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AN INFINITE FAMILY OF NON-ISOMORPHIC C*-ALGEBRAS WITH IDENTICAL K-THEORY

ANDREW S. TOMS

Abstract. We exhibit a countably infinite family of simple, separable, nuclear, and mutually non-isomorphic C*-algebras which agree on K-theory and traces. The algebras do not absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially, answering a question of N. C. Phillips. They are also pairwise shape and Morita equivalent, confirming a conjecture from our earlier work. The distinguishing invariant is the radius of comparison, a non-stable invariant of the Cuntz semigroup.

1. Introduction

In 1997 J. Villadsen gave an example of a simple, separable, and nuclear C*-algebra whose ordered K_0-group exhibited pathological behaviour, answering a long-standing open question. Since his discovery, those working on the structure of nuclear C*-algebras have come to realise that there is, apparently, a dichotomy in their field: algebras are either amenable to classification via K-theory, or so wild as to resist most attempts to organise them. Villadsen's example was simply the first of the "wild" algebras.

There are several proposed characterisations of those simple, separable, and nuclear C*-algebras which will prove amenable to classification. The property of absorbing the Jiang-Su algebra $\mathcal{Z}$ tensorially — being *$\mathcal{Z}$-stable* — is chief among them. The main reason for this is the existence of several examples due first to Rørdam ([13]) and later the author ([16], [17]), exhibiting pairs of simple, separable, and nuclear C*-algebras which agree on K-theory and traces but are non-isomorphic. The algebras are, in each case, distinguished by the fact that one of them is not $\mathcal{Z}$-stable. They leave open the following question, posed by N. Christopher Phillips:

> Are there simple, separable, nuclear, and non-$\mathcal{Z}$-stable C*-algebras which agree on K-theory and traces, yet are non-isomorphic?

The idea is this: eliminate the possibility that the failure of K-theory to be a complete invariant for simple, separable, and nuclear C*-algebras is due simply to the need to differentiate between $\mathcal{Z}$-stable and non-$\mathcal{Z}$-stable algebras. Phillips' question is a specific instance of a more general question, namely, to what extent can separable, nuclear, and non-isomorphic C*-algebras be similar? In the sequel we give a positive answer to Phillips' question, and make some interesting progress on the more general question, too.

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The major difficulty in answering Phillips' question lies in the fact that non-$\mathcal{Z}$-stable C*-algebras are difficult to control. We cannot, to date, elicit prescribed K-theoretic data from them. Also, if one manages to produce two such algebras with the same K-theory and tracial state space, how can one tell them apart? In the sequel we overcome these challenges by studying a non-stable invariant of the Cuntz semigroup — the *radius of comparison* — introduced in [18]. We obtain:

Theorem. *There is a simple, separable, nuclear, and non-$\mathcal{Z}$-stable C*-algebra A such that for any natural numbers $m \neq n$ we have:*

(i) $\mathrm{M}_n(A) \ncong \mathrm{M}_m(A)$;
(ii) $\mathrm{M}_n(A)$ *and* $\mathrm{M}_m(A)$ *are shape and (evidently) Morita equivalent;*
(iii) $\mathrm{M}_n(A)$ *and* $\mathrm{M}_m(A)$ *have the same scaled ordered* K*-theory and tracial state space.*

The algebras of the theorem thus agree on stable isomorphism invariants, and on homotopy invariant functors which commute with sequential inductive limits. They moreover agree on every version of non-commutative dimension for C*-algebras: the real, stable, tracial topological, and decomposition ranks. The Morita equivalence of simple, separable, nuclear, and non-isomorphic C*-algebras with identical K-theory and traces is a new phenomenon, and so our theorem constitutes a considerable strengthening of [17, Theorem 1.1].

Our main theorem confirms [18, Conjecture 6.10]. When this conjecture was made, there were two main obstacles to its confirmation. The first was the lack of a good formula for the radius of comparison of a commutative C*-algebra, a problem since overcome (see [19]). The second was the lack of a lower semicontinuity result (with respect to inductive limits) for the radius of comparison. We provide such a result in the sequel; this is the main technical innovation of the paper.

Our paper is organised as follows: in Section 2 we review Cuntz comparison, dimension functions, and the radius of comparison; Section 3 studies the behaviour of the radius of comparison with respect to inductive limits; Section 4 contains the proof of the main theorem.

2. Preliminaries

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. If $A = \mathbb{C}$, then we simply write M_n. Let $\mathrm{M}_\infty(A)$ denote the algebraic limit of the direct system $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathrm{M}_\infty(A)_+$ (resp. $\mathrm{M}_n(A)_+$) denote the positive elements in $\mathrm{M}_\infty(A)$ (resp. $\mathrm{M}_n(A)$). Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$||v_n b v_n^* - a|| \stackrel{n\to\infty}{\longrightarrow} 0.$$

We say that a and b are *Cuntz equivalent* (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. This relation is an equivalence relation, and we write $\langle a \rangle$ for the equivalence class of a. The set

$$W(A) := \mathrm{M}_\infty(A)_+ / \sim$$

becomes a positively ordered Abelian monoid when equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \precsim b.$$

In the sequel, we refer to this object as the *Cuntz semigroup* of A. The Grothendieck enveloping group of $W(A)$ is denoted $\mathrm{K}_0^*(A)$.

Given $a \in \mathrm{M}_\infty(A)_+$ and $\epsilon > 0$, we denote by $(a-\epsilon)_+$ the element of $C^*(a)$ corresponding (via the functional calculus) to the function

$$f(t) = \max\{0, t-\epsilon\}, \ t \in \sigma(a).$$

(Here $\sigma(a)$ denotes the spectrum of a.) The proposition below collects some facts about Cuntz subequivalence due to Kirchberg and Rørdam.

Proposition 2.1 (Kirchberg-Rørdam ([10]), Rørdam ([14]))**.** *Let A be a C*-algebra, and $a, b \in A_+$.*

(i) $(a-\epsilon)_+ \precsim a$ *for every* $\epsilon > 0$.
(ii) *The following are equivalent:*
  (a) $a \precsim b$*;*
  (b) *for all* $\epsilon > 0$*,* $(a-\epsilon)_+ \precsim b$*;*
  (c) *for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $(a-\epsilon)_+ \precsim (b-\delta)_+$.
(iii) *If* $\epsilon > 0$ *and* $||a-b|| < \epsilon$*, then* $(a-\epsilon)_+ \precsim b$.

Now suppose that A is unital and stably finite, and denote by $\mathrm{QT}(A)$ the space of normalised 2-quasitraces on A (v. [2, Definition II.1.1]). Let $S(W(A))$ denote the set of additive and order preserving maps s from $W(A)$ to $\mathbb{R}^+$ having the property that $s(\langle 1_A \rangle) = 1$. Such maps are called *states*. Given $\tau \in \mathrm{QT}(A)$, one may define a map $s_\tau : \mathrm{M}_\infty(A)_+ \to \mathbb{R}^+$ by

$$s_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}). \tag{1}$$

This map is lower semicontinous, and depends only on the Cuntz equivalence class of a. It moreover has the following properties:

(i) if $a \precsim b$, then $s_\tau(a) \leq s_\tau(b)$;
(ii) if a and b are mutually orthogonal, then $s_\tau(a+b) = s_\tau(a) + s_\tau(b)$;
(iii) $s_\tau((a-\epsilon)_+) \nearrow s_\tau(a)$ as $\epsilon \to 0$.

Thus, s_τ defines a state on $W(A)$. Such states are called *lower semicontinuous dimension functions*, and the set of them is denoted $\mathrm{LDF}(A)$. $\mathrm{QT}(A)$ is a simplex ([2, Theorem II.4.4]), and the map from $\mathrm{QT}(A)$ to $\mathrm{LDF}(A)$ defined by (1) is bijective and affine ([2, Theorem II.2.2]). A *dimension function* on A is a state on $\mathrm{K}_0^*(A)$, assuming that the latter has been equipped with the usual order coming from the Grothendieck map. The set of dimension functions is denoted $\mathrm{DF}(A)$. $\mathrm{LDF}(A)$ is a (generally proper) face of $\mathrm{DF}(A)$. If A has the property that $a \precsim b$ whenever $s(a) < s(b)$ for every $s \in \mathrm{LDF}(A)$, then we say that A has *strict comparison of positive elements*.

Finally, we recall the definition of the radius of comparison.

Definition 2.2 (Definition 6.1, [18])**.** Say that A has r-comparison if whenever one has positive elements $a, b \in \mathrm{M}_\infty(A)$ such that

$$s(\langle a \rangle) + r < s(\langle b \rangle), \ \forall s \in \mathrm{LDF}(A),$$

then $\langle a \rangle \leq \langle b \rangle$ in $W(A)$. Define the radius of comparison of A, denoted $\mathrm{rc}(A)$, to be

$$\inf\{r \in \mathbb{R}^+ \mid (W(A), \langle 1_A \rangle) \text{ has } r-\text{comparison}\}$$

if it exists, and ∞ otherwise.

3. The radius of comparison and inductive limits

In this section we prove that the radius of comparison is lower semicontinuous with respect to inductive limits provided that the limit algebra is simple and the inductive sequence is an "efficient decomposition" with respect to traces. We then show that if, as conjectured by Blackadar and Handelman, one has $\mathrm{LDF}(A)$ weak-$*$ dense in $\mathrm{DF}(A)$ for unital and stably finite A, then this "efficient decomposition" assumption is unneccessary.

Our first lemma is a state-theoretic analogue of the implication (i) $\Rightarrow$ (iii) of Proposition 2.1.

Lemma 3.1. *Let A be a simple, unital, and stably finite C^*-algebra. Let $a, b \in \mathrm{M}_\infty(A)_+$ and $r > 0$ be such that*

$$s(a) + r < s(b), \ \forall s \in \mathrm{LDF}(A).$$

Then, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$s((a-\epsilon)_+) + r < s((b-\delta)_+), \ \forall s \in \mathrm{LDF}(A).$$

Proof. Write s_τ for the element of $\mathrm{LDF}(A)$ induced by $\tau \in \mathrm{QT}(A)$, and suppose first that $\langle a \rangle = \langle p \rangle$ for some projection $p \in \mathrm{M}_\infty(A)$. Then, the map $\tau \mapsto s_\tau(a)$ defines a continuous function on $\mathrm{QT}(A)$. By [11, Proposition 2.7], the map $\tau \mapsto s_\tau(b)$ is lower semicontinuous on $\mathrm{QT}(A)$ for any $b \in \mathrm{M}_\infty(A)_+$. Thus, the map $\tau \mapsto s_\tau(b) - s_\tau(a) - r$ is lower semicontinuous and strictly positive, and so achieves a minimum value $\eta > 0$ on the compact set $\mathrm{QT}(A)$. For each $\tau \in \mathrm{QT}(A)$ there is an open neighbourhood U_τ of τ such that

$$|s_\gamma(a) - s_\tau(a)| < \eta/3, \ \forall \gamma \in U_\tau.$$

We have that $s((b-\delta)_+) \nearrow s(b)$ as $\delta \to 0$, for every $s \in \mathrm{LDF}(A)$. Choose, then, for each $\tau \in \mathrm{QT}(A)$, a $\delta_\tau > 0$ such that

$$s_\tau(b) - s_\tau((b-\delta_\tau)_+) < \eta/3.$$

The lower semicontinuity of $\gamma \mapsto s_\gamma((b-\delta_\tau)_+)$ ensures that for each $\tau \in \mathrm{QT}(A)$, there is an open neighbourhood V_τ of τ such that

$$s_\gamma((b-\delta_\tau)_+) > s_\tau((b-\delta_\tau)_+) - \eta/3, \ \forall \gamma \in V_\tau.$$

Put $W_\tau = U_\tau \cap V_\tau$. Then,

$$s_\gamma((b-\delta_\tau)_+) > s_\gamma(a) + r \geq s_\gamma((a-\epsilon)_+) + r, \ \forall \gamma \in W_\tau.$$

The W_τ cover the compact set $\mathrm{QT}(A)$ and so there exist $\tau_1, \ldots, \tau_n \in \mathrm{QT}(A)$ such that

$$\mathrm{QT}(A) = W_{\tau_1} \cup \cdots \cup W_{\tau_n}.$$

Put $\delta = \min\{\delta_{\tau_1}, \ldots, \delta_{\tau_n}\}$. Then, since

$$s_\gamma((b-\delta)_+) \geq s_\gamma((b-\delta_{\tau_i})_+), \ \forall \gamma \in \mathrm{QT}(A), \ 1 \leq i \leq n,$$

we have

$$s((a-\epsilon)_+) + r < s((b-\delta)_+), \ \forall s \in \mathrm{LDF}(A),$$

as desired.

Now suppose that $\langle a\rangle \neq \langle p\rangle$ for any projection $p \in \mathrm{M}_\infty(A)$. Then, by a functional calculus argument, 0 is not an isolated point of the spectrum $\sigma(a)$ of a. The simplicity of A implies that each quasitrace on A is faithful, so for each $\epsilon > 0$ we have

$$s((a-\epsilon)_+) < s(a),\ \forall s \in \mathrm{LDF}(A).$$

Each $\tau \in \mathrm{QT}(A)$ is implemented on $C^*(a)$ by a probability measure μ_τ on $\sigma(a)$. Put

$$A_\tau = \{\gamma \in \mathrm{QT}(A) \mid \mu_\gamma([\epsilon,\infty)\cap\sigma(a)) \geq s_\tau(a)\}.$$

If $\gamma \notin A_\tau$, then

$$s_\gamma((a-\epsilon)_+) = \mu_\gamma((\epsilon,\infty)\cap\sigma(a)) \leq \mu_\gamma([\epsilon,\infty)\cap\sigma(a)) < s_\tau(a).$$

Clearly, $\tau \notin A_\tau$. Let (γ_n) be a convergent sequence in A_τ with limit γ. This implies that $\mu_{\gamma_n} \to \mu_\gamma$ in measure on $\sigma(a)$. By Portmanteau's Theorem, this implies that $\mu_\gamma(C) \geq \limsup \mu_{\gamma_n}(C)$ for every closed subset of $\sigma(a)$. Since $[\epsilon,\infty)\cap\sigma(a)$ is closed we have

$$\mu_\gamma([\epsilon,\infty)\cap\sigma(a)) \geq \limsup \mu_{\gamma_n}([\epsilon,\infty)\cap\sigma(a)) \geq s_\tau(a),$$

whence $\gamma \in A_\tau$ and A_τ is closed. Thus $U_\tau := A_\tau^c$ is an open neighbourhood of τ, and

$$s_\gamma((a-\epsilon)_+) < s_\tau(a),\ \forall \gamma \in U_\tau.$$

Find (as above) for each $\tau \in \mathrm{QT}(A)$ a $\delta_\tau > 0$ such that $s_\tau((b-\delta_\tau)_+) > s_\tau(a)+r$. Apply the lower semicontinuity of $\gamma \mapsto s_\gamma((b-\delta_\tau)_+)$ to find an open neighbourhood V_τ of τ such that

$$s_\gamma((b-\delta_\tau)_+) > s_\tau(a)+r,\ \forall \gamma \in V_\tau.$$

Put $W_\tau = U_\tau \cap V_\tau$. Then,

$$s_\gamma((a-\epsilon)_+) + r < s_\tau(a) + r < s_\gamma((b-\delta_\tau)_+),\ \forall \gamma \in W_\tau.$$

Use the compactness argument above to find $\delta > 0$ such that

$$s((a-\epsilon)_+) + r < s((b-\delta)_+),\ \forall s \in \mathrm{LDF}(A). \qquad \square$$

The next lemma is due to M. Rørdam. Its proof can be found in [19].

Lemma 3.2 (Rørdam, [15])**.** *Let A be a C^*-algebra and $\{A_i\}_{i\in I}$ a collection of C^*-subalgebras whose union is dense. Then, for every $a \in \mathrm{M}_\infty(A)_+$ and $\epsilon > 0$ there exists $i \in \mathbb{N}$ and $\tilde{a} \in \mathrm{M}_\infty(A_i)$ such that*

$$(a-\epsilon)_+ \precsim \tilde{a} \precsim (a-\epsilon/2)_+ \precsim a$$

in $\mathrm{M}_\infty(A)$.

Proposition 3.3. *Let $A = \lim_{i\to\infty}(A_i,\phi_i)$ be a simple, unital, and stably finite C^*-algebra with ϕ_i injective. Suppose this decomposition satisfies the following property: given $\epsilon > 0$ and $a, b \in \mathrm{M}_\infty(\bigcup_{i=1}^\infty A_i)_+$ such that*

$$s(a) < s(b),\ \forall s \in \mathrm{LDF}(A),$$

there is a $j \in \mathbb{N}$ such that

$$s(a) < s(b) + \epsilon,\ \forall s \in \mathrm{LDF}(A_j).$$

Then,

$$\mathrm{rc}(A) \leq \liminf_{i\to\infty} \mathrm{rc}(A_i).$$

Proof. The theorem is trivial if

$$\liminf \operatorname{rc}(A_i) = \infty,$$

so suppose that

$$r := \liminf \operatorname{rc}(A_i) < \infty.$$

Passing to a subsequence if necessary, we assume that $(\operatorname{rc}(A_i))_{i=1}^{\infty}$ is decreasing.

Let there be given $a, b \in \mathrm{M}_n(A)_+ \hookrightarrow \mathrm{M}_\infty(A)_+$ and $m > r$ such that

$$s(a) + m < s(b),\ \forall s \in \mathrm{LDF}(A).$$

By [18, Proposition 6.3], it will suffice to prove that $a \precsim b$. Let $\epsilon > 0$ be given, and use Lemma 3.1 to find a $\delta > 0$ such that

$$s((a - \epsilon/2)_+) + m < s((b - \delta)_+),\ \forall s \in \mathrm{LDF}(A).$$

Find, using Lemma 3.2, a positive element $\tilde{a} \in A_j$, some $j \in \mathbb{N}$, such that

$$||(a - \epsilon/2)_+ - \tilde{a}|| < \epsilon/2$$

and

$$(a - \epsilon)_+ \precsim \tilde{a} \precsim (a - \epsilon/2)_+.$$

Put $\epsilon^{'} = \min\{\epsilon, \delta\}$, and find a positive element $\tilde{b}$ in some A_j (we may assume that it is the same A_j that contains $\tilde{a}$) such that $||b - \tilde{b}|| < \epsilon^{'}$ and $(b - \delta)_+ \precsim \tilde{b} \precsim b$. Finally, assume that $j \in \mathbb{N}$ has been chosen large enough to ensure that $\operatorname{rc}(A_j) < m$. We now have

$$s_\gamma(\tilde{a}) + m \leq s_\gamma((a - \epsilon/2)_+) + m < s_\gamma((b - \delta)_+) \leq s_\gamma(\tilde{b}),\ \forall \gamma \in \mathrm{QT}(A).$$

Choose $\eta > 0$ such that $m - \eta > \operatorname{rc}(A_j)$. By our hypothesis we may assume, upon increasing j if necessary, that

$$s_\gamma(\tilde{a}) + m - \eta < s_\gamma(\tilde{b}),\ \forall \gamma \in \mathrm{QT}(A_j).$$

Since $\operatorname{rc}(A_j) < m - \eta$, we conclude that

$$(a - \epsilon)_+ \precsim \tilde{a} \precsim \tilde{b} \precsim b;$$

ϵ was arbitrary, and the proposition follows. □

The reader may wonder whether the "extra" hypothesis of Proposition 3.3 — the "efficient decomposition" hypothesis alluded to at the beginning of this subsection — can be removed. Indeed, if a and b as in the proposition are projections, then there is always a natural number j such that $s_\tau(a) = \tau(a) < \tau(b) = s_\tau(b)$ for each $\tau \in \mathrm{QT}(A_j)$. (To the best of our knowledge, this was first observed by Blackadar in [1].) We point out why this argument does not carry over to the setting of positive elements and lower semicontinuous dimension functions after proving the next lemma. We thank Wilhelm Winter for pointing out the ultrafilter argument used in the proof.

Lemma 3.4. *Let $A = \lim_{i\to\infty}(A_i, \phi_i)$ be simple and unital, with each A_i stably finite and each ϕ_i injective. Suppose that $a, b \in \mathrm{M}_\infty(\bigcup_{i=1}^{\infty} A_i)_+$ are such that*

$$s(a) < s(b),\ \forall s \in \mathrm{DF}(A).$$

Then, there is some $j \in \mathbb{N}$ such that $a, b \in \mathrm{M}_\infty(A_j)_+$ and

$$s(a) < s(b),\ \forall s \in \mathrm{DF}(A_j).$$

Proof. Let B be the C*-algebra consisting of all bounded sequences

$$(a_1, a_2, a_3, \ldots),$$

where $a_i \in A_i$, and let $I \subseteq B$ be the closed two-sided ideal of sequences such that $a_i \to 0$ as $i \to \infty$. Then, there is a $*$-monomorphism $\iota : A \to B/I$ given by

$$a \mapsto (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, a, \phi_i(a), \phi_{i+1}(\phi_i(a)), \ldots) + I$$

on $A_i \subseteq A$, $i \in \mathbb{N}$, and extended by continuity.

We may assume, by truncating our inductive sequence if necessary, that $a, b \in A_1$. Suppose, contrary to our desired conclusion, that for each $i \in \mathbb{N}$ there exists a dimension function $d_i \in \mathrm{DF}(A_i)$ satisfying

$$d_i(a) \geq d_i(b).$$

Let ω be a free ultrafilter on $\mathbb{N}$, and let s be the map given by

$$s(a_1, a_2, \ldots) = \lim_{\omega} d_i(a_i).$$

It is straightforward to check that $s \in \mathrm{DF}(A)$. But then $s(b) \geq s(a)$, contrary to our assumption. □

The free ultrafilter approach in Lemma 3.4 can be applied after replacing the d_i with lower semicontinuous dimension functions $s_{\tau_i} \in \mathrm{LDF}(A_i)$, but it is then unclear whether the resulting dimension function is lower semicontinuous. Alternatively, one can use the τ_i themselves to define, via the free ultrafilter, a faithful trace τ on A, but it is then unclear whether $s_\tau(a) \geq s_\tau(b)$. At issue is an interchanging of the limit over ω and the limit appearing in the definition of a lower semicontinuous dimension function. In any case, there is no obvious way to prove the lemma upon substituting lower semicontinuous dimension functions for dimension functions. But this difficulty vanishes if $\mathrm{LDF}(A)$ is dense in $\mathrm{DF}(A)$ whenever A is unital and stably finite.

Theorem 3.5. *Let $A = \lim_{i\to\infty}(A_i, \phi)$ be unital and simple, with each A_i stably finite and each ϕ_i injective. Also suppose that* $\mathrm{LDF}(A)$ *is dense in* $\mathrm{DF}(A)$. *Then,*

$$\mathrm{rc}(A) \leq \liminf_{i\to\infty} \mathrm{rc}(A_i).$$

Proof. As in the proof of Proposition 3.3, we assume that

$$r := \liminf_{i\to\infty} \mathrm{rc}(A_i) < \infty.$$

Let $\epsilon > 0$ and $a, b \in A_+$ be given. For any $d \in \mathrm{DF}(A)$, there is a sequence $(s_{\tau_i})_{i=1}^\infty$ in $\mathrm{LDF}(A)$ converging to d. It follows that $d(a) \leq d(b) < d(b) + \epsilon$ whenever

$$s_\tau(a) < s_\tau(b), \forall s_\tau \in \mathrm{LDF}(A).$$

By Lemma 3.4 there is some $i \in \mathbb{N}$ such that $d(a) < d(b) + \epsilon$ for every $d \in \mathrm{DF}(A_i) \supseteq \mathrm{LDF}(A_i)$. Thus, A satisfies the hypotheses of Proposition 3.3, and the theorem follows. □

4. The main result

Theorem 4.1. *There is a simple, separable, and nuclear C*-algebra A such that for any natural numbers $n \neq m$ one has:*

(i) $\mathrm{M}_n(A) \ncong \mathrm{M}_m(A)$;
(ii) $\mathrm{M}_n(A)$ *and* $\mathrm{M}_m(A)$ *agree on the Elliott invariant;*
(iii) $\mathrm{M}_n(A)$ *and* $\mathrm{M}_m(A)$ *are shape equivalent;*
(iv) $\mathrm{M}_n(A)$ *is non-$\mathcal{Z}$-stable, and has stable rank one, real rank one, and property (SP);*
(v) $(V(\mathrm{M}_n(A)), [1_{\mathrm{M}_n(A)}]) \cong (\mathbb{Q}^+, 1)$; *in particular,* $\mathrm{K}_0(\mathrm{M}_n(A))$ *is a divisible and weakly unperforated partially ordered group.*

Before proceeding with the proof we briefly recall some terminology. Let X and Y be compact Hausdorff spaces, and let $m, n \in \mathbb{N}$ be such that $m|n$. Recall that a $*$-homomorphism

$$\phi : \mathrm{M}_m(\mathrm{C}(X)) \to \mathrm{M}_n(\mathrm{C}(Y))$$

is called *diagonal* if

$$\phi(f) = \bigoplus_{i=1}^{n/m} f \circ \lambda_i,$$

where each $\lambda_i : Y \to X$ is continuous. The λ_i are called *eigenvalue maps*.

Proof. A will be constructed as per the general framework set out in [20], and will be identical to the construction in the proof of [17, Theorem 1.1]. It is necessary, however, to recall the details of the construction, as they are essential to proving that $\mathrm{rc}(A)$ is finite and non-zero.

Put $X = [-1,1]^3$. Put $X_1 = X \times X$, and put $X_{i+1} = (X_i)^{n_i}$ — the n_i-fold Cartesian product of X_i with itself. Let $\pi_i^j : X_{i+1} \to X_i$, $1 \leq j \leq n_i$, be the co-ordinate projections. Let A_i be the homogeneous C*-algebra $\mathrm{M}_{m_i} \otimes \mathrm{C}(X_i)$, where m_i is a natural number to be specified, and let $\phi_i : A_i \to A_{i+1}$ be the $*$-homomorphism given by

$$\phi_i(a)(x) = \mathrm{diag}\left(a \circ \pi_i^1(x), \ldots, a \circ \pi_i^{n_i}(x), a(x_i^1), \ldots, a(x_i^i)\right), \ \forall x \in X_{i+1},$$

where $x_i^1, \ldots, x_i^i \in X_i$ are to be specified. Put $A = \lim_{i \to \infty}(A_i, \phi_i)$, and define

$$\phi_{i,j} := \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let $\phi_{i\infty} : A_i \to A$ be the canonical map. Assume that the n_i have been chosen so that

$$\prod_{i=1}^{\infty} \frac{n_i}{i + n_i} \neq 0,$$

and that the $x_i^1, \ldots, x_i^i$ have been chosen to ensure that A is simple (this can always be arranged; cf. [20]). Finally, assume that our choice of the n_i is such that every natural number divides some m_i. Put $m_1 = 2$.

(i). We will first prove that the inductive sequence (A_i, ϕ_i) satisfies the hypotheses of Proposition 3.3, i.e., that given $\epsilon > 0$ and $a, b \in \mathrm{M}_\infty(\bigcup_{i=1}^\infty A_i)_+$ such that

$$s(a) < s(b), \ \forall s \in \mathrm{LDF}(A),$$

there is an $i \in \mathbb{N}$ such that

$$s(a) < s(b) + \epsilon, \ \forall s \in \mathrm{LDF}(A_i).$$

We will accomplish this by finding for each $\epsilon > 0$ an $i \in \mathbb{N}$ such that the following holds: for any $\tau \in \mathrm{T}(A_i)$ there is some $\gamma \in \mathrm{T}(A)$ satisfying

$$\phi^{\sharp}_{i\infty}(\gamma) = (1-\lambda)\tau + \lambda\eta$$

for some λ such that $0 < \frac{\lambda}{1-\lambda} < \epsilon$ and $\eta \in \mathrm{T}(A_i)$. To see that this will suffice, take any $\tau \in \mathrm{T}(A_i)$ and $a, b \in \mathrm{M}_\infty(\bigcup_{i=1}^\infty A_i)_+$ such that

$$s(a) < s(b),\ \forall s \in \mathrm{LDF}(A).$$

Then, we have

$$\begin{aligned} s_\tau(a) &\leq s_\tau(a) + \frac{\lambda}{1-\lambda} s_\eta(a) \\ &= \frac{1}{1-\lambda} s_{(1-\lambda)\tau+\lambda\eta}(a) \\ &= \frac{1}{1-\lambda} s_{\phi^{\sharp}_{i\infty}(\gamma)}(a) \\ &= \frac{1}{1-\lambda} s_\gamma(a) \\ &< \frac{1}{1-\lambda} s_\gamma(b) \\ &= \frac{1}{1-\lambda} s_{\phi^{\sharp}_{i\infty}(\gamma)}(b) \\ &= s_\tau(b) + \frac{\lambda}{1-\lambda} s_\eta(b) \\ &< s_\tau(b) + \epsilon. \end{aligned}$$

Let $\epsilon > 0$ be given. For any $i \in \mathbb{N}$, $\tau \in \mathrm{T}(A_i)$, and $j > i$, let $\tau_{i,j} \in \mathrm{T}(A_j)$ be the trace corresponding to the product measure

$$\underbrace{\tau \times \cdots \times \tau}_{n_{i+1}n_{i+2}\cdots n_j \text{ times}} .$$

If $i \leq k < j$, then define $\tau_k^j := \phi^{\sharp}_{k,j}(\tau_{i,j})$. A staightforward calculation shows that $\tau_k^j = (1-\lambda_k^j)\tau_{i,k} + \lambda_k^j \eta_k^j$ for some $\eta_k^j \in \mathrm{T}(A_k)$ and $0 < \lambda_k^j < 1$. In fact, if $N_{k,j}$ denotes the number of eigenvalue maps of $\phi_{k,j}$ which are co-ordinate projections onto X_k, then

$$\lambda_k^j = \frac{N_{k,j}}{\mathrm{mult}(\phi_{k,j})}.$$

By construction $N_{k,j}/\mathrm{mult}(\phi_{k,j})$ is a decreasing sequence with a strictly positive limit $1-\lambda_k$. By increasing i if necessary, we may assume that $\lambda_i/(1-\lambda_i) < \epsilon$. We claim that for each $k \geq i$,

$$(1-\lambda_k^j)\tau_{i,k} + \lambda_k^j \eta_k^j \overset{j\to\infty}{\longrightarrow} (1-\lambda_k)\tau_{i,k} + \lambda_k \eta_k$$

for some $\eta_k \in \mathrm{T}(A_k)$. It will suffice to prove that η_k^j is a Cauchy sequence in j. Let $R_{k,j}$ be the multiset whose elements are the points of X_k which appear as point evaluations in $\phi_{k,j}$. We have that

$$\eta_k^j = \frac{1}{|R_{k,j}|} \sum_{x \in R_{k,j}} ev_x;$$

η_k^j is a finite convex combination of extreme traces on A_k. $R_{k,j+1}$ is formed by taking the union (with multiplicity) of $\mathrm{mult}(\phi_{j,j+1})$ copies of $R_{k,j}$ and some other multiset S_{j+1}. Thus,

$$\begin{aligned}\eta_k^{j+1} &= \frac{1}{|R_{k,j+1}|}\sum_{x\in R_{k,j+1}} ev_x \\ &= \frac{\mathrm{mult}(\phi_{j,j+1})}{\mathrm{mult}(\phi_{j,j+1})|R_{j,k}|+|S_{j+1}|}\sum_{x\in R_{k,j}} ev_x \\ &\quad +\frac{1}{\mathrm{mult}(\phi_{j,j+1})|R_{j,k}|+|S_{j+1}|}\sum_{x\in S_{j+1}} ev_x \\ &= \frac{\mathrm{mult}(\phi_{j,j+1})|R_{j,k}|}{\mathrm{mult}(\phi_{j,j+1})|R_{j,k}|+|S_{j+1}|}\eta_k^j \\ &\quad +\frac{1}{\mathrm{mult}(\phi_{j,j+1})|R_{jk}|+|S_{j+1}|}\sum_{x\in S_{j+1}} ev_x.\end{aligned}$$

It follows that for any normalised element $f \in A_k$, we have that

$$|\eta_k^j(f) - \eta_k^{j+1}(f)| \leq \frac{2|S_{j+1}|}{\mathrm{mult}(\phi_{j,j+1})|R_{j,k}|+|S_{j+1}|}.$$

By construction, the right hand side vanishes as $j \to \infty$, proving the claim.

We now have that for each $k \in \mathbb{N}$, the sequence τ_k^j converges as $j \to \infty$. Call the limit τ_k. Since $\tau_k^j = \phi^\sharp_{k,k+1}(\tau^j_{k+1})$, we have that $\tau_k = \phi^\sharp_{k,k+1}(\tau_{k+1})$ for every $k \geq i$. It follows that the sequence $(\tau_i, \tau_{i+1}, \ldots)$ defines a point in limit of the inverse system $(\mathrm{T}(A_i), \phi_i^\sharp)$, i.e., a point in $\mathrm{T}(A)$. Thus,

$$\tau_i = (1-\lambda_i)\tau + \lambda_i\eta_i$$

is the image of some $\gamma \in \mathrm{T}(A)$ under the map $\phi^\sharp_{i\infty}$. Since $0 < \lambda_i/(1-\lambda_i) < \epsilon$, we have established the hypotheses of Proposition 3.3 for the inductive system (A_i, ϕ_i). It is clear from our construction and [19, Theorem 4.2] that $\mathrm{rc}(A_i) < 10$, $\forall i \in \mathbb{N}$. Proposition 3.3 then shows that $\mathrm{rc}(A) < \infty$. The proof of [17, Theorem 1.1] shows that $\mathrm{rc}(A) > 0$, so that $\mathrm{rc}(A)$ is finite and non-zero.

Now let $m \neq n$ be natural numbers. [18, Proposition 6.2, (i)] shows that

$$\mathrm{rc}(\mathrm{M}_n(A)) = \mathrm{rc}(A)/n \neq \mathrm{rc}(A)/m = \mathrm{rc}(\mathrm{M}_m(A)),$$

whence $\mathrm{M}_n(A) \ncong \mathrm{M}_m(A)$, as desired.

(ii). By the contractibility of X_i we have $\mathrm{K}_0(A_i) \cong \mathbb{Z}$, $\mathrm{K}_1(A_i) = 0$, and $\mathrm{K}_0(\phi_i)(1) = m_i$. It follows from our assumption on the m_is that

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]) \cong (\mathbb{Q}, \mathbb{Q}^+, 1),$$

and this same isomorphism clearly holds for any matrix algebra over A.

Since $\mathrm{K}_0(\mathrm{M}_n(A)) \cong \mathbb{Q}$, there is a unique pairing between traces and K_0. Thus, all of the non-stable information in the Elliott invariant of $\mathrm{M}_n(A)$ is independent of n. The remaining elements of the Elliott invariant are stable isomorphism invariants, and are thus also independent of n.

(iii). We will prove that A and $\mathrm{M}_2(A)$ are shape equivalent. The argument for $\mathrm{M}_n(A)$ and $\mathrm{M}_m(A)$ is similar.

By compressing the inductive sequence (A_i, ϕ_i) if necessary, we may assume that $2m_i | m_{i+1}$ for all $i \in \mathbb{N}$. To prove that A and $\mathrm{M}_2(A)$ are shape equivalent, it will suffice to find sequences of $*$-homomorphisms $(\eta_i)_{i=1}^{\infty}$ and $(\mu_i)_{i=1}^{\infty}$ making the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_1"] \arrow[d, "\eta_1"] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\eta_2"] & A_3 \arrow[r] \arrow[d, "\eta_3"] & \cdots \arrow[r] & A \\
\mathrm{M}_2 \otimes A_1 \arrow[r, "\mathrm{id}\otimes\phi_1"] \arrow[ur, "\mu_1"] & \mathrm{M}_2 \otimes A_2 \arrow[r, "\mathrm{id}\otimes\phi_2"] \arrow[ur, "\mu_2"] & \mathrm{M}_2 \otimes A_3 \arrow[r] \arrow[ur, "\mu_3"] & \cdots \arrow[r] & \mathrm{M}_2 \otimes A
\end{tikzcd}$$

commute up to homotopy. Fix a point $y_i \in X_i$ for every $i \in \mathbb{N}$, and define

$$\eta_i(f) = f(y_i) \oplus f(y_i); \quad \mu_i(g) = \bigoplus_{l=1}^{\frac{n_{i+1}}{2n_i}} g(y_i).$$

Then, for every $i \in \mathbb{N}$, the maps $\phi_i, \mathrm{id}_{\mathrm{M}_2} \otimes \phi_i, \mu_i \circ \eta_i$, and $\eta_i \circ \mu_{i-1}$ are diagonal. They have the form

$$f \mapsto \bigoplus_{j=1}^{k} f \circ \lambda_j,$$

where the λ_j are continuous maps from the spectrum of the target algebra to the spectrum of the source algebra. The latter are always pairwise homotopic in our setting, since each X_i is contractible. It follows that any two diagonal maps from A_i to A_{i+1}, or from $\mathrm{M}_2(A_i)$ to $\mathrm{M}_2(A_{i+1})$, are homotopic, and so our diagram commutes up to homotopy as required.

(iv). That the stable rank and the real rank of A are one follows from [20, Proposition 10] and the discussion following it. Both the stable and real rank are monotone decreasing under taking tensor products with full matrix algebras, with the caveat that in both cases an algebra with non-minimal rank is not Morita equivalent to an algebra with minimal rank ([12, Theorem 3.6] and [3]). It follows that every matrix algebra over A has stable rank one and real rank one.

Villadsen proves in [20] that his choice of points $x_i^1, \ldots, x_i^i \in X_i$, $i \in \mathbb{N}$, ensures that for every open $V \subseteq X_j$, there is a $k > j$ such that the composed map

$$\phi_{jk} := \phi_{k-1} \circ \cdots \circ \phi_j$$

has an eigenvalue map which is a point evaluation with range in V. It follows that the hereditary subalgebra generated by the image $\phi_{i\infty}(a)$ of some non-zero positive $a \in A_i$ contains a non-zero projection. Now let $a \in A$ be positive and non-zero. Choose $\epsilon > 0$ such that $\epsilon \ll ||a||$, and find a positive element $\tilde{a}$ in some A_i such that $||a - \tilde{a}|| < \epsilon$. Then, $(\tilde{a} - \epsilon)_+ \precsim a$, and $(\tilde{a} - \epsilon)_+$ is a non-zero positive element of A_i. Let p be a non-zero projection contained in the hereditary subalgebra of A generated by $(\tilde{a} - \epsilon)_+$. Then, $p \precsim a$ by transitivity. It follows from the proof of [11, Proposition 2.2] that $\overline{aAa}$ contains a non-zero projection, whence A and matrix algebras over it all have property (SP).

Every simple, unital, exact, finite, and $\mathcal{Z}$-stable C*-algebra has $\mathrm{rc} = 0$ by [14, Corollary 4.6], whence $\mathrm{M}_n(A)$ is not $\mathcal{Z}$-stable for any $n \in \mathbb{N}$.

(v). That

$$(V(\mathrm{M}_n(A)), [1_{\mathrm{M}_n(A)}]) \cong (\mathbb{Q}^+, 1)$$

follows from our calculation of $\mathrm{K}_0(\mathrm{M}_n(A))$ in (ii) and the fact that A has stable rank one. $\square$

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References

[1] Blackadar, B.: *Traces on simple AF C^*-algebras*, J. Funct. Anal. **38** (1980), 156-168 MR587906 (82a:46062)
[2] Blackadar, B., and Handelman, D.:*Dimension Functions and Traces on C^*-algebras*, J. Funct. Anal. **45** (1982), 297-340 MR650185 (83g:46050)
[3] Brown, L. G. and Pedersen, G. K.: *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131-149 MR1120918 (92m:46086)
[4] Brown, N., Perera, F., and Toms, A. S.: *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, to appear in J. Reine Angew. Math., arXiv preprint math.OA/0609182 (2006)
[5] Cuntz, J.: *Dimension Functions on Simple C^*-algebras*, Math. Ann. **233** (1978), 145-153 MR0467332 (57:7191)
[6] Elliott, G. A.: *The classification problem for amenable C^*-algebras*, Proc. ICM '94, Zurich, Switzerland, Birkhauser Verlag, Basel, Switzerland, 922-932 MR1403992 (97g:46072)
[7] Gong, G., Jiang, X. and Su, H.: *Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras*, Canad. Math. Bull. **43** (2000), 418-426 MR1793944 (2001k:46086)
[8] Husemoller, D.: Fibre Bundles, McGraw-Hill, New York, 1966 MR0229247 (37:4821)
[9] Jiang, X. and Su, H.: *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), 359-413 MR1680321 (2000a:46104)
[10] Kirchberg, E. and Rørdam, M.: *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637-666 MR1759891 (2001k:46088)
[11] Perera, F. and Toms, A. S.: *Recasting the Elliott conjecture*, Math. Ann. **338** (2007), 669-702 MR2317934
[12] Rieffel, M. A.: *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46** (1983), 301-333 MR693043 (84g:46085)
[13] Rørdam, M.: *A simple C^*-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), 109-142 MR2020420 (2005m:46096)
[14] Rørdam, M.: *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Int. J. Math. **15** (2004), 1065-1084 MR2106263 (2005k:46164)
[15] Rørdam, M., private communication
[16] Toms, A. S.: *On the independence of* K-*theory and stable rank for simple C^*-algebras*, J. Reine Angew. Math. **578** (2005), 185-199 MR2113894 (2005k:46189)
[17] Toms, A. S.: *On the classification problem for nuclear C^*-algebras*, Ann. of Math. (2) **167** (2008), 1059-1074.
[18] Toms, A. S.: *Flat dimension growth for C^*-algebras*, J. Funct. Anal. **238** (2006), 678-708 MR2253738 (2007j:46098)
[19] Toms, A. S.: *Stability in the Cuntz semigroup of a commutative C^*-algebra*, Proc. London Math. Soc. **96** (2008), 1-25.
[20] Villadsen, J.: *Simple C^*-algebras with perforation*, J. Funct. Anal. **154** (1998), 110-116 MR1616504 (99j:46069)

Department of Mathematics and Statistics, York University, 4700 Keele St., Toronto, Ontario, Canada M3J 1P3
E-mail address: atoms@mathstat.yorku.ca