Then,* $\operatorname{drr}(A) \leqslant \liminf_{i\to\infty} \operatorname{drr}(A_i)$.

**Proof.** There is a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that

$$\lim_{k\to\infty} \operatorname{drr}(A_{n_k}) = \liminf_{i\to\infty} \operatorname{drr}(A_i)$$

in the extended reals, and $A = \lim_{k\to\infty}(A_{n_k}, \phi_{n_k})$. Assuming the notation from (2) for the $A_{n_k}$s, we have

$$\operatorname{drr}(A_{n_k}) = \max_{1\leqslant l\leqslant m_{n_k}} \left\{ \frac{\dim(X_{n_k,l})}{\operatorname{rank}(p_{n_k,l})} \right\} \xrightarrow{k\to\infty} \liminf_{i\to\infty} \operatorname{drr}(A_i).$$

This gives $\operatorname{drr}(A) \leqslant \liminf_{i\to\infty} \operatorname{drr}(A_i)$ by definition. □

We conclude this section by noting a connection between the dimension–rank ratio and Rieffel's stable rank for $C^*$-algebras [22]. Let $\operatorname{sr}(A)$ denote the stable rank of a $C^*$-algebra $A$, and let $\lceil x \rceil$ (respectively $\lfloor x \rfloor$) denote the least (respectively greatest) integer greater (respectively less) than $x \in \mathbb{R}$. Consider the following formula, established by Nistor in [19]:

$$\operatorname{sr}\bigl(p\bigl(\mathrm{C}(X) \otimes \mathcal{K}\bigr)p\bigr) = \left\lceil \frac{\lfloor \dim(X)/2 \rfloor}{\operatorname{rank}(p)} \right\rceil + 1 \tag{7}$$

whenever $X$ is a compact Hausdorff space and $p \in \mathrm{C}(X) \otimes \mathcal{K}$ is a projection of constant rank. Clearly, the right-hand side is all but equal to $2\operatorname{drr}(p(\mathrm{C}(X) \otimes \mathcal{K})p)$, with any difference owing to the fact that the dimension–rank ratio need not be an integer. This observation leads to:

**Proposition 2.5.** *Let $A$ be an unital AH algebra. Then,*

$$\operatorname{drr}(A) \geqslant \frac{\operatorname{sr}(A)}{2} - 1.$$

**Proof.** The proposition is trivial if $\operatorname{sr}(A) = 1, 2$.

Suppose that $\operatorname{sr}(A) < \infty$. Theorem 5.1 of [19] states that if $A = \lim_i (A_i, \phi_i)$ is an inductive limit algebra where $\operatorname{sr}(A_i) \leqslant n$, $\forall i \in \mathbb{N}$, then $\operatorname{sr}(A) \leqslant n$. Thus, we may assume that regardless of the decomposition $A = \lim_{i \to \infty}(A_i, \phi_i)$, one has $\operatorname{sr}(A_i) \geqslant \operatorname{sr}(A)$. If the $A_i$ are direct sums of homogeneous building blocks as in Eq. (2), then by (7) above we have

$$\left\lceil \frac{\lfloor \dim(X_{i,l})/2 \rfloor}{\operatorname{rank}(p_{i,l})} \right\rceil + 1 \geqslant \operatorname{sr}(A)$$

for some $1 \leqslant l \leqslant m_i$. Straightforward calculation yields $\operatorname{drr}(A_i) \geqslant (\operatorname{sr}(A) - 2)/2$, so that $\limsup_{i \to \infty} \operatorname{drr}(A_i) \geqslant (\operatorname{sr}(A) - 2)/2$. Since the decomposition of $A$ was arbitrary, we conclude that $\operatorname{drr}(A) \geqslant (\operatorname{sr}(A) - 2)/2$.

The case of $\operatorname{sr}(A) = \infty$ is similar. $\square$

## 3. Ordered K-theory

In this section we establish connections between the dimension–rank ratio and the ordered K-theory of AH algebras. We examine first the case of a homogeneous $C^*$-algebra with spectrum a compact metric space of finite covering dimension.

**Theorem 3.1.** (See Husemoller [13, Chapter 8, Theorems 1.2 and 1.5].) *Let $X$ be a compact metric space of covering dimension $n \in \mathbb{N}$, and let $\gamma, \omega$ be complex vector bundles over $X$.*

(i) *If $\gamma$ and $\omega$ are stably isomorphic and the fibre dimension of $\gamma$ is greater than or equal to $\lceil n/2 \rceil$, then $\gamma$ and $\omega$ are isomorphic.*
(ii) *If the fibre dimension of $\gamma$ exceeds that of $\omega$ by an amount greater than or equal to $\lceil n/2 \rceil$, then $\omega$ is isomorphic to a sub-bundle of $\gamma$.*

Making the identifications

$$\mathrm{K}_0\bigl(p\bigl(\mathrm{C}(X) \otimes \mathcal{K}\bigr)p\bigr) \equiv \mathrm{K}_0\bigl(\mathrm{C}(X)\bigr) \equiv \mathrm{K}^0(X),$$

we recast Theorem 3.1 in terms of K-theory (this is standard fare). Let $p, r \in \mathrm{M}_\infty(\mathrm{C}(X))$ be projections, and let $[p], [r]$ denote their $\mathrm{K}_0$-classes. Then parts (i) and (ii) of Theorem 3.1 are equivalent to the following two statements, respectively:

(i) if $[p] = [r]$ and $\mathrm{rank}(p) \geqslant \lceil \dim(X)/2 \rceil$, then $p$ and $r$ are Murray–von Neumann equivalent;
(ii) if $\mathrm{rank}(p) - \mathrm{rank}(r) \geqslant \lceil \dim(X)/2 \rceil$, then $r$ is Murray–von Neumann equivalent to a subprojection of $p$ ($r \prec p$). In particular $[p] - [r] \in \mathrm{K}_0(\mathrm{C}(X))^+$.

Let $A$ be an unital stably finite $C^*$-algebra, and let $\mathrm{QT}(A)$ denote the compact convex set of normalised quasi-traces on $A$. (A deep theorem of Haagerup [12] asserts that every quasi-trace on an unital and exact $C^*$-algebra $A$ is a trace. Thus, when $A$ is exact, unital, and stably finite, we identify $\mathrm{QT}(A)$ with the space $\mathrm{T}(A)$ of normalised traces on $A$.) We recall three familiar concepts in the K-theory of $C^*$-algebras:

(i) If projections $p, q \in \mathrm{M}_\infty(A)$ are Murray–von Neumann equivalent whenever $[p] = [q] \in \mathrm{K}_0 A$, then $A$ is said to have *cancellation of projections* (or simply *cancellation*).
(ii) If the condition that $\tau(p) < \tau(q)$ for every $\tau \in \mathrm{QT}(A)$ implies that $p$ is Murray–von Neumann equivalent to a subprojection of $q$, then we say that $A$ has (FCQ)—$A$ satisfies *Blackadar's second fundamental comparability question*.
(iii) If, given elements $x_1, x_2, y_1, y_2$ in a partially ordered Abelian group $(G, G^+)$ such that $x_i \leqslant y_j$, $i, j \in \{1, 2\}$, there exists $z \in G$ such that $x_i \leqslant z \leqslant y_j$, $i, j \in \{1, 2\}$, then we say that $G$ has the *Riesz interpolation property* (or simply *interpolation*).

Our next definition generalises these notions and another besides.

**Definition 3.2.** Let $A$ be an unital and stably finite $C^*$-algebra, $p, q \in \mathrm{M}_\infty(A)$ projections, and $r \geqslant 0$.

(i) Say that $A$ has $r$-cancellation if $p$ and $q$ are Murray–von Neumann equivalent whenever $[p] = [q]$ and

$$\tau(p) = \tau(q) > r, \quad \forall \tau \in \mathrm{QT}(A).$$

(ii) Say that $A$ has $r$-(FCQ) if $p$ is Murray–von Neumann equivalent to a subprojection of $q$ whenever

$$\tau(p) + r < \tau(q), \quad \forall \tau \in \mathrm{QT}(A).$$

(iii) Let $(G, G^+, u)$ be a partially ordered Abelian group with distinguished order unit $u$ and state space $S(G)$. Let $r > 0$. Say that $G$ has $r$-interpolation if whenever one has elements $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leqslant y_j$, $i, j \in \{1, 2\}$, and

$$s(x_i) + r < s(y_j), \quad i, j \in \{1, 2\}, \ \forall s \in S(G),$$

then there exists $z \in G$ such that $x_i \leqslant z \leqslant y_j$, $i, j \in \{1, 2\}$.

(iv) Let $(M, u)$ be a positive ordered semigroup with distinguished strong order unit $u$ and state space $S(M)$. Let $r > 0$ and $x, y \in M$. Say that $M$ has *$r$-strict comparison* if

$$s(x) + r < s(y), \quad \forall s \in S(M),$$

implies that $x \leqslant y$ in $M$.

We will prove that the elements of Definition 3.2 are connected naturally to the dimension–rank ratio.

To prepare the next proposition, recall that a positive ordered semigroup $(M, \leqslant)$ is said to have an *algebraic order* if whenever one has $x, y \in M$ such that $x \leqslant y$, then there is $z \in M$ such that $x + z = y$. $M$ is said to be *cancellative* if whenever one has elements $x, y, z \in M$ such that $x + z = y + z$, then $x = y$ (cf. [9]).

**Proposition 3.3.** *Let $(M, v)$ be a positive ordered semigroup with distinguished strong order unit $v$. Suppose that the order on $M$ is algebraic, and that $M$ is cancellative. Let $G$ be the Grothendieck enveloping group of $M$. Let $\iota : M \to G$ denote the Grothendieck map, and put $G^+ = \iota(M)$, $u = \iota(v)$. Let $S(G)$ denote the state space of $G$.*

*Let $r > 0$ and $x, y \in G$. Then,*

$$s(x) + r < s(y), \quad \forall s \in S(G),$$

*implies that $x \leqslant y$ in $G$ if and only if $(M, v)$ has $r$-strict comparison.*

**Proof.** Our hypotheses on $M$ imply that $(M, v) \cong (\iota(M), \iota(v)) = (G^+, u)$, whence $(G^+, u)$ has $r$-strict comparison if and only if $(M, v)$ does.

We may identify $S(G)$ and $S(G^+)$, whence the forward implication follows from restricting to $G^+$. (There is a subtle point here: states on partially ordered Abelian groups are merely positive homomorphisms into the reals which take the order unit to $1 \in \mathbb{R}$, whereas states on ordered Abelian semigroups are, in addition, order preserving. We are using the fact that $\iota(M) \cong M$ whenever $M$ is algebraically ordered and cancellative to make our identification of state spaces [9]. We are grateful to F. Perera for pointing this out to us.)

Now suppose that $(G^+, u)$ has $r$-strict comparison. Let $x, y \in G$ and write

$$x = x_+ - x_-, \qquad y = y_+ - y_-,$$

where $x_+, x_-, y_+, y_- \in G^+$. If

$$s(x) + r < s(y), \quad \forall s \in S(G),$$

then

$$s(x_+ + y_-) + r < s(y_+ + x_-), \quad \forall s \in S(G) \equiv S\big(G^+\big),$$

whence $x_+ + y_- \leqslant y_+ + x_-$ in $G^+$. It follows that $x \leqslant y$, as desired. $\square$

In light of the proposition above, we will say that a partially ordered Abelian group $(G, G^+, u)$ such that $G^+ \cong \iota(G^+)$ has $r$-strict comparison whenever $(G^+, u)$ does; this definition makes sense for the ordered $\mathrm{K}_0$-group of an unital and stably finite $C^*$-algebra.

**Proposition 3.4.** *Let $A$ be an unital and stably finite $C^*$-algebra, and $(G, G^+, u)$ a partially ordered group as in the statement of Proposition 3.3. Then, the following sets are closed*:

(i) $A_1 := \{r \in \mathbb{R} \mid A \text{ has } r\text{-cancellation}\}$;
(ii) $A_2 := \{r \in \mathbb{R} \mid A \text{ has } r\text{-(FCQ)}\}$;
(iii) $A_3 := \{r \in \mathbb{R} \mid G \text{ has } r\text{-strict comparison}\}$;
(iv) $A_4 := \{r \in \mathbb{R} \mid G \text{ has } r\text{-interpolation}\}$.

**Proof.** For each $i \in \{1, 2, 3, 4\}$ one has that $s \in A_i$ whenever $s > r$ and $r \in A_i$, so it will suffice to prove that $\alpha_i := \inf(A_i) \in A_i$. The proof of each case follows a common thread.

For (i), let there be given projections $p, q \in \mathrm{M}_\infty(A)$ such that

$$[p] = [q], \qquad \tau(p) = \tau(q) > \alpha_1, \quad \forall \tau \in \mathrm{QT}(A).$$

The map $\tau \mapsto \tau(p)$ on $\mathrm{QT}(A)$ is continuous and $\mathrm{QT}(A)$ is compact, so this map achieves a minimum value $\delta > \alpha_1$. Since $\delta \in A_1$, we conclude that $p$ and $q$ are Murray–von Neumann equivalent, as desired.

For (ii), let there be given projections $p, q \in \mathrm{M}_\infty(A)$ such that

$$\tau(p) + \alpha_2 < \tau(q), \quad \forall \tau \in \mathrm{QT}(A).$$

The map $\tau \mapsto \tau(q) - \tau(p)$ is continuous on the compact space $\mathrm{QT}(A)$, and so achieves a minimum value $\delta > \alpha_2$. Thus,

$$\tau(p) + \delta < \tau(q), \quad \forall \tau \in \mathrm{QT}(A).$$

Since $\delta \in A_2$, the desired conclusion follows.

For (iii), let $x, y \in G^+$ be such that

$$s(x) + \alpha_3 < s(y), \quad \forall s \in S(G).$$

The map

$$s \mapsto s(y) - s(x)$$

is strictly positive and continuous, and the space $S(G)$ is compact (cf. [9, Proposition 6.2]). Thus, this map achieves a minimum value $\delta > \alpha_3$. We now have

$$s(x) + \delta < s(y), \quad \forall s \in S(G).$$

Since $\delta \in A_3$, the desired conclusion follows.

For (iv), let there be given elements $x_1, x_2, y_1, y_2 \in G$ satisfying

$$s(x_i) + \alpha_4 < s(y_j), \quad i, j \in \{1, 2\},\ \forall s \in S(G).$$

For each pair $(i, j)$, $i, j \in \{1, 2\}$, there exists $r_{i,j} > 0$ such that

$$s(x_i) + \alpha_4 + r_{i,j} < s(y_j), \quad \forall s \in S(G).$$

Put $\delta = \alpha_4 + \min\{r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}\}$. Now

$$s(x_i) + \delta < s(y_j), \quad i, j \in \{1,2\},\ \forall s \in S(G),$$

and $\delta \in A_4$. We conclude that there is an interpolating element $z \in G$ such that $x_i \leqslant z \leqslant y_j$, $\forall i, j \in \{1,2\}$. □

Definition 3.2 can be used to summarise the natural connections between the K-theory of homogeneous $C^*$-algebras and their dimension–rank ratios.

**Proposition 3.5.** *Let $A = p(\mathrm{C}(X) \otimes \mathcal{K})p$, where $X$ is a compact metric space of finite covering dimension. Then*:

(i) *$A$ has* $(\mathrm{drr}(A)/2)$*-cancellation*;
(ii) *$A$ has* $(\mathrm{drr}(A)/2)$-(FCQ);
(iii) *$\mathrm{K}_0 A$ has* $(\mathrm{drr}(A) + 1/\mathrm{rank}(p))$*-interpolation*;
(iv) *$\mathrm{K}_0 A^+$ has* $(\mathrm{drr}(A)/2)$*-strict comparison.*

**Proof.** (i), (ii), and (iv) are straightforward: combine Definition 3.2 with Theorem 3.1(ii). We prove (iii), which is slightly more involved.

Let $s$ denote the unique (geometric) state on $\mathrm{K}_0 A$, and recall that for a projection $r \in \mathrm{M}_\infty(A)$ we have

$$s\big([r]\big) = \frac{\mathrm{rank}(r)}{\mathrm{rank}(p)}.$$

For the remainder of the proof, let $r, q \in \mathrm{M}_\infty(A)$ be projections.

Assume that we are given four elements $x_1, x_2, y_1, y_2 \in \mathrm{K}_0 A$ such that $x_i \leqslant y_j$, $i, j \in \{1,2\}$, and

$$s(x_i) + \mathrm{drr}(A) + 1/\mathrm{rank}(p) < s(y_j), \quad i, j \in \{1,2\}. \tag{8}$$

Every element $x \in \mathrm{K}_0 A$ can be written as a difference of $\mathrm{K}_0$-classes of projections, say $x = [q] - [r]$. The difference $\mathrm{rank}(q) - \mathrm{rank}(r)$ is commonly referred to as the *virtual dimension* of $x$. We will let $\mathrm{rank}(x)$ denote this virtual dimension, thus extending the notion of rank to all of $\mathrm{K}_0 A$. With this notation we have

$$s(x) = \frac{\mathrm{rank}(x)}{\mathrm{rank}(p)}, \quad \forall x \in \mathrm{K}_0 A.$$

We may now rewrite (8) above as

$$\frac{\mathrm{rank}(x_i)}{\mathrm{rank}(p)} + \frac{\dim(X)}{\mathrm{rank}(p)} + \frac{1}{\mathrm{rank}(p)} < \frac{\mathrm{rank}(y_j)}{\mathrm{rank}(p)},$$

which yields

$$\mathrm{rank}(y_j) - \mathrm{rank}(x_i) > \dim(X) + 1, \quad i, j \in \{1,2\}.$$

Let $z$ be any element of $\mathrm{K}_0 A$ such that

$$\mathrm{rank}(z) = \max\big\{\mathrm{rank}(x_1), \mathrm{rank}(x_2)\big\} + \big\lceil \dim(X)/2 \big\rceil.$$

Then

$$\mathrm{rank}(z - x_i), \mathrm{rank}(y_j - z) \geqslant \big\lceil \dim(X)/2 \big\rceil, \quad i, j \in \{1, 2\},$$

and $z$ is the desired interpolating element by Theorem 3.1(ii). □

We shall see below that Proposition 3.5 can be generalised to the setting of general unital AH algebras, provided that the algebras have ordered $\mathrm{K}_0$-groups which admit a unique state.

**Example 3.6.** While part (iii) of Proposition 3.5 gives a positive real $r$ such that the algebra $A$ as in the hypotheses has $r$-interpolation, it is not immediately clear that there may be a nonzero lower bound on the set of all such reals. But be one there may. Consider, for any natural number $n > 1$, the $C^*$-algebra $A = \mathrm{M}_n(\mathrm{C}(S^{2n}))$. Clearly, $\mathrm{drr}(A) = 2$. The ordered $\mathrm{K}_0$-group of $A$ is well known: it is isomorphic as a group to $\mathbb{Z} \oplus \mathbb{Z}$; the first co-ordinate is generated by the $\mathrm{K}^0$-class $[\theta_1]$ of the trivial line bundle $\theta_1$; the second co-ordinate is generated by the difference $[\xi] - [\theta_n]$, where $\xi$ is the bundle corresponding to the $n$-dimensional Bott projection and $\theta_n$ is the trivial bundle of fibre dimension $n$; the positive cone $\mathrm{K}_0 A^+$ is

$$\big\{(x, y) \mid y = 0 \text{ and } x \geqslant 0\big\} \cup \big\{(x, y) \mid x \geqslant n\big\}.$$

Put

$$x_1 = 0 \oplus 0, \qquad x_2 = 0 \oplus 1, \qquad y_1 = n \oplus 0, \qquad y_2 = n \oplus 1.$$

With the description of $\mathrm{K}_0 A^+$ in hand, one checks easily that

$$x_i \leqslant y_j, \quad i, j \in \{1, 2\}$$

in $\mathrm{K}_0 A$, yet there is no $z \in \mathrm{K}_0 A$ which interpolates these four elements. The unique geometric state $s$ on $\mathrm{K}_0 A$ returns the rank of a $\mathrm{K}_0$ element divided by $n$, whence

$$s(x_i) + r < s(y_j), \quad \forall r < 1,\ i, j \in \{1, 2\}.$$

Thus, $\mathrm{K}_0 A$ does not have $r$-interpolation for any $r < \mathrm{drr}(A)/2$.

**Lemma 3.7.** *Let $A \cong \lim_{i \to \infty}(A_i, \phi_i)$ be an unital AH algebra, where each $A_i$ is homogeneous with connected spectrum. Then, $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$ is a simple partially ordered Abelian group admitting a unique state.*

**Proof.** For each $i \in \mathbb{N}$ write

$$A_i = p_i\big(\mathrm{C}(X_i) \otimes \mathcal{K}\big) p_i,$$

where $X_i$ is a compact connected Hausdorff space, and $p_i \in \mathrm{C}(X_i) \otimes \mathcal{K}$ is a projection. As noted following Definition 2.1, the $X_i$ may be assumed to have finite covering dimension (cf. [1,10]). $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is a partially ordered Abelian group for every stably finite $A$ (cf. [2, Chapter 6, Section 3]).

There is a unique (geometric) state on $\mathrm{K}_0 A_i$ which returns the normalised rank of a projection corresponding to a positive $\mathrm{K}_0$-class, and is extended to all of $\mathrm{K}_0 A_i$ by linearity. By [9, Proposition 6.14], $S(\mathrm{K}_0 A)$ is the inverse limit of the $S(\mathrm{K}_0 A_i)$s, whence $\mathrm{K}_0 A$ admits a unique state.

It remains to prove that $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is a simple ordered group, i.e., that every non-zero positive element is an order unit. It will suffice to prove that each $(\mathrm{K}_0(A_i), \mathrm{K}_0(A_i)^+)$ is a simple ordered group. Each element of $\mathrm{K}_0(A_i) = \mathrm{K}^0(X_i)$ corresponds to a difference $x = [q] - [p]$, where $q, p \in \mathrm{M}_\infty(A_i)$ are projections. Let $y = [e] - [f] \in \mathrm{K}_0 A_i$, where $e, f \in \mathrm{M}_\infty(A_i)$ are projections. If $x$ is positive, then $\mathrm{rank}(q) > \mathrm{rank}(p)$. In particular, there exists $n \in \mathbb{N}$ such that

$$\mathrm{rank}(nx) - \big(\mathrm{rank}\big([e]\big) - \mathrm{rank}\big([f]\big)\big) \geqslant \big\lceil \dim(X_i)/2 \big\rceil,$$

so $ny \geqslant y$ by Theorem 3.1(ii), and $x$ is an order unit. □

We will need the following result to prove our next lemma:

**Theorem 3.8.** (See Goodearl [9, Proposition 4.16].) *Let $(G, G^+, u)$ be a non-zero partially ordered Abelian group with distinguished order unit. If $G$ admits a unique state $s$, then for any $x \in G^+$ one has*

$$\begin{aligned} s(x) &= \inf\{l/n \mid l, n \in \mathbb{N} \text{ and } nx \leqslant lu\} \\ &= \sup\{k/m \mid k \in \mathbb{Z}^+,\ m \in \mathbb{N},\ \text{and } ku \leqslant mx\}. \end{aligned}$$

**Lemma 3.9.** *Let $A$ be an unital AH algebra, and suppose that $\mathrm{K}_0(A)$ admits a unique state. Let there be given a decomposition of $A$ as in Eqs.* (1) *and* (2) *and a tolerance $\epsilon > 0$. Then, for any $x \in \mathrm{K}_0(A)^+$ there exists $j \in \mathbb{N}$ such that*:

(i) *$x$ has a pre-image $x_j \in \mathrm{K}_0(A_j)$;*
(ii) *if*

$$A_j = \bigoplus_{l=1}^{m_j} p_{j,l}\big(\mathrm{C}(X_{j,l}) \otimes \mathcal{K}\big) p_{j,l},$$

*and $s_l$ denotes the state on $\mathrm{K}_0(A_j)$ which is equal to the* (*unique*) *geometric state $g_l$ on $\mathrm{K}_0(p_{j,l}(\mathrm{C}(X_{j,l}) \otimes \mathcal{K}) p_{j,l})$ and zero on the other direct summands of $A_j$, then*

$$\big|s_l(x) - s(x)\big| < \epsilon, \quad 1 \leqslant l \leqslant m_j.$$

**Proof.** By truncating the given inductive sequence for $A$, we may assume that $x$ has a pre-image in every $A_i$, $i \in \mathbb{N}$.

Using Theorem 3.8, find non-negative integers $r, n, k, m$ such that

$$r/n - s(x) < \epsilon/2, \qquad s(x) - k/m < \epsilon/2,$$

$nx \leqslant r[1_A]$, and $k[1_A] \leqslant mx$ inside $\mathrm{K}_0(A)$. The last two inequalities must hold already in some $A_j$, and, since $\mathrm{K}_0(A_j)$ has the direct sum order coming from the summands

$$\mathrm{K}_0\big(p_{j,l}\big(\mathrm{C}(X_{j,l}) \otimes \mathcal{K}\big)p_{j,l}\big),$$

they will still hold upon restricting to any such summand. Let $x_l$ denote the restriction of $x$ to $\mathrm{K}_0(p_{j,l}(\mathrm{C}(X_{j,l}) \otimes \mathcal{K})p_{j,l})$. We have

$$nx_l \leqslant r[p_{j,l}], \quad k[p_{j,l}] \leqslant mx_l, \quad 1 \leqslant l \leqslant m_j.$$

Since the geometric state $g_l$ on $\mathrm{K}_0(p_{j,l}\mathrm{M}_{k_{j,l}}(\mathrm{C}(X_{j,l}))p_{j,l})$ preserves order, we conclude that

$$k/m \leqslant g_l(x_l) \leqslant r/n.$$

Since $g_l(x_l) = s_l(x)$, the lemma follows. □

**Theorem 3.10.** *Let $A$ be an unital AH algebra with* $\mathrm{drr}(A) < \infty$*, and suppose that* $\mathrm{K}_0(A)$ *admits a unique state $s$. Then*:

(i) *$A$ has* $(\mathrm{drr}(A)/2)$*-cancellation*;
(ii) *$A$ has* $(\mathrm{drr}(A)/2)$-(FCQ);
(iii) $\mathrm{K}_0A$ *has* $(\mathrm{drr}(A)/2)$*-strict comparison.*

*If, in addition, $A$ is simple, then*

(iv) $\mathrm{K}_0(A)$ *has* $(\mathrm{drr}(A))$*-interpolation.*

**Proof.** We prove that $A$ has $(\mathrm{drr}(A)/2+\epsilon)$-cancellation, $(\mathrm{drr}(A)/2+\epsilon)$-(FCQ), $(\mathrm{drr}(A)/2+\epsilon)$-strict comparison, and $(\mathrm{drr}(A)+\epsilon)$-interpolation for every $\epsilon > 0$; the theorem then follows from Proposition 3.4. Let $\epsilon > 0$ be given.

For (i), let there be given projections $p, q \in \mathrm{M}_\infty(A)$ such that $[p] = [q]$ and

$$\tau(p) = \tau(q) = s\big([p]\big) = s\big([q]\big) > \mathrm{drr}(A)/2 + \epsilon, \quad \forall \tau \in \mathrm{T}(A).$$

Fix a decomposition $A = \lim_{i \to \infty}(A_i, \phi_i)$ where

$$A_i = \bigoplus_{l=1}^{m_i} p_{i,l}\big(\mathrm{C}(X_{i,l}) \otimes \mathcal{K}\big)p_{i,l}$$

and

$$\max_{1 \leqslant l \leqslant m_i} \left\{ \frac{\dim(X_{j,l})}{\mathrm{rank}(p_{j,l})} \right\} \leqslant \mathrm{drr} + \epsilon/2, \quad \forall i \in \mathbb{N}.$$

Use Lemma 3.9 to find $j \in \mathbb{N}$ such that $p$ and $q$ have pre-images at the level of $\mathrm{K}_0$ (which are projections) $\tilde{p}$ and $\tilde{q}$, respectively, in $\mathrm{M}_\infty(A_j)$ with the properties that $[\tilde{p}] = [\tilde{q}]$, $s([\tilde{p}]) = s([p])$, and

$$\big|s_l\big([\tilde{p}]\big) - s\big([\tilde{p}]\big)\big| = \big|s_l\big([\tilde{q}]\big) - s\big([\tilde{q}]\big)\big| < \frac{s(p) - \mathrm{drr}(A)}{4}, \quad 1 \leqslant l \leqslant m_j.$$

Since $s_l([\tilde{p}])$ represents the normalised rank of $\tilde{p}$ restricted to the direct summand $p_{j,l}(\mathrm{C}(X_{j,l})\otimes \mathcal{K})p_{j,l}$ of $A_j$, we conclude that this restriction is in the stable range of $\mathrm{K}_0(p_{j,l}(\mathrm{C}(X_{j,l})\otimes\mathcal{K})p_{j,l})$ (and similarly for the restriction of $\tilde{q}$). Thus, the said restrictions, having the same class in $\mathrm{K}_0$, are Murray–von Neumann equivalent by Theorem 3.1. It follows that $\tilde{p}$ and $\tilde{q}$ are Murray–von Neumann equivalent, whence so are $p$ and $q$. This shows that $A$ has $(\mathrm{drr}(A)/2+\epsilon)$-cancellation. Since $\epsilon$ was arbitrary, this proves (i).

For (ii)–(iv) we will retain the decomposition of $A$ from the proof of (i); for (ii) and (iii) we will retain as the pre-images $\tilde{p}$ and $\tilde{q}$ of $p$ and $q$ above, with the property that

$$\bigl|s_l\bigl([\tilde{p}]\bigr)-s\bigl([\tilde{p}]\bigr)\bigr|,\quad \bigl|s_l\bigl([\tilde{q}]\bigr)-s\bigl([\tilde{q}]\bigr)\bigr|<\frac{s(p)-\mathrm{drr}(A)}{4},\quad 1\leqslant l\leqslant m_j.$$

For (ii), let there be given projections $p,q\in \mathrm{M}_\infty(A)$ such that

$$\tau(p)+\mathrm{drr}(A)/2+\epsilon<\tau(q),\quad \forall\tau\in \mathrm{T}(A).$$

Since $\mathrm{K}_0A$ has a unique state $s$, the statement above is equivalent to

$$s\bigl([p]\bigr)+\mathrm{drr}(A)/2+\epsilon<s\bigl([q]\bigr).$$

Find pre-images $\tilde{p}$ and $\tilde{q}$ as before. Then, the virtual dimension of the restriction of $[\tilde{q}]-[\tilde{p}]$ to a direct summand $p_{j,l}(\mathrm{C}(X_{j,l})\otimes\mathcal{K})p_{j,l}$ of $A_j$ is in the stable range of $\mathrm{K}_0(p_{j,l}(\mathrm{C}(X_{j,l})\otimes\mathcal{K})p_{j,l})$, whence the said restriction is positive. The direct sum of these restrictions, namely, $[\tilde{q}]-[\tilde{p}]$ itself, is then positive. Write $[\tilde{q}]=[\tilde{p}]+[r]$ for some projection $r\in \mathrm{M}_\infty(A_j)$. Since

$$s\bigl([\tilde{p}]\bigr)+s\bigl([r]\bigr)=s\bigl([\tilde{q}]\bigr)=s\bigl([q]\bigr)>\mathrm{drr}(A)/2+\epsilon,$$

we conclude by (i) that $\tilde{p}\oplus r$ and $\tilde{q}$ are Murray–von Neumann equivalent. It follows that $\tilde{p}$ is equivalent to a subprojection of $\tilde{q}$, and similarly for $p$ and $q$. This proves that $A$ has $(\mathrm{drr}(A)/2+\epsilon)$-(FCQ), and so proves (ii).

$\mathrm{K}_0A$ has $(\mathrm{drr}(A)/2+\epsilon)$-strict comparison if and only if the same is true of the semigroup $(\mathrm{K}_0A^+,[1_A])$. The latter condition is equivalent to the statement that for $[p],[q]\in \mathrm{K}_0A^+$ such that

$$s\bigl([p]\bigr)+\mathrm{drr}(A)/2+\epsilon<s\bigl([q]\bigr),$$

one has $[p]\leqslant[q]$. This, in turn, follows from (i), proving (iii).

For (iv), we must prove that for any $x_1,x_2,y_1,y_2\in \mathrm{K}_0A$ such that

$$x_i\leqslant y_j,\quad s(x_i)+\mathrm{drr}(A)+\epsilon<s(y_j),\quad i,j\in\{1,2\},$$

there exists $z\in \mathrm{K}_0A$ such that $x_i\leqslant z\leqslant y_j$, $i,j\in\{1,2\}$. We may assume that $x_1=0$ and put $x_2=x$, for convenience—(iv) then follows by translating $z$.

Fix projections $p_{y_1},p_{y_2},p_x^+,p_x^-\in \mathrm{M}_\infty(A)$ such that

$$y_1=[p_{y_1}],\qquad y_2=[p_{y_2}];\qquad x=\bigl[p_x^+\bigr]-\bigl[p_x^-\bigr].$$

Find, as in the proof of (i), some $j \in \mathbb{N}$ such that $p_{y_1}, p_{y_2}, p_x^+$, and $p_x^-$ have pre-images (at the level of $\mathrm{K}_0$) $\tilde{p}_{y_1}, \tilde{p}_{y_2}, \tilde{p}_x^+$, and $\tilde{p}_x^-$ (all projections), respectively, in $\mathrm{M}_\infty(A_j)$, with the property that

$$\left|s_l([q]) - s([q])\right| < \frac{\epsilon}{4}, \quad \forall q \in \{p_{y_1}, p_{y_2}, p_x^+, p_x^-\},\ 1 \leqslant l \leqslant m_j.$$

We may assume, by the simplicity of $A$, that $j$ has also been chosen large enough to ensure that $1/\operatorname{rank}(p_{j,l}) \ll \epsilon/4$, $1 \leqslant l \leqslant m_j$.

Fix a summand $A_{j,l} = p_{j,l}(\mathrm{C}(X_{j,l}) \otimes \mathcal{K})p_{j,l}$ of $A_j$. This, by Proposition 3.5(iii), has

$$\frac{\dim(X_{j,l})}{\operatorname{rank}(p_{j,l})} + \frac{1}{\operatorname{rank}(p_{j,l})} \leqslant \operatorname{drr}(A) + \epsilon/2 + \epsilon/4 = \operatorname{drr}(A) + 3\epsilon/4$$

interpolation. The restrictions of $\tilde{p}_{y_1}, \tilde{p}_{y_2}, \tilde{p}_x^+$, and $\tilde{p}_x^-$ to $A_{j,l}$ are such that:

$$\operatorname{drr}(A) + 3\epsilon/4 < s_l([\tilde{p}_{y_k}|_{A_{j,l}}]), \quad k \in \{1,2\};$$
$$s_l([\tilde{p}_x^+|_{A_{j,l}}] - [\tilde{p}_x^-|_{A_{j,l}}]) + \operatorname{drr}(A) + 3\epsilon/4 < s_l([\tilde{p}_{y_k}|_{A_{j,l}}]), \quad k \in \{1,2\}.$$

It follows that there exists $z_l \in \mathrm{K}_0 A_{j,l}$ such that

$$0,\ [\tilde{p}_x^+|_{A_{j,l}}] - [\tilde{p}_x^-|_{A_{j,l}}] \leqslant z_l \leqslant [\tilde{p}_{y_1}|_{A_{j,l}}],\ [\tilde{p}_{y_2}|_{A_{j,l}}].$$

Thus,

$$0,\ [\tilde{p}_x^+] - [\tilde{p}_x^-] \leqslant \bigoplus_{l=1}^{m_j} z_l \leqslant [\tilde{p}_{y_1}],\ [\tilde{p}_{y_2}]$$

in $\mathrm{K}_0 A_j$, and, upon taking images in $\mathrm{K}_0 A$ and setting $z = \mathrm{K}_0(\phi_{j\infty})(\bigoplus_{l=1}^{m_j} z_l)$,

$$0,\ x \leqslant z \leqslant y_1,\ y_2,$$

as desired. □

## 4. A classification result

Clearly, slow dimension growth implies $\operatorname{drr} = 0$. This begs the obvious question:

**Question 4.1.** *Does* $\operatorname{drr}(A) = 0$ *imply that $A$ has slow dimension growth for every simple unital AH algebra $A$?*

The next theorem and corollary provide a positive answer to Question 4.1 in the case of simple algebras with real rank zero. It is plausible that this positive answer will extend to simple algebras of real rank one, too. Recall that a simple partially ordered Abelian group $(G, G^+)$ is said to be *weakly unperforated* if $mx > 0$ for some $m \in \mathbb{N}$ and $x \in G$ implies that $x > 0$ [2, Chapter 6].

**Theorem 4.2.** *Let $A$ be an unital AH algebra such that* $\mathrm{drr}(A) = 0$, *and suppose that* $\mathrm{K}_0A$ *is a simple ordered group. Then,* $\mathrm{K}_0A$ *is weakly unperforated, and $A$ has cancellation.*

**Proof.** Suppose that $mx > 0$ for some $m \in \mathbb{N}$ and $x \in G$. Since $\mathrm{K}_0A$ is a simple ordered group, there exists $n \in \mathbb{N}$ such that $nmx > [1_A] \in \mathrm{K}_0A$. Since $\mathrm{drr}(A) = 0$, we may choose an injective decomposition

$$A \cong \lim_{i\to\infty}\left(A_i := \bigoplus_{l=1}^{m_i} p_{i,l}\big(\mathrm{C}(X_{i,l})\otimes\mathcal{K}\big)p_{i,l}, \phi_i\right)$$

with the property that for every $i \in \mathbb{N}$,

$$\max_{1\leqslant l\leqslant m_i}\left\{\frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})}\right\} < \frac{1}{nm}.$$

Find a pre-image $x_i \in \mathrm{K}_0A_i$ of $x$ such that $nmx_i > [1_{A_i}]$. Write $x_i = [p] - [q]$ for projections $p$ and $q$ in $\mathrm{M}_\infty(A_i)$. Let $\mathrm{Sp}(\cdot)$ denote spectrum of a $C^*$-algebra. Upon restricting to any direct summand $B$ of $A_i$ corresponding to a connected component of $\mathrm{Sp}(A_i)$ one has

$$\mathrm{rank}(p|_B) - \mathrm{rank}(q|_B) > \frac{\mathrm{rank}(1_B)}{nm} \geqslant \dim\big(\mathrm{Sp}(B)\big).$$

It follows from Theorem 3.1 that $[p|_B] - [q|_B] \in \mathrm{K}_0B^+$, whence $x_i$ and its image $x \in \mathrm{K}_0A$ are positive. Thus, $\mathrm{K}_0A$ is weakly unperforated.

Now suppose that we are given projections $p, q \in \mathrm{M}_\infty(A)$ such that $[p] = [q] \in \mathrm{K}_0A$. Since $\mathrm{K}_0A$ is a simple ordered group, every positive element is an order unit. Hence, there exists some $m \in \mathrm{N}$ such that $m[p] = m[q] \geqslant [1_A]$. Find an injective decomposition for $A$ as above, with the property that for every $i \in \mathbb{N}$,

$$\max_{1\leqslant l\leqslant m_i}\left\{\frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})}\right\} < \frac{1}{m}.$$

Find projections $p_i, q_i \in \mathrm{M}_\infty(A_i)$, some $i \in \mathbb{N}$, such that $p_i$ is a pre-image of $p$, $q_i$ is a pre-image of $q$, $[p_i] = [q_i] \in \mathrm{K}_0A_i$, and $m[p_i] \geqslant [1_{A_i}] \in \mathrm{K}_0A_i$. Now, upon restricting to any direct summand $B$ of $A_i$ corresponding to a connected component of the spectrum of $A_i$ one has

$$\mathrm{rank}(p_i|_B),\ \mathrm{rank}(q_i|_B) > \frac{\mathrm{rank}(1_B)}{m} \geqslant \dim\big(\mathrm{Sp}(B)\big).$$

It follows that $p_i|_B$ and $q_i|_B$ are in the stable range of $\mathrm{K}_0B$, whence they are Murray–von Neumann equivalent by Theorem 3.1. It follows that $p_i$ and $q_i$ are Murray–von Neumann equivalent, and so are $p$ and $q$. Thus, $A$ has cancellation. □

This is the natural point at which to prove the next corollary, but its statement refers to the almost unperforation of the Cuntz semigroup $W(A)$; we have yet to remind the reader of this notion. As we will have occasion to discuss this notion in depth in Section 6, we defer our definition until then.

**Corollary 4.3.** *Let $A$ be a simple unital AH algebra of real rank zero. Then, the following are equivalent*:

(i) $\mathrm{drr}(A) = 0$;
(ii) *$A$ is tracially AF*;
(iii) *$A$ has slow dimension growth*;
(iv) *$A$ is $\mathcal{Z}$-stable*;
(v) *$W(A)$ is almost unperforated and* $\mathrm{sr}(A) = 1$.

**Proof.** The equivalence of (ii), (iii), (iv), and (v) is [31, Theorem 3.13], and is the work of many hands, including M. Dădărlat, G. Elliott, G. Gong, H. Lin, M. Rørdam, W. Winter, and the author.

If $A$ has slow dimension growth, then $\mathrm{drr}(A) = 0$ by definition. Thus, (iii) implies (i).

We now prove that (i) implies (ii). It follows from Theorem 4.2 that $\mathrm{K}_0(A)$ is weakly unperforated and has cancellation of projections. Combining this with real rank zero yields stable rank one for $A$ [2, Proposition 6.5.2]. That $A$ is tracially AF then follows from [17]. □

Corollary 4.3 allows us to view the dimension–rank ratio as a measure of dimension growth which extends the existing notion of slow dimension growth. The condition $\mathrm{drr} = 0$ is a more natural way to view slow dimension growth, since it has higher analogues in the form of non-zero dimension–rank ratios. As promised, the dimension–rank ratio is insensitive to differences between $\mathcal{Z}$-stable algebras, provided that they are simple and of real rank zero.

## 5. The range of the dimension–rank ratio

It is clear from Theorem 2.3 that the dimension–rank ratio may take any finite, nonnegative, and rational value. In fact, more is true:

**Theorem 5.1.** *Let $c \in \mathbb{R}^+ \cup \{\infty\}$. There exists a simple, unital AH algebra $A_c$ such that $\mathrm{K}_0 A_c$ admits a unique state and $\mathrm{drr}(A_c) = c$. Moreover, the stable rank of $A_c$ is one.*

**Proof.** We address the extreme cases first. The case $c = 0$ is straightforward: any UHF algebra has $\mathrm{drr} = 0$. For $c = \infty$, we use an existing example due to Villadsen. In [32], Villadsen constructs several simple unital AH algebras whose $\mathrm{K}_0$-groups admit a unique state $s$. One of these, say $A$, has unbounded perforation in its ordered $\mathrm{K}_0$-group—for every $n \in \mathbb{N}$, there is a non-positive element $x_n \in \mathrm{K}_0 A$ such that $s(x_n) \geqslant n$. No matter how one decomposes $A$ as an inductive limit of direct sums of homogeneous $C^*$-algebras—as

$$A = \lim_{i \to \infty} \left( A_i := \bigoplus_{l=1}^{m_i} p_{i,l} \big( \mathrm{C}(X_{i,l}) \otimes \mathcal{K} \big) p_{i,l}, \phi_i \right),$$

say—one will always have $x_n$ arising in the $\mathrm{K}_0$-group of $A_j$ for all $j$ greater than or equal to some $j_0 \in \mathbb{N}$. Since $\mathrm{K}_0 A$ admits a unique state, we may apply Lemma 3.9 to conclude that for any $\epsilon > 0$, there is some $j \geqslant j_0$ with the following property: the restriction $x_{n,l}$ of $x_n$ to the $\mathrm{K}_0$-group of the direct summand $p_{j,l}(\mathrm{C}(X_{j,l}) \otimes \mathcal{K}) p_{j,l}$ of $A_j$ satisfies

$$\big| s_l(x_{n,l}) - s(x_n) \big| < \epsilon, \quad \forall 1 \leqslant l \leqslant m_j.$$

Since

$$s_l(x_{n,l}) = \frac{\mathrm{rank}(x_{n,l})}{\mathrm{rank}(p_{j,l})},$$

we have

$$\left|\mathrm{rank}(x_{n,l}) - s(x_n)\cdot \mathrm{rank}(p_{j,l})\right| < \epsilon \cdot \mathrm{rank}(p_{j,l})$$

and

$$\mathrm{rank}(x_{n,l}) \geqslant \big(s(x_n) - \epsilon\big)\cdot \mathrm{rank}(p_{j,l}) \geqslant (n-\epsilon)\cdot \mathrm{rank}(p_{j,l}).$$

It follows that from Theorem 3.1(ii) (rephrased in K-theoretic terms) that

$$\frac{\dim(X_{j,l})}{\mathrm{rank}(p_{j,l})} > \frac{n-1}{2}, \quad \forall 1 \leqslant l \leqslant m_j.$$

Since $n$ was arbitrary, we conclude that no matter the decomposition, $\limsup_{i\to\infty} \mathrm{drr}(A_i) = \infty$; $\mathrm{drr}(A) = \infty$ by definition.

Now suppose that $c \in \mathbb{R}^+\backslash\{0\}$. We construct $A_c$ by methods similar to those of [32]. $A_c$ will be the limit of an inductive sequence $(B_i, \phi_i)$, where

$$B_i = \mathrm{M}_{n_i}\big(\mathrm{C}(X_i)\big), \quad X_i = \big(\mathrm{S}^2\big)^{m_1 m_2 \ldots m_i},$$

and $n_i, m_i \in \mathbb{N}$ are to be specified.

Choose $m_1$ and $n_1$ so that $m_1/n_1 > c/2$. We have $X_{i+1} = (X_i)^{m_{i+1}}$ by construction. Let

$$\pi_i^j : (X_i)^{m_{i+1}} \to X_i, \quad 1 \leqslant j \leqslant m_{i+1},$$

be the co-ordinate projections. Define a map $\phi_i : B_i \to B_{i+1}$ by

$$\phi_i(f)(x) = \mathrm{diag}\big(f\circ\pi_i^1(x), \ldots, f\circ\pi_i^{m_{i+1}}(x), f\big(x_i^1\big), \ldots, f\big(x_i^{s_{i+1}}\big)\big),$$

where $s_{i+1} \in \mathbb{N}$ and the $x_i^1, \ldots, x_i^{s_{i+1}} \in X_i$ are to be specified. Suppose that for $i \leqslant k$ we have chosen the parameters in our construction inductively so that

$$\frac{c}{2} < \frac{m_1 m_2 \ldots m_k}{n_k} < \frac{c}{2} + \frac{1}{2^k}. \tag{9}$$

We have

$$\frac{m_1 m_2 \ldots m_{k+1}}{n_{k+1}} = \frac{m_1 m_2 \ldots m_{k+1}}{n_k(m_{k+1} + s_{k+1})} = \frac{m_1 m_2 \ldots m_k}{n_k}\cdot\frac{m_{k+1}}{m_{k+1} + s_{k+1}},$$

by construction.

We may then choose $m_{k+1}$ and $s_{k+1} \neq 0$ to satisfy

$$\frac{c}{2} < \frac{m_1 m_2 \ldots m_{k+1}}{n_{k+1}} < \frac{c}{2} + \frac{1}{2^{k+1}}, \tag{10}$$

whence (9) holds for all $k \in \mathbb{N}$. Theorem 1 of [32] shows that the points $x_{i-1}^1, \ldots, x_{i-1}^{s_{i-1}} \in X_{i-1}$, $i \in \mathbb{N}$, may be chosen in a manner which makes the (unital) limit algebra $A_c = \lim_{i\to\infty}(B_i, \phi_i)$ simple. By (10) we have

$$\lim_{i\to\infty} \frac{\dim(X_i)}{n_i} = \lim_{i\to\infty} \frac{m_1 m_2 \ldots m_i}{n_i} = 2\left(\frac{c}{2}\right) = c,$$

whence $\mathrm{drr}(A_c) \leqslant c$.

In order to conclude that $\mathrm{drr}(A_c) = c$, we must prove that any other decomposition of $A$ satisfies

$$\liminf_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(\mathrm{p_{i,l}})} \right\} \geqslant c.$$

To this end we will employ the ordered $\mathrm{K}_0$-group of $A_c$.

Let $\xi$ denote the Hopf line bundle over $\mathrm{S}^2$, and $\theta_l$ the trivial vector bundle of complex fibre dimension $l \in \mathbb{N}$ over an arbitrary compact Hausdorff space $X$. In [32] it is proved that the $\mathrm{K}^0(X_i)$-class

$$y_i := \left[\xi^{\times m_1 m_2 \ldots m_i}\right] - [\theta_1]$$

is not positive in either $\mathrm{K}^0(X_i)$ or $\mathrm{K}_0(A_c)$. By Lemma 3.7, $\mathrm{K}_0(A_c)$ admits a unique state $s$, which is realised on $B_i$ as the normalised geometric state—the state which returns the virtual dimension of a $\mathrm{K}^0(X_i)$-class divided by $n_i$. Thus,

$$s(y_i) = \frac{m_1 m_2 \ldots m_i - 1}{n_i} \xrightarrow{i\to\infty} \frac{c}{2}.$$

Suppose that there exists an injective decomposition $A_c = \lim_{i\to\infty}(A_i, \phi_i)$ with

$$A_i := \bigoplus_{l=1}^{m_i} p_{i,l}\big(\mathrm{C}(Y_{i,l}) \otimes \mathcal{K}\big) p_{i,l}$$

and such that

$$\liminf_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(Y_{i,l})}{\mathrm{rank}(\mathrm{p}_{i,l})} \right\} < c.$$

By compressing the inductive sequence in this decomposition we may assume that

$$\lim_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(Y_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} < c.$$

Choose $i_0 \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(Y_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} < c - \epsilon, \quad \forall i \geqslant i_0.$$

Choose $j \geqslant i_0$ large enough so that $s(y_j) > (c-\epsilon)/2$ and $y_j \in \mathrm{K}_0(A_j)$. Put

$$y_j^l = y_j|_{\mathrm{K}_0(p_{j,l}(\mathrm{C}(Y_{i,l})\otimes\mathcal{K})p_{j,l})}, \quad 1 \leqslant l \leqslant m_j,$$

so that $y_j = \bigoplus_l y_j^l$. Applying Lemma 3.9, we may have that

$$s(y_j^l) > \frac{c-\epsilon}{2}, \quad 1 \leqslant l \leqslant m_j.$$

This, in turn, implies that the virtual dimension of each $y_j^l$ is greater than $\lceil \dim(Y_{j,l})/2 \rceil$, whence each $y_j^l$ is positive in $\mathrm{K}_0(p_{j,l}(\mathrm{C}(Y_{i,l})\otimes\mathcal{K})p_{j,l})$. But then $y_j$ must be positive, contradicting our choice of $y_j$.

That $A_c$ has stable rank one follows from [32, Lemma 9 and Proposition 10], upon noticing that the general construction of $A_c$ is of the type described in Section 2 of the same paper. □

**Corollary 5.2.** *Let $c \in \mathbb{R}^+$. Then, with $A_c$ as in Theorem* 5.1*, we have*

$$\inf\{s \in \mathbb{R} \mid \mathrm{K}_0 A_c \text{ has } s\text{-strict comparison}\} = \frac{\mathrm{drr}(A)}{2} = \frac{c}{2}.$$

**Proof.** $A_c$ has a $\mathrm{K}_0$-group which admits a unique state by Lemma 3.7. We may thus apply Theorem 3.10 to conclude that $\mathrm{K}_0 A_c$ has $(\mathrm{drr}(A)/2)$-strict comparison. This proves the corollary if $c = 0$.

If $c > 0$, then $\mathrm{K}_0 A_c$ does not have $s$-strict comparison for any $s < c/2$. Indeed, the element $y_i = [\xi^{\times m_1 m_2 \ldots m_i}] - [\theta_1]$ is not positive in $\mathrm{K}_0 A_i$, and neither is its image in $\mathrm{K}_0 A_c$. Applying the geometric state on $\mathrm{K}_0 A_i$, one has

$$s([\theta_1]) = \frac{1}{n_i}; \qquad s([\xi^{\times m_1 m_2 \ldots m_i}]) = \frac{m_1 m_2 \ldots m_i}{n_i}.$$

Choosing $i$ large enough so that $c/2 - 1/n_i > s$ we have

$$s([\theta_1]) + s < s([\xi^{\times m_1 m_2 \ldots m_i}]),$$

yet $[\theta_1] \nleqslant [\xi^{\times m_1 m_2 \ldots m_i}]$. The corollary follows. □

**Corollary 5.3.** *The class of simple, unital and non-$\mathcal{Z}$-stable AH algebras is uncountable.*

**Proof.** The algebra $A_c$ of Theorem 5.1 has a perforated ordered $\mathrm{K}_0$-group for each $c \neq 0$. Theorem 1 of [11] states that a simple, unital, finite, and $\mathcal{Z}$-stable $C^*$-algebra has a weakly unperforated ordered $\mathrm{K}_0$-group, whence the $A_c$s in question are non-$\mathcal{Z}$-stable. □

The pairwise non-isomorphic algebras constructed in the proof of Theorem 5.1 are difficult to distinguish from one another without using the dimension–rank ratio. Straightforward calculation shows that, for each $c \neq 0$, $\mathrm{T}(A_c)$ is a Bauer simplex with extreme boundary homeomorphic to $(S^2)^\infty$, and $\mathrm{K}_1 A_c = 0$. Computing the ordered group $\mathrm{K}_0 A_c$ is not feasible—the order structure on $\mathrm{K}_0(S^2)^n$ is not known for general $n$.

## 6. Abstracting the dimension–rank ratio

The dimension–rank ratio functions well as an invariant tailored for the study of unital and non-$\mathcal{Z}$-stable AH algebras, so it is natural to ask whether there exists an invariant defined for any unital and stably finite $C^*$-algebra which recovers (or is at least closely related to) the dimension–rank ratio upon restricting to the subclass of unital AH algebras. In this section, we present a candidate for such an invariant.

One could, in light of Corollary 5.2, be forgiven for wondering briefly if the extended real

$$\inf\{s \mid \mathrm{K}_0 A \text{ has } s\text{-strict comparison}\}$$

might be the invariant we seek. The algebra $\mathrm{C}([0,1]^n)$ dispels this notion: its $\mathrm{K}_0$-group has comparison, yet $\mathrm{drr}(\mathrm{C}([0,1]^n)) = n$. There is, however, a different version of ordered K-theory, whose prospects for recovering the dimension–rank ratio are distinctly better than those of the $\mathrm{K}_0$-group.

Let $A$ be a $C^*$-algebra. We recall the definition of the Cuntz semigroup $W(A)$ from [5]. (Our synopsis is essentially that of [25].) Let $\mathrm{M}_n(A)^+$ denote the positive elements of $\mathrm{M}_n(A)$, and let $\mathrm{M}_\infty(A)^+$ be the disjoint union $\bigcup_{i=n}^\infty \mathrm{M}_n(A)^+$. For $a \in \mathrm{M}_n(A)^+$ and $b \in \mathrm{M}_m(A)^+$ set $a \oplus b = \mathrm{diag}(a,b) \in \mathrm{M}_{n+m}(A)^+$, and write $a \precsim b$ if there is a sequence $\{x_k\}$ in $\mathrm{M}_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Put $W(A) = \mathrm{M}_\infty(A)^+/\sim$, and let $\langle a\rangle$ be the equivalence class containing $a$. Then, $W(A)$ is a positive ordered Abelian semigroup when equipped with the relations:

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle, \quad \langle a\rangle \leqslant \langle b\rangle \quad \Longleftrightarrow \quad a \precsim b, \quad a,b \in \mathrm{M}_\infty(A)^+.$$

The relation $\precsim$ reduces to Murray–von Neumann comparison when $a$ and $b$ are projections and $A$ is stably finite.

In the case of a stably finite $C^*$-algebra $A$, the Cuntz semigroup may be thought of as a generalised version of the semigroup of Murray–von Neumann equivalence classes of projections in $\mathrm{M}_\infty(A)$. If $A$ is unital, then we scale $W(A)$ with $\langle 1_A\rangle$. Let $S(W(A))$ denote the set of additive and order preserving maps from $W(A)$ to $\mathbb{R}^+$ having the property that $s(\langle 1_A\rangle) = 1$, $\forall s \in S(W(A))$. Such maps are called *states*. Given $\tau \in \mathrm{QT}(A)$, one may define a map $s_\tau : \mathrm{M}_\infty(A)^+ \to \mathbb{R}^+$ by

$$s_\tau(a) = \lim_{n\to\infty} \tau\big(a^{1/n}\big). \tag{11}$$

This map is lower semicontinuous, and defines a state on $W(A)$. Such maps are called *lower semicontinuous dimension functions*, and the set of them is denoted $\mathrm{LDF}(A)$. $\mathrm{QT}(A)$ is a simplex [3, Theorem II.4.4], and the map from $\mathrm{QT}(A)$ to $\mathrm{LDF}(A)$ defined by (11) is bijective and affine [3, Theorem II.2.2].

**Definition 6.1.** Let $A$ be an unital and stably finite $C^*$-algebra, and let $r > 0$.

(i) Say that $A$ has $r$-comparison if whenever one has positive elements $a, b \in \mathrm{M}_\infty(A)$ such that

$$s\big(\langle a\rangle\big) + r < s\big(\langle b\rangle\big), \quad \forall s \in \mathrm{LDF}(A),$$

then $\langle a\rangle \leqslant \langle b\rangle$ in $W(A)$.

(ii) Define the radius of comparison of $A$, denoted $\mathrm{rc}(A)$, to be

$$\inf\{r \in \mathbb{R}^+ \mid (W(A), \langle 1_A \rangle) \text{ has } r\text{-comparison}\}$$

if it exists, and $\infty$ otherwise.

We summarise some properties of the radius of comparison which are more or less immediate from its definition. (Compare with Propositions 2.2 and 3.5.)

**Proposition 6.2.** *Let $A$, $B$ be unital and stably finite $C^*$-algebras. Then*:

(i) $\mathrm{rc}(A \oplus B) = \max\{\mathrm{rc}(A), \mathrm{rc}(B)\}$;
(ii) *if $k \in \mathbb{N}$, then* $\mathrm{rc}(\mathrm{M}_k(A)) = (1/k)\mathrm{rc}(A)$;
(iii) *if $I$ is an ideal of $A$, $\pi : A \to A/I$ is the quotient map, and*

$$\pi^\sharp : \mathrm{QT}(A/I) \to \mathrm{QT}(A)$$

*is surjective, then* $\mathrm{rc}(A/I) \leqslant \mathrm{rc}(A)$;
(iv) *$A$ has* $\mathrm{rc}(A)$-(FCQ);
(v) $(\mathrm{K}_0 A^+, [1_A])$ *has* $\mathrm{rc}(A)$-*strict comparison.*

**Proof.** For (i), use the fact that $\mathrm{LDF}(A \oplus B)$ is the convex hull of $\mathrm{LDF}(A)$ and $\mathrm{LDF}(B)$ to obtain $\mathrm{rc}(A \oplus B) \leqslant \max\{\mathrm{rc}(A), \mathrm{rc}(B)\}$. The reverse inequality will follow from (iii).

(ii) is straightforward from Definition 6.1.

For (iii), let there be given positive elements $a, b \in \mathrm{M}_\infty(A/I)$ such that

$$s(\langle a \rangle) + r < s(\langle b \rangle), \quad \forall s \in \mathrm{LDF}(A/I), \text{ some } r > \mathrm{rc}(A).$$

We may find positive elements $\tilde{a}, \tilde{b} \in \mathrm{M}_\infty(A)$ such that $\pi(\tilde{a}) = a$ and $\pi(\tilde{b}) = b$ (lift to self-adjoint elements and apply the functional calculus). Let $d_\tau \in \mathrm{LDF}(A)$ be a state corresponding to a normalised quasi-trace $\tau$ on $A$. Then, $\tau = \pi^\sharp(\eta)$ for some $\eta \in \mathrm{QT}(A/I)$ by assumption, and

$$d_\tau(\langle \tilde{a} \rangle) = \lim_{n\to\infty} (\tau(\tilde{a}^{1/n})) = \lim_{n\to\infty} \eta(a^{1/n}) = s_\eta(\langle a \rangle)$$

for the state $s_\eta$ corresponding to some $\eta \in \mathrm{QT}(A/I)$. It follows that

$$d_\tau(\langle \tilde{a} \rangle) + r < d_\tau(\langle \tilde{b} \rangle), \quad \forall d \in \mathrm{LDF}(A), \text{ some } r > \mathrm{rc}(A),$$

whence $\tilde{a} \precsim \tilde{b}$ in $W(A)$. This implies the existence of a sequence $(v_k) \subseteq \mathrm{M}_\infty(A)$ such that

$$v_k^* \tilde{b} v_k \xrightarrow{k\to\infty} \tilde{a}.$$

Applying $\pi$ to the expression above shows that $a \precsim b$ in $W(A/I)$, as desired.

(iv) and (v) follow from the fact that there is an order unit preserving order embedding of the scaled ordered semigroup $(V(A), [1_A])$ of Murray–von Neumann equivalence classes of projections in $\mathrm{M}_\infty(A)$ into $(W(A), \langle 1_A \rangle)$ whenever $A$ is stably finite (cf. [26]). □

The next proposition is the analogue of Proposition 3.4 for the radius of comparison.

**Proposition 6.3.** *Let $A$ be an unital and stably finite $C^*$-algebra for which every $\tau \in \mathrm{QT}(A)$ is faithful. Then, the set*

$$B := \{r \in \mathbb{R}^+ \mid W(A) \text{ has } r\text{-comparison}\}$$

*is closed. In other words, $W(A)$ has $\mathrm{rc}(A)$-comparison.*

**Proof.** If $B = \emptyset$, then it is closed; suppose that $B \neq \emptyset$. As in the proof of Proposition 3.4, we need only prove that $\alpha := \inf(B) \in B$. Let there be given $a, b \in W(A)$ satisfying

$$s(a) + \alpha < s(b), \quad \forall s \in \mathrm{LDF}(A).$$

Suppose first that $a = \langle p \rangle$ for some projection $p \in \mathrm{M}_\infty(A)$. Then, the map $\gamma_a : \mathrm{QT}(A) \to \mathbb{R}^+$ given by $\gamma_a(s) = s(a)$ is continuous. By [21, Proposition 2.7], the map $\gamma_b : \mathrm{QT}(A) \to \mathbb{R}^+$ given by $\gamma_b(s) = s(a)$ is lower semicontinuous. It follows that $\gamma_b - \gamma_a$ is lower semicontinuous and strictly positive on $\mathrm{QT}(A)$. Since $\mathrm{QT}(A)$ is compact, $\gamma_b - \gamma_a$ achieves a lower bound $\delta > 0$, whence

$$s(a) + \alpha + \delta/2 < s(b), \quad \forall s \in \mathrm{LDF}(A).$$

$W(A)$ has $(\alpha + \delta/2)$-comparison, and so $a \leqslant b$, as desired.

Now suppose that $a$ is not Cuntz equivalent to any projection. We will abuse notation by identifying $a$ with one of its representatives in $\mathrm{M}_\infty(A)_+$. By the functional calculus, we conclude that 0 is not an isolated point of the spectrum of $a$. Viewing $a$ as the function $f(t) = t$ on its spectrum, we denote by $(a - \epsilon)_+$ the function $\max\{0, f(t) - \epsilon\}$ on the spectrum of $a$. By [15, Proposition 2.6], proving that $a \leqslant b$ is equivalent to proving that $\langle (a - \epsilon)_+ \rangle \leqslant b$, $\forall \epsilon > 0$. Let $g_\epsilon(t) \in C^*(a)$ be a function supported on $(0, \epsilon) \cap \sigma(a)$ $(\neq \emptyset)$, where $\sigma(a)$ denotes the spectrum of $a$. Since $g_\epsilon(t) + (a - \epsilon)_+ \leqslant f(t) = a$, we have

$$s(a) - s((a - \epsilon)_+) \geqslant s(g_\epsilon), \quad \forall s \in \mathrm{LDF}(A).$$

Let $\mathrm{supp}(\cdot)$ denotes the support of a function. Each $s \in \mathrm{LDF}(A)$ is implemented on $C^*(a)$ by a probability measure $\mu_s$ in the following sense: for any $d \in C^*(a)$, $s(d) = \mu_s(\mathrm{supp}(d))$. Moreover, our assumption about the faithfulness of quasitraces on $A$ implies that $\mu_s(U) > 0$ for every open subset $U$ of $\sigma(a)$. Thus, the map $\gamma_{g_\epsilon} : \mathrm{QT}(A) \to \mathbb{R}^+$ given by

$$\gamma_{g_\epsilon}(s) = s(g_\epsilon) = \mu_s((0, \epsilon) \cap \sigma(a))$$

is strictly positive, and, as above, lower semicontinuous. It follows that $\gamma_{g_\epsilon}$ achieves a lower bound on $\mathrm{QT}(A)$, say $\delta_\epsilon$. Now

$$s((a - \epsilon)_+) + \alpha + \delta_\epsilon/2 < s(b), \quad \forall s \in \mathrm{LDF}(A).$$

$W(A)$ has $(\alpha + \delta_\epsilon/2)$-comparison for every $\epsilon > 0$, whence $\langle (a - \epsilon)_+ \rangle \leqslant b$, $\forall \epsilon > 0$. □

Recall that $W(A)$ is said to be *almost unperforated* if $x \leqslant y$ in $W(A)$ whenever $mx \leqslant ny$ for natural numbers $m > n$ [26].

**Proposition 6.4.** *Let $A$ be an unital and stably finite $C^*$-algebra for which every $\tau \in \mathrm{QT}(A)$ is faithful. If $\mathrm{rc}(A) = 0$, then $W(A)$ is almost unperforated.*

**Proof.** Let $m > n$ be natural numbers, and $x, y \in W(A)$ such that $mx \leqslant ny$. For any $s \in \mathrm{LDF}(A)$ we have the following string of inequalities:

$$0 \leqslant n \cdot s(y) - m \cdot s(x),$$
$$0 \leqslant n\big(s(y) - s(x)\big) - (m-n)s(x),$$
$$\frac{(m-n)s(x)}{n} \leqslant s(y) - s(x).$$

The map $\gamma : \mathrm{QT}(A) \to \mathrm{R}^+$ given by $s \mapsto s(x)$ is thus strictly positive (since each $\tau \in \mathrm{QT}(A)$ is faithful) and lower semicontinuous [21, Proposition 2.7]. Since $\mathrm{QT}(A)$ is compact, $\gamma$ achieves a minimum value $\delta > 0$. Now $s(x) + \delta/2 < s(y)$, $\forall s \in \mathrm{LDF}(A)$. Since $\mathrm{rc}(A) = 0$, $W(A)$ has $(\delta/2)$-comparison. We conclude that $x \leqslant y$ in $W(A)$, as desired. □

**Theorem 6.5.** (See Rørdam [26, Corollary 4.6].) *Let $A$ be a simple, unital, exact, stably finite $C^*$-algebra with $W(A)$ almost unperforated. Then, $W(A)$ has* 0*-comparison, and* $\mathrm{rc}(A) = 0$.

Combining Proposition 6.4, Theorem 6.5, and Corollary 4.3, we conclude that $\mathrm{drr} = 0$ and $\mathrm{rc} = 0$ are equivalent for simple and infinite-dimensional AH algebras of real rank zero and stable rank one. We shall see in Corollary 6.7 that if a semi-homogeneous algebra $A$ has $W(A)$ almost unperforated and spectrum a CW-complex, then the dimension of its spectrum, and hence its dimension–rank ratio, is at most four; by Proposition 6.4, this conclusion holds a fortiori if the said semi-homogeneous algebra has $\mathrm{rc} = 0$. If every finite-dimensional representation of $A$ is large, then $\mathrm{rc} = 0$ implies that $\mathrm{drr} \approx 0$. $\mathrm{drr}(A) = 0$ implies that the spectrum of $A$ is zero-dimensional; $W(A)$ is then almost unperforated by [20, Theorem 3.4]. Taken together, these results show the condition $\mathrm{rc} = 0$ to be an appropriate abstraction of the condition $\mathrm{drr} = 0$.

**Theorem 6.6.** *Let $X$ be a CW-complex of finite dimension $n$, $p \in \mathrm{C}(X) \otimes \mathcal{K}$ a projection, and let $m$ be the greatest nonnegative integer such that $2m < n$. Then,*

$$\mathrm{rc}\big(p\big(\mathrm{C}(X) \otimes \mathcal{K}\big)p\big) \geqslant \frac{m-1}{\mathrm{rank}(p)}.$$

**Proof.** The theorem is trivial if $m \leqslant 1$, so suppose that $m \geqslant 2$. Choose an $n$-cell of $X$, say $E$. There is a subset $A$ of $E^\circ$ homeomorphic to $(-1,1)^n$. Let $\psi : A \to (-1,1)^{2m+1}$ be the projection onto the first $2m+1$ co-ordinates of $A$, and let $d$ be the usual Euclidean metric on $\mathrm{Im}(\psi) = (-1,1)^{2m+1}$. Put

$$Y := \big\{(x_1, \ldots, x_{2m+1}) \in \mathrm{Im}(\psi) \mid d\big((x_1, \ldots, x_{2m+1}), (0, \ldots, 0)\big) = 1/2\big\}$$

and

$$S := \big\{(x_1, \ldots, x_{2m+1}) \in \mathrm{Im}(\psi) \mid 1/3 < d\big((x_1, \ldots, x_{2m+1}), (0, \ldots, 0)\big) < 2/3\big\}.$$

Let $r : S \to Y$ be the projection along rays emanating from $(0, \ldots, 0) \in \mathrm{Im}(\psi)$. Put $O = \psi^{-1}(S)$ and $\pi = r \circ \psi$. We now have a closed subset $Y$ of $E^\circ$ homeomorphic to $S^{2m}$, an open set $O$ such that $E^\circ \supseteq O \supseteq Y$, and a continuous map $\pi : O \to Y$ such that $\pi|Y = \mathrm{id}_Y$.

Recalling the description of $(\mathrm{K}^0 S^{2m}, \mathrm{K}^0 S^{2m+})$ from the example following Proposition 3.5, let $\xi_m$ be a complex vector bundle over $Y$ whose $\mathrm{K}^0$-class corresponds to $m \oplus 1 \in \mathbb{Z} \oplus \mathbb{Z} \cong \mathrm{K}^0 S^{2m}$. $\xi_m$ can be realised inside $M_{2m}(\mathrm{C}(X))$. If $\theta_1$ is the trivial complex line bundle over $Y$, then the class $[\theta_1]$ corresponds to the element $1 \oplus 0 \in \mathrm{K}^0 S^{2m}$ and is clearly not dominated by $[\xi_m]$. Let $f : X \to [0,1]$ be a continuous function which vanishes off $O$ and takes the value 1 at every point in the closure of some open set $V \supseteq Y$ such that $\overline{V} \subseteq O$. Define positive functions $a, b \in \mathrm{M}_{2m}(\mathrm{C}(X))$ by

$$a(x) = f(x)\pi^*(\xi_m); \qquad b(x) = f(x)\pi^*(\theta_1).$$

We may think of $a$ and $b$ as being contained in $\mathrm{M}_k(p(\mathrm{C}(X) \otimes \mathcal{K})p)$ for some sufficiently large $k \in \mathbb{N}$.

We claim that $\langle b \rangle \nleqslant \langle a \rangle$ in $W(p(\mathrm{C}(X) \otimes \mathcal{K})p)$. Indeed, since $f(y) = 1$, $\forall y \in Y$, and $\pi|Y = \mathrm{id}_Y$, we have that

$$a(y) = \xi_m(y), \quad b(y) = \theta_1(y), \quad \forall y \in Y.$$

$\langle b \rangle \leqslant \langle a \rangle$ implies that $\langle b|Y \rangle \leqslant \langle a|Y \rangle$ in $W(\mathrm{C}(Y))$, but the second inequality contradicts the fact that $\theta_1$ is not Murray–von Neumann equivalent to a subprojection of $\xi_m$ (remember that the Cuntz equivalence relation reduces to Murray–von Neumann equivalence on projections in a stably finite algebra). The claim follows.

Choose a continuous function $g : X \to [0,1]$ such that $g$ is identically zero on $Y$, and identically one on the complement of $V$. Define a positive element $v := g \cdot \theta_n$. Since $v$ is zero on $Y$, the argument of the preceding paragraph shows that

$$\langle b \rangle \nleqslant \langle a \oplus v \rangle.$$

The lower semicontinuous dimension functions on $A = p(\mathrm{C}(X) \otimes \mathcal{K})p$ correspond to normalised traces on $A$. This correspondence may be viewed as follows: each normalised trace $\tau$ corresponds to a probability measure $\mu_\tau$ on $X$, and the dimension function $d_\tau$ is given by

$$d_\tau\big(\langle a \rangle\big) = \int_X \frac{\mathrm{rank}(a)(x)}{\mathrm{rank}(p)}\, d\mu_\tau.$$

Let $\tau \in \mathrm{T}A$ be given. We have

$$\begin{aligned}
\mathrm{rank}(p) \cdot d_\tau\big(\langle a \oplus v \rangle\big) &= \int_X \mathrm{rank}(a \oplus v)(x)\, d\mu_\tau \\
&= \int_{X \backslash V} n + \mathrm{rank}(a)(x)\, d\mu_\tau + \int_{V \backslash Y} \mathrm{rank}(v)(x) + m\, d\mu_\tau + \int_Y m\, d\mu_\tau \\
&\geqslant n\mu_\tau(X \backslash V) + m\mu_\tau(V \backslash Y) + m\mu_\tau(Y) \geqslant m
\end{aligned}$$

and

$$\mathrm{rank}(p) \cdot d_\tau\big(\langle b \rangle\big) = \int_X \mathrm{rank}(b)(x)\, d\mu_\tau = \int_O d\mu_\tau \leqslant 1.$$

Thus, for any $s \in \mathrm{LDF}(A)$ we have

$$s\big(\langle b\rangle\big) + \frac{m-1}{\mathrm{rank}(p)} \leqslant s\big(\langle a \oplus v\rangle\big)$$

while $\langle b\rangle \nleqslant \langle a \oplus v\rangle$. The proposition follows. □

The lower bound on $\mathrm{rc}(p(\mathrm{C}(X) \otimes \mathcal{K})p)$ in Theorem 6.6 is close to $\mathrm{drr}(A)/2$, particularly when $\dim(X)$ and $\mathrm{rank}(p)$ are large. If a simple unital AH algebra $B$ has $\mathrm{drr}(B) > 0$, then the dimensions of the spectra of its building blocks and the ranks of the units of these building blocks must tend toward infinity, regardless of the injective decomposition chosen. Thus, the bound of Theorem 6.6 applied to these building blocks will be all but equal to one half of their respective dimension–rank ratios. One can obtain a lower bound in the spirit of Theorem 6.6 for the algebras of Theorem 5.1.

The proof of Theorem 6.6 yields:

**Corollary 6.7.** *Let $A$ be a semi-homogeneous $C^*$-algebra with spectrum a CW-complex. If $W(A)$ is almost unperforated, then the dimension of the spectrum of $A$ is at most four.*

**Proof.** We prove the contrapositive. Retain the notation used in the proof of Theorem 6.6. Suppose that the dimension of the spectrum of $A$ is at least five. Construct $a$ and $b$ as in the proof of Theorem 6.6, and notice that $a = \pi^*(\xi_2)$. Theorem 3.1 shows that

$$\xi_2 \oplus \xi_2 \oplus \xi_2 \cong \theta_4 \oplus \eta$$

for some complex vector bundle $\eta$ over $Y$. In other words, there is a partial isometry $v \in \mathrm{M}_\infty(\mathrm{C}(Y))$ such that

$$v^*(\xi_2 \oplus \xi_2 \oplus \xi_2)v = \theta_4.$$

Let $(g_k)$ be a self-adjoint approximate unit for $\mathrm{C}(O)$. Put $w_k = g_k \cdot \pi^*(v)$. Then,

$$\begin{aligned} w_k^*(a \oplus a \oplus a)w_k &= g_k\pi^*\big(v^*(\xi_2 \oplus \xi_2 \oplus \xi_2)v\big)g_k = g_k\pi^*(\theta_4)g_k \\ &= g_k\left(\bigoplus_{j=1}^{4}\pi^*(\theta_1)\right)g_k = \bigoplus_{j=1}^{4} g_k b g_k \xrightarrow{k\to\infty} \bigoplus_{j=1}^{4} b. \end{aligned}$$

This is precisely the statement that $4\langle b\rangle \leqslant 3\langle a\rangle$. $\langle b\rangle \nleqslant \langle a\rangle$ by the proof of Theorem 6.6, and the corollary follows. □

**Proposition 6.8.** *For any $r \in \mathbb{R}^+$, there is a simple unital AH algebra $A$ such that* $\mathrm{rc}(A) \geqslant r = \mathrm{drr}(A)/2$.

**Proof.** There is nothing to prove when $r = 0$, so fix $r > 0$. For a $C^*$-algebra $A$, let $V(A)$ denote the semigroup of Murray–von Neumann equivalence classes of projections in $\mathrm{M}_\infty(A)$. The algebra $A_{2r}$ constructed in the proof of Theorem 5.1 has $\mathrm{drr} = 2r$ and stable rank one. It follows that there is an order unit preserving order isomorphism

$$\big(\mathrm{K}_0(A_{2r})^+, [1_{A_{2r}}]\big) \cong \big(V(A_{2r}), [1_{A_{2r}}]\big).$$

Since $A_{2r}$ is stably finite, there is an order unit preserving order embedding of $(V(A_{2r}), [1_{A_{2r}}])$ into $(W(A_{2r}), \langle 1_{A_{2r}}\rangle)$. The proof of Corollary 5.2 shows that for any $t < r$, there are projections $p, q \in \mathrm{M}_\infty(A)$ such that $[p] \nleqslant [q]$, yet $s(p) + t < s(q)$ for the unique state $s \in S(\mathrm{K}_0(A_{2r}), [1_{A_{2r}}])$.

Let $d_\tau \in \mathrm{LDF}(A_{2r})$ be induced by $\tau \in \mathrm{T}(A)$. $d_\tau$ gives rise to an element of $S(\mathrm{K}_0(A_{2r})$, and so agrees with $s$ on the image of $\mathrm{K}_0(A_{2r})^+$ in $W(A_{2r})$. In particular,

$$d_\tau\big(\langle p\rangle\big) + t < d_\tau\big(\langle q\rangle\big), \quad \forall d_\tau \in \mathrm{LDF}(A_{2r}).$$

The existence of an order unit preserving order embedding

$$\iota : \big(\mathrm{K}_0(A_{2r})^+, [1_{A_{2r}}]\big) \to \big(W(A_{2r}), \langle 1_{A_{2r}}\rangle\big)$$

implies that $\langle p\rangle$ is not less than $\langle q\rangle$ in $W(A_{2r})$, whence $\mathrm{rc}(A_{2r}) \geqslant t$; $t$ was arbitrary, and the proposition follows. □

One wants an upper bound on the radius of comparison of $A = p(\mathrm{C}(X) \otimes \mathcal{K})p$ of the form

$$\mathrm{rc}(A) \leqslant K\,\mathrm{drr}(A), \quad K > 0, \tag{12}$$

where $X$ is a CW-complex, $p \in \mathrm{C}(X)$ is a projection, and $K$ is independent of our choice of $X$ and $p$. (This bound holds already in the case $\mathrm{drr}(A) = 0$ by Theorem 3.4 of [20].) This would complete the confirmation of the radius of comparison as the correct abstraction of the dimension–rank ratio. Applied to the algebras of Theorem 6.6, it would show that the radius of comparison roughly determines the dimension rank ratio. Philosophically, asking for the bound in (12) is reasonable—it amounts to asking for stability properties in the Cuntz semigroup analogous to the stability properties of vector bundles (cf. Theorem 3.1):

**Question 6.9.** *Does there exist a constant $K > 0$ such that for any compact Hausdorff space $X$ and any positive elements $a, b \in \mathrm{M}_\infty(\mathrm{C}(X))$ satisfying*

$$\mathrm{rank}(b)(x) - \mathrm{rank}(a)(x) \geqslant K \dim(X), \quad \forall x \in X,$$

*one has $a \precsim b$ in $W(\mathrm{C}(X))$?*

It follows more or less directly from Theorem 3.1(ii), that Question 6.9 has a positive answer upon restricting to positive elements whose rank functions take at most two values, one of which is zero, but this partial result does not address the essential difficulties of the question. Nevertheless, an affirmative answer seems likely. To generate interest in Question 6.9, we outline an application of a positive answer to it.

**Conjecture 6.10.** *There exists a simple, unital, separable, and nuclear $C^*$-algebra $A$ of stable rank one such that*

$$A \ncong \mathrm{M}_n(A), \quad \text{some } n \in \mathbb{N},$$

*yet*

$$\big(V(A), [1_A]\big) \cong \big(V\big(\mathrm{M}_n(A)\big), [1_{\mathrm{M}_n(A)}]\big) \cong \big(\mathbb{Q}^+, 1\big).$$

The algebras $A$ and $\mathrm{M}_n(A)$ thus constitute a particularly strong counterexample to Elliott's classification conjecture for simple, separable, and nuclear $C^*$-algebras (cf. [24]).

**Sketch of proof.** The proof of [28, Theorem 1.1], contains a construction of a simple unital AH algebra $A$ of stable rank one which has the properties that $\mathrm{rc}(A) > 1/2$, and

$$\big(V(A), [1_A]\big) \cong \big(\mathbb{Q}^+, 1\big).$$

Explicitly (cf. [28]), one has

$$A = \lim_{i \to \infty} \big(\mathrm{M}_{n_i}\big(\mathrm{C}\big([0,1]^{m_i}\big)\big), \phi_i\big),$$

where $m_i \leqslant n_i$. If Question 6.9 has a positive answer, then we may conclude that $\mathrm{rc}(A) \leqslant K$. Choose $n > 2K$. Then,

$$\big(V\big(\mathrm{M}_n(A)\big), [1_{\mathrm{M}_n(A)}]\big) \cong \big(\mathbb{Q}^+, 1\big),$$

but $\mathrm{rc}(\mathrm{M}_n(A)) < 1/2$ by part (ii) of Proposition 6.2. It follows that $A \ncong \mathrm{M}_n(A)$, as desired. □

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