MINIMAL DYNAMICS AND K-THEORETIC RIGIDITY: ELLIOTT'S CONJECTURE

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ABSTRACT. Let X be a compact infinite metric space of finite covering dimension and $\alpha : X \to X$ a minimal homeomorphism. We prove that the crossed product $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ absorbs the Jiang-Su algebra tensorially and has finite nuclear dimension. As a consequence, these algebras are determined up to isomorphism by their graded ordered K-theory under the necessary condition that their projections separate traces. This result applies, in particular, to those crossed products arising from uniquely ergodic homeomorphisms.

0. INTRODUCTION

0.1 From its earliest days the theory of operator algebras has been entwined with dynamics, and some of the most important developments in the subject revolve around this interaction. The group-measure space construction of Murray and von Neumann provided the first examples of non-type-I factors; recently we have seen connections with orbit equivalence and associated rigidity phenomena in the remarkable works of Popa and Giordano et al. (see [16] and [6], respectively).

In this article we consider crossed product C*-algebras arising from topological dynamical systems, and prove a theorem concerning the degree to which they are determined by their graded ordered K-theory.

THEOREM: *Let $\mathcal{C}$ denote the class of C*-algebras having the following properties:*

   (i) *$A \in \mathcal{C}$ has the form $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ for some compact, infinite, finite-dimensional metric space X and minimal homeomorphism $\alpha : X \to X$;*
   (ii) *the projections of A separate traces.*

If $A, B \in \mathcal{C}$ and there is a graded ordered isomorphism $\phi : \mathrm{K}_*(A) \to \mathrm{K}_*(B)$, then there is a $*$-isomorphism $\Phi : A \to B$.

This result was conjectured by G. A. Elliott in 1990 as part of his wider program to classify separable amenable C*-algebras. The hypotheses of minimality for α, finite-dimensionality for X, and the separation of traces by projections are all known to be necessary; the necessity of finite-dimensionality for X was established recently by Giol and Kerr in [5]. If one imposes unique ergodicity on α, then condition (ii) is unnecessary. Our result is the culmination of a sequence of earlier important

results due to Elliott and Evans ([3]), H. Lin and Phillips ([10]), and the second named author ([25, 26]).

0.2 Our proof does not employ a decomposition for $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ as an inductive limit of type I C*-algebras; rather, it implies that one exists! Using [21], we reduce the problem to a setting similar to the one considered in [10], where an abundance of projections was assumed. This reduction is made by applying a general classification result developed by the second named author in [26], the key hypothesis of which is the condition that the C*-algebras considered all absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially. (This condition is necessary for K-theoretic rigidity results akin to Theorem 0.1, both in general and in the case of tracial algebras (see [18] and [22], respectively). We refer the reader to [4] and to [20] for a complete discussion of the Jiang-Su algebra and its relevance to Elliott's program.) The bulk of our effort is concentrated on proving the following result.

THEOREM: *Let X be a compact, infinite, finite-dimensional metric space and $\alpha : X \to X$ a minimal homeomorphism. It follows that*

$$(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}) \otimes \mathcal{Z} \cong \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}.$$

Notice that we do not require projections to separate traces in this theorem. It is conjectured that Theorem 0.1 continues to hold in the absence of condition (ii), provided that one augments the invariant K_* by the simplex of tracial states (here identified with the α-invariant measures on X). We expect that Theorem 0.2 will prove crucial to the solution of this conjecture, too.

0.3 Theorem 0.2 actually follows immediately from a second result.

THEOREM: *Let X be a compact, infinite, finite-dimensional metric space and $\alpha : X \to X$ a minimal homeomorphism. It follows that the nuclear dimension of $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is at most $2\dim(X) + 1$.*

We explain a bit more about this in the remark below–technology has improved dramatically since this article was first written. We note that the C*-algebras of Theorem 0.1 have nuclear dimension at most 2, and that the same is most likely true of the algebras considered in Theorem 0.3. This improved bound, however, relies on the existence of a special inductive limit decomposition for the crossed product, and this, in turn, relies on the classification theorem itself.

0.4 REMARK: The authors would like to point out that this article has changed substantially in one important regard since it was originally posted as a preprint: we have discarded a direct proof of $\mathcal{Z}$-stability for the crossed products we consider, appealing instead to the second author's proof of $\mathcal{Z}$-stability in [28] (a proof which only requires finite nuclear dimension). This is efficient, natural, and better represents the state of the art in the classification theory of nuclear separable C*-algebras.

0.5 Our paper is organised as follows: Section 1 collects some preliminary results, in Section 2 we establish simplicity and finite decomposition rank for some natural subalgebras of $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$, and in Sections 3 and 4 we prove Theorems 0.3, 0.2, and 0.1, respectively. We thank the referee for several useful suggestions.

1. Preliminaries

1.1 PROPOSITION: *Let $\mathcal{G} \subset \mathcal{C}([0,1])$ be a finite subset of positive functions. Then, for any $\eta > 0$ there is $\delta > 0$ such that the following holds:*

If A is a C^*-algebra and $h \in A$ is a positive element of norm at most one, then if $b \in A$ satisfies

$$\|[b,h]\| < \delta,$$

this implies

$$\|[b,f(h)]\| < \eta$$

for all $f \in \mathcal{G}$.

PROOF: It will suffice, by uniform density, to consider the case where $\mathcal{G}$ is a finite set of polynomials. Let k be the largest degree of any $f \in \mathcal{G}$. Observe that if $\|[b,h]\| < \delta$, then for any natural number $i \leq k$ we have

$$bh^i \approx_\delta hbh^{i-1} \approx_\delta h^2bh^{i-2} \approx_\delta \cdots \approx_\delta h^ib,$$

where $\approx_\delta$ denotes the relation of being at norm distance strictly less than δ. It follows that $\|[b,h^i]\| < i\delta \leq k\delta$.

Let $f(x) = a_0 + a_1x + \cdots a_kx^k$ be a generic element of $\mathcal{G}$, and let $\eta > 0$ be given. We set $M_f = \max\{1, |a_0|, |a_1|, \ldots, |a_k|\}$ and

$$M = \max_{f(x) \in \mathcal{G}} M_f.$$

With $\delta = \eta/(k^2M)$ we compute:

$$
\begin{aligned}
\|[b,f(h)]\| &= \left\| \sum_{i=0}^{k} [b, a_ih^i] \right\| \\
&\leq M \left\| \sum_{i=0}^{k} [b, h^i] \right\| \\
&\leq M \sum_{i=0}^{k} \|[b, h^i]\| \\
&\leq Mk(k\delta) \\
&< \eta
\end{aligned}
$$

∎

1.2 PROPOSITION: *Let C be a C^*-algebra, $\mathcal{G} \subset C$ a self-adjoint subset generating C as a C^*-algebra, $\mathcal{E} \subset C$ finite and $\theta > 0$. Then, there is $\beta > 0$ and a finite set $\mathcal{G}' \subseteq \mathcal{G}$ such that the following holds:*

If B is another C^*-algebra and $\gamma : C \to B$ is a $*$-homomorphism, then if $b \in B$ has norm at most one and satisfies

$$\|[b, \gamma(g)]\| < \beta$$

for $g \in \mathcal{G}'$, we have

$$\|[b, \gamma(e)]\| < \theta$$

for $e \in \mathcal{E}$.

PROOF: Let $\theta > 0$ be given. Since $\mathcal{G}$ generates C, there is a natural number k such that each $e \in \mathcal{E}$ is at norm distance strictly less than $\theta/4$ from a sum of at most k monomials of the form

$$\alpha x_1 x_2 \cdots x_l,$$

where $x_j \in \mathcal{G}$, $\alpha \in \mathbb{C}$, and $l \leq k$. We denote this sum by g_e, so that $\|e - g_e\| < \theta/4$, and we let $\mathcal{G}'$ denote the set of all elements of $\mathcal{G}$ which occur in a monomial of g_e as e ranges over $\mathcal{E}$. Now, for any C*-algebra B and *-homomorphism $\gamma : C \to B$ we have

$$
\begin{aligned}
\|[b, \gamma(e)]\| &= \|[b, \gamma(g_e) - \gamma(e - g_e)]\| \\
&\leq \|[b, \gamma(g_e)]\| + \|[b, \gamma(e - g_e)]\| \\
&\leq \|[b, \gamma(g_e)]\| + 2\|b\| \cdot \|\gamma(e - g_e)\| \\
&< \|[b, \gamma(g_e)]\| + \theta/2.
\end{aligned}
$$

Thus, to complete the proof of the Proposition, we must show that there is $\beta > 0$ such that for any γ as above we have the following statement: if

$$\|[b, \gamma(g)]\| < \beta$$

for every $g \in \mathcal{G}'$, then

$$\|[b, \gamma(g_e)]\| < \theta/2$$

for every $e \in \mathcal{E}$. Since $\|[b, g_e/\eta]\| < \theta/(2\eta)$ implies $\|[b, g_e]\| < \theta/2$ for any $\eta > 0$, we may scale the g_es and assume that the elements of $\mathcal{G}'$ have norm at most one.

Let $\gamma : C \to B$ be given, and let

$$\alpha x_1 x_2 \cdots x_l$$

be a monomial appearing in the sum which constitutes some g_e. Set $y_j = \gamma(x_j)$, and note that $\|y_j\| \leq 1$. The assumption $\|[b, y]\| < \beta$ for every $y \in \gamma(\mathcal{G}')$ then yields

$$
\begin{aligned}
\|(y_1 y_2 \cdots y_{j-1} b y_j \cdots y_l) - (y_1 y_2 \cdots y_j b y_{j+1} \cdots y_l)\| &= \|(y_1 \cdots y_j)[b, y_j](y_{j+1} \cdots y_l)\| \\
&\leq \|[b, y_j]\| \\
&< \beta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
y_1 y_2 \cdots y_l b &\approx_\beta y_1 y_2 \cdots y_{l-1} b y_l \\
&\approx_\beta y_1 y_2 \cdots y_{l-2} b y_{l-1} y_l \\
&\cdots \\
&\approx_\beta b y_1 y_2 \cdots y_l,
\end{aligned}
$$

and so

$$\|[b, y_1 y_2 \cdots y_l]\| < k\beta.$$

We have

$$\gamma(g_e) = \sum_{j=1}^{k} \alpha_j y_{j,1} y_{j,2} \cdots y_{j,l_j},$$

where $\alpha_j \in \mathbb{C}$ and $y_{j,t} \in \gamma(\mathcal{G}')$. (Recall that k does not depend on e, and that $l_j \leq k$ regardless of i.) Set $M_e = \max\{1, |\alpha_1|, \ldots, |\alpha_k|\}$, and

$$M = \max_{e \in \mathcal{E}} M_e$$

Set $\beta = \theta/(2k^2 M)$, and observe that β does not depend on γ. Now, we compute:

$$
\begin{aligned}
\|[b, \gamma(g_e)]\| &= \left\| \sum_{j=1}^{k} [b, \alpha_j y_{j,1} y_{j,2} \cdots y_{j,l_j}] \right\| \\
&\leq M \left\| \sum_{j=1}^{k} [b, y_{j,1} y_{j,2} \cdots y_{j,l_j}] \right\| \\
&\leq M \sum_{j=1}^{k} \|[b, y_{j,1} y_{j,2} \cdots y_{j,l_j}]\| \\
&< Mk(k\beta)/2 = \theta/2,
\end{aligned}
$$

as required.

$\blacksquare$

2. LOCAL SIMPLICITY

2.1 The arguments in this and the next section will employ the notion of a recursive subhomogeneous C*-algebra. We refer the reader to [15] for the basic definitions and terminology related to these algebras. We will also use heavily certain natural subalgebras of crossed product C*-algebras, which arise as follows. Let X be a compact metric space, $\alpha : X \to X$ a homeomorphism, Y a closed subset of X, and u the unitary implementing the action of α in $A = \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$; we define

$$
(1) \qquad A_Y = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \backslash Y)).
$$

2.2 PROPOSITION: *Let X be an infinite, compact, metrizable, finite-dimensional space with a minimal homeomorphism α. If x_0, $x_1 \in X$ have disjoint orbits under α, then the subalgebras $A_{\{x_0\}}$, $A_{\{x_0,x_1\}}$ and $A_{\{x_1\}}$ are all simple and have finite decomposition rank. Moreover, any algebra of the form A_Y for closed $Y \subseteq X$ has nuclear dimension at most $\dim(X)$.*

PROOF: Let us first prove the statement about finite decomposition rank. Use Y to denote any of $\{x_0\}$, $\{x_1\}$, or $\{x_0, x_1\}$. Let Y_m be a decreasing sequence of closed subsets of X such that $\mathrm{int}(Y_m) \neq \emptyset$ and $\cap_{m=1}^{\infty} Y_m = Y$. It is immediate that

$$
A_Y = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \backslash Y)) = \lim_{m \to \infty} \mathrm{C}^*(\mathcal{C}_0(X), u\mathcal{C}(X \backslash Y_m)) = \lim_{m \to \infty} A_{Y_m}
$$

(see (1)). It is proved in [12] that A_{Y_m} is a recursive subhomogeneous C*-algebra of topological dimension at most $\dim(X)$. It then follows from the main theorem of [27] that each A_{Y_m} has decomposition rank at most $\dim(X)$. Decomposition rank is lower semicontinuous with respect to inductive limits, whence A_Y has decomposition rank at most $\dim(X)$, too (see [8]).

The simplicity of A_Y is established by [10, Proposition 2.5] in the case that Y is a singleton. We adapt their proof to establish the simplicity of $A_{\{x_0,x_1\}}$. Let $I \subseteq A_{\{x_0,x_1\}}$ be a nonzero ideal. It follows that $\mathcal{C}(X) \cap I$ is an ideal of $\mathcal{C}(X)$, and therefore has the form $\mathcal{C}_0(U)$ for some open subset U. Explicitly,

$$
U = \{x \in X \mid \exists f \in \mathcal{C}(X) \cap I : f(x) \neq 0\}.
$$

The proof of [10, Proposition 2.5] shows that $U \neq \emptyset$.

We claim that $\alpha^{-1}(U\backslash\{\alpha(x_0), \alpha(x_1)\}) \subseteq U$. To see this, fix $z \in U\backslash\{\alpha(x_0), \alpha(x_1)\}$. Choose $f \in \mathcal{C}(X) \cap I$ such that $f(z) \neq 0$, and choose $g \in \mathcal{C}_0(X\backslash\{x_0, x_1\})$ such that $g(\alpha^{-1}(z)) \neq 0$. Now $ug \in A_{\{x_0, x_1\}}$, and so

$$(ug)^* f(ug) = \bar{g}u^* fug = |g|^2 (f \circ \alpha)$$

is in $\mathcal{C}(X) \cap I$. Since $|g|^2(f \circ \alpha)$ is not zero at $\alpha^{-1}(z)$, we see that $\alpha^{-1}(z) \in U$.

We also claim that $\alpha(U\backslash\{x_0, x_1\}) \subseteq U$. Let $z \in U\backslash\{x_0, x_1\}$, and choose $f \in \mathcal{C}(X) \cap I$ and $g \in \mathcal{C}_0(X\backslash\{x_0, x_1\})$ to be nonzero at z. Now $ug \in A_{\{x_0, x_1\}}$, and so

$$(ug)f(ug)^* = u(|g|^2 f)u^* = (|g|^2 f) \circ \alpha^{-1}$$

is in $\mathcal{C}(X) \cap I$. Since $(|g|^2 f) \circ \alpha^{-1}$ is not zero at $\alpha(z)$, we see that $\alpha(z) \in U$.

Set $Z = X\backslash U$. We will prove that $Z = \emptyset$ by contradiction.

First suppose that $z \in Z$ and $z \notin \mathrm{Orb}(x_0) \cup \mathrm{Orb}(x_1)$. Suppose that $\alpha^k(z) \in U$ for some k. If $k > 0$ then, since $\alpha^k(z) \neq \alpha(x_0), \alpha(x_1)$, we use the first of the two claims above to conclude that $\alpha^{k-1}(z) \in U\backslash\{\alpha(x_0), \alpha(x_1)\}$. Iterating this procedure k times yields $z \in U$, contradicting $z \in Z$. We conclude that $\alpha^k(z) \in Z$. Similarly, suppose that $\alpha^k(z) \in U$ for some $k < 0$. Since $\alpha^k(z) \neq x_0, x_1$, we can use the second of the two claims above to conclude that $\alpha^{k+1}(z) \in U\backslash\{x_0, x_1\}$. Iterating this procedure k times yields $z \in U$, contradicting $z \in Z$. We conclude again that $\alpha^k(z) \in Z$. We have proved that $\mathrm{Orb}(z) \subseteq Z$, but this contradicts the minimality of α. We conclude that if $z \in Z$, then $z \in \mathrm{Orb}(x_0) \cup \mathrm{Orb}(x_1)$.

Suppose, without loss of generality, that $z = \alpha^k(x_0)$. Assume that $k > 0$, and that $\alpha^n(x_0) \in U$ for some $n > k$. Using $\mathrm{Orb}(x_0) \cap \mathrm{Orb}(x_1) = \emptyset$ and $n > 1$ we see that in fact $\alpha^n(x_0) \in U\backslash\{\alpha(x_0), \alpha(x_1)\}$. Applying the first of our two claims above, we see that $\alpha^{n-1}(x_0) \in U\backslash\{\alpha(x_0), \alpha_(x_1)\}$. Iterating this argument yields $\alpha^k(x_0) \in U$, a contradiction, so we must have $\{\alpha^n(x_0) \mid n \geq k\} \subseteq Z$. This, however, violates minimality. Similarly, suppose that $k \leq 0$, and that $\alpha^n(x_0) \in U$ for some $n < k$. Using $\mathrm{Orb}(x_0) \cap \mathrm{Orb}(x_1) = \emptyset$ and $n < 0$ we see that in fact $\alpha^n(x_0) \in U\backslash\{x_0, x_1\}$. Applying the second of our two claims above, we see that $\alpha^{n+1}(x_0) \in U\backslash\{x_0, x_1\}$. Iterating this argument yields $\alpha^k(x_0) \in U$, a contradiction, so we conclude that $\{\alpha^n(x_0) \mid n \leq k\} \subseteq Z$. This again violates minimality.

Since every possible choice of $z \in Z$ leads to a contradiction, we conclude that $Z = \emptyset$, $U = X$, and so $1_A \in I$. It follows that $A_{\{x_0, x_1\}}$ is simple, as desired.

∎

3. NUCLEAR DIMENSION

3.1 Recall the following definition from [29].

DEFINITION: *A C^*-algebra A has nuclear dimension at most n, $\dim_{\mathrm{nuc}} A \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)$ of finite-dimensional c.p. approximations for A (i.e., F_λ are finite dimensional C^*-algebras, and $\psi_\lambda : A \to F_\lambda$ and $\varphi_\lambda : F_\lambda \to A$ are completely positive maps for all λ) such that*

   (i) *$\varphi_\lambda \circ \psi_\lambda(a) \to a$ uniformly on finite subsets of A*
   (ii) *$\|\psi_\lambda\| \leq 1$ for each λ*

(iii) *each F_λ decomposes into $n + 1$ ideals, $F_\lambda = F_\lambda^{(0)} \oplus \ldots \oplus F_\lambda^{(n)}$, such that $\varphi_\lambda|_{F_\lambda^{(i)}}$ is an order zero contraction (i.e., it is c.p.c. and preserves orthogonality) for $i = 0, \ldots, n$.*

3.2 PROPOSITION: *Let X be an infinite, compact, metrizable, space with a minimal homeomorphism α. It follows that for any $\delta > 0$ and $\mathcal{F} \subset \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ finite, there are a positive normalized element $h \in \mathcal{C}(X)$ and distinct points $x_0, x_1 \in X$ such that*

$$h(x_0) = 0, \, h(x_1) = 1$$

and

$$\|[h, b]\| < \delta$$

for all $b \in \mathcal{F}$. We may moreover arrange that $\mathrm{Orb}(\mathrm{x}_0) \cap \mathrm{Orb}(x_1) = \emptyset$.

PROOF: Let u be the unitary operator in $A := \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ implementing α. Applying Proposition 1.2 with $\gamma = \mathrm{id}_A$, $\mathcal{G} = \{\mathcal{C}(X), u, u^*\}$, $\mathcal{E} = \mathcal{F}$, and $\theta = \delta$, we see that there is $0 < \eta < \delta$ such that if $h \in \mathcal{C}(X)$ is positive with the property that

$$\|[h, u^*]\| = \|[h, u]\| < \eta,$$

then $\|[h, b]\| < \delta$ for every $b \in \mathcal{F}$. Thus, to establish the commutator estimate of this Proposition, we need only show that h can be chosen to commute with u to within η.

Set $n = \lceil 1/\eta \rceil + 1$, and fix some $x_0 \in X$. By minimality and the fact that X is infinite, we can find an open neighbourhood U of x_0 such that the sets $\alpha^i(U)$ are pairwise disjoint for $i \in \{0, 1, \ldots, 2n\}$. Let $f \in \mathcal{C}_0(U)$ be a positive function of norm one with the property that $f \equiv 1$ on an open neighbourhood V of x_0 such that $\overline{V} \subseteq U$. Now define $h \in \mathcal{C}_0(U \cup \alpha(U) \cup \cdots \cup \alpha^{2n}(U))$ via the formula

$$h|_{\alpha^i(U)} = \begin{cases} (i/n)(f \circ \alpha^{-i}), & i \leq n \\ (2 - i/n)(f \circ \alpha^{-i}), & n < i \leq 2n \end{cases}.$$

Our assumptions on X and α imply that there is more than one orbit under α. By minimality we can find $x_1 \in \alpha^n(V)$ such that $\mathrm{Orb}(\mathrm{x}_0) \cap \mathrm{Orb}(\mathrm{x}_1) = \emptyset$, so that $h(x_1) = 1$ and $h(x_0) = 0$. Clearly, h is positive and of norm one.

It remains to prove that $\|uh - hu\| < \eta$. This is equivalent to showing that

$$\|uhu^* - h\| < \eta.$$

Now

$$uhu^* = h \circ \alpha^{-1} \in \mathcal{C}_0(U \cup \alpha(U) \cup \cdots \cup \alpha^{2n}(U))$$

is given by the formula

$$uhu^*|_{\alpha^i(U)} = \begin{cases} 0, & i = 2n \\ (2 - (i+1)/n)(f \circ \alpha^{-i}), & n - 1 < i \leq 2n - 1 \\ ((i+1)/n)(f \circ \alpha^{-i}), & 0 \leq i \leq n - 1 \end{cases},$$

and comparing this with the formula for h we see that

$$\|uhu^* - h\| = (1/n)\|f\| = 1/n < \eta,$$

as required. ∎

3.3 THEOREM: *Let X be an infinite, compact, metrizable, finite-dimensional space with a minimal homeomorphism α. Then,*

$$\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}) \leq 2 \dim X + 1.$$

PROOF: Set $A = \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$, and let there be given a finite subset $\mathcal{F} = \{a_1, \ldots, a_n\}$ of A and a tolerance $\epsilon > 0$. We must find a c.p. approximation for $\mathcal{F}$ to within ϵ which has the form described in Definition 3.1.

Since $\mathcal{C}(X)$ and the implementing unitary u together generate A, there is a finite set $\mathcal{F}' = \{a_1', \ldots, a_n'\}$ with the following properties:

(i) $\|a_i - a_i'\| < \epsilon/24$, for each $i \in \{1, \ldots, n\}$;
(ii) there exists $k \in \mathbb{N}$ such that each a_i' is the sum of at most k monomials of the form

$$f_1 u^{j_1} f_2 u^{j_2} \cdots f_k u^{j_k}, \ \ j_l \in \{-1, 0, 1\}, \ f_l \in \mathcal{C}(X), \ 1 \leq l \leq k.$$

We assume, without loss of generality, that the f_l are contractions.

Apply Proposition 3.2 to the singleton $\{u\}$ and a value of δ so small that the resulting function h has the property that

$$\|[u^*, h^{i/k}]\| = \|[u, h^{i/k}]\| < \epsilon/(24k^2)$$

and

$$\|[u^*, (1-h)^{i/k}]\| = \|[u, (1-h)^{i/k}]\| < \epsilon/(24k^2)$$

for each $i \in \{1, \ldots, k\}$ (this is possible by Proposition 1.1). Now for any monomial as in (ii) above, we have

$$(2) \qquad \|(f_1 u^{j_1} f_2 u^{j_2} \cdots f_k u^{j_k})h - (f_1 u^{j_1} h^{1/k} f_2 h^{1/k} u^{j_2} \cdots f_k u^{j_k} h^{1/k})\| < \epsilon/24k,$$

where we have inserted $h^{1/k}$ to the left of u^{j_l} if $j_l = -1$, and to the right if $j_l = 0, 1$. To see why this is so, observe that we may pass from

$$(3) \qquad\qquad\qquad (f_1 u^{j_1} f_2 u^{j_2} \cdots f_k u^{j_k})h$$

to

$$(4) \qquad\qquad\qquad (f_1 u^{j_1} h^{1/k} f_2 h^{1/k} u^{j_2} \cdots f_k u^{j_k} h^{1/k})$$

in at most k steps, each of which involves passing a function of the form $h^{l/k}$ from one side of a u or u^* to the other. Keep in mind that $h^{l/k}$ commutes with each f_t for all sensible l and t. Each such step has a cost in norm which is less than or equal to

$$\|[u^*, h^{i/k}]\| = \|[u, h^{i/k}]\| < \epsilon/(24k^2).$$

Each of f_l, $h^{1/k} u^{-1}$, and $u h^{1/k}$ is contained in

$$A_{\{x_0\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \backslash \{x_0\})).$$

The monomial of (4) is a product of such elements, hence also in $A_{\{x_0\}}$. Since each $a_i' h$ is a sum of at most k monomials as in (3) above, and since each such monomial is at distance strictly less that $\epsilon/(24k)$ from a monomial as in (4), we conclude that $a_i' h$ is at distance at most $\epsilon/24$ from an element $b_i^{(0)}$ of $A_{\{x_0\}}$. Set $\mathcal{G}_0 = \{b_1^{(0)}, \ldots, b_n^{(0)}\}$. An argument similar to the one just presented shows that $a_i'(1-h)$ is at distance at most $\epsilon/24$ from some

$$b_i^{(1)} \in A_{\{x_1\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \backslash \{x_1\})).$$

Set $\mathcal{G}_1 = \{b_1^{(1)}, \ldots, b_n^{(1)}\}$. Using the fact that $\|a_i - a_i'\| < \epsilon/24$, and also that we may assume that h commutes with each a_i to within an arbitrarily small tolerance, we conclude that

$$\|h^{1/2}a_i h^{1/2} - b_i^{(0)}\|, \ \|(1-h)^{1/2}a_i(1-h)^{1/2} - b_i^{(1)}\| < \epsilon/12.$$

From Proposition 2.2, we know that both $A_{\{x_0\}}$ and $A_{\{x_1\}}$ have decomposition rank $d = \dim(X) < \infty$. We can therefore find finite-dimensional C*-algebras $F^{(0)}$ and $F^{(1)}$ and c.p. contractions $\phi^{(j)} : A_{\{x_j\}} \to F^{(j)}$ and $\psi^{(j)} : F^{(j)} \to A_{\{x_j\}}$ with the property that

$$\|\psi^{(j)} \circ \phi^{(j)}(b_i^{(j)}) - b_i^{(j)}\| < \epsilon/12, \ j \in \{0,1\}, \ i \in \{1, \ldots, n\}.$$

We may moreover take the $\psi^{(j)}$ to be d-decomposable in the sense of [8]. By Arveson's Extension Theorem, the $\phi^{(j)}$ may be extended to c.p. contractions

$$\overline{\phi^{(j)}} : A \to F^{(j)}, \ j = 0, 1,$$

and we can thus define a c.p. contraction $\overline{\phi} : A \to F^{(0)} \oplus F^{(1)}$ by the formula

$$\overline{\phi}(x) = \overline{\phi^{(0)}}\left(h^{1/2}xh^{1/2}\right) \oplus \overline{\phi^{(1)}}\left((1-h)^{1/2}x(1-h)^{1/2}\right).$$

We also define a c.p. map $\overline{\psi} : F^{(0)} \oplus F^{(1)} \to A$ by the formula

$$\overline{\psi}(x) = \psi^{(0)}(x) + \psi^{(1)}(x),$$

assuming that $\psi^{(0)}|_{F^{(1)}} \equiv 0$ and $\psi^{(1)}|_{F^{(0)}} \equiv 0$. By the d-decomposability of the $\psi^{(j)}$, we can write

$$F^{(j)} = F_0^{(j)} \oplus F_1^{(j)} \oplus \cdots \oplus F_d^{(j)}$$

so that $\psi^{(j)}|_{F_l^{(j)}}$ is an order zero contraction for each $l \in \{0, 1, \ldots, d\}$ and $j \in \{0, 1\}$. It follows immediately that the restriction of $\overline{\psi}$ to any of the direct summands of

$$F_0^{(0)} \oplus F_1^{(0)} \oplus \cdots \oplus F_d^{(0)} \oplus F_0^{(1)} \oplus F_1^{(1)} \oplus \cdots \oplus F_d^{(1)} (= F^{(0)} \oplus F^{(1)})$$

is an order zero contraction, so that $\overline{\psi}$ is $(2d+1)$-decomposable.

Now we estimate:

$$
\begin{aligned}
&\|\overline{\psi} \circ \overline{\phi}(a_i) - a_i\| \\
=\ &\|\psi^{(0)} \circ \overline{\phi^{(0)}}(h^{1/2}a_i h^{1/2}) + \psi^{(1)} \circ \overline{\phi^{(1)}}((1-h)^{1/2}a_i(1-h)^{1/2}) - a_i\| \\
<\ &\|\psi^{(0)} \circ \phi^{(0)}(b_i^{(0)}) + \psi^{(1)} \circ \phi^{(1)}(b_i^{(1)}) - a_i\| + \epsilon/6 \\
\leq\ &\|b_i^{(0)} + b_i^{(1)} - a_i\| + \epsilon/3 \\
<\ &\|a_i'h + a_i'(1-h) - a_i\| + \epsilon/2 \\
=\ &\|a_i' - a_i\| + \epsilon/2 \\
<\ &\epsilon
\end{aligned}
$$

Since $\|\overline{\phi}\| \leq 1$ and $\overline{\psi}$ is $(2d+1)$-decomposable, we conclude that the nuclear dimension of A is at most $2d+1$, as desired.

∎

3.4

COROLLARY: *Let X be an infinite, compact, metrizable, finite-dimensional space with a minimal homeomorphism α. It follows that $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable.*

Proof. Finite nuclear dimension implies $\mathcal{Z}$-stability by Corollary 6.3 of [28].   □

4. Classification by K-theory: Elliott's conjecture

4.1 Our proof of Theorem 0.1 will be an application of a general classification result, due to the second named author (Theorem 7.1 of [26]). To avoid the task of deriving a succinct and useful form of this theorem for our purpose (although such derivation is not difficult), we state here a generalisation of [26, Theorem 7.1] due to Lin and Niu ([9]). First we fix some notation: if l is a prime, then set

$$\mathfrak{U}_l := \bigotimes_{i=1}^{\infty} \mathrm{M}_l.$$

THEOREM: [26, 9] *Let A, B be unital simple separable nuclear C*-algebras which absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially. Suppose that for any prime l, the tensor products $\mathfrak{U}_l \otimes A$ and $\mathfrak{U}_l \otimes B$ have tracial rank zero and satisfy the Universal Coefficient Theorem. Also suppose that there is a graded ordered isomorphism $\phi : \mathrm{K}_*(A) \to \mathrm{K}_*(B)$. It follows that there is a $*$-isomorphism $\Phi : A \to B$.*

4.2 The crossed products considered in Theorem 0.1 are simple unital separable nuclear C*-algebras satisfying the Universal Coefficient Theorem, and were shown in Corollary 3.4 to be $\mathcal{Z}$-stable. Thus, we see that the proof of Theorem 0.1 is reduced to the problem of showing that for any prime l and C*-algebra A as in the said theorem, the tensor product $\mathfrak{U}_l \otimes A$ has tracial rank zero.

This task will be accomplished in two steps: First, we will employ known results to show that, under the hypotheses of Theorem 0.1 and with notation as in 2.1, algebras of the form $\mathfrak{U}_l \otimes A_{\{y\}}$ are TAF. This is the content of Proposition 4.3 below.

Second, we need to conclude that if $\mathfrak{U}_l \otimes A_{\{y\}}$ is TAF, then so is $\mathfrak{U}_l \otimes (\mathcal{C}(X) \rtimes_\alpha \mathbb{Z})$. This will be a special case of a result by Strung and the second named author, Theorem 4.4; we will outline the proof at least in the special case we need, cf. Proposition 4.7 below.

4.3 So, let us turn to the first step of 4.2.

PROPOSITION: *Let X be a compact infinite metric space of finite covering dimension, and $\alpha : X \to X$ a minimal homeomorphism. Suppose further that the projections of $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ separate traces. It follows that for any prime l and any $y \in X$, the C*-algebra $B_y := \mathfrak{U}_l \otimes A_{\{y\}}$ has tracial rank zero.*

PROOF: By Proposition 2.2, $A_{\{y\}}$ is a simple, $\mathcal{Z}$-stable C*-algebra with decomposition rank at most $d := \dim(X)$. The properties of the decomposition rank, established in [8], include the following: decomposition rank is insensitive to taking tensor products with full matrix algebras over $\mathbb{C}$; decomposition rank is lower semicontinuous with respect to inductive limits. Since $\mathfrak{U}_l$ is an inductive limit of full matrix algebras over $\mathbb{C}$, we conclude that B_y has decomposition rank at most d. The inclusion $A_{\{y\}} \hookrightarrow A$ induces isomorphisms $\gamma_1 : \mathrm{K}_0(A_{\{y\}}) \to \mathrm{K}_0(A)$ and $\gamma_2 : \mathrm{T}(A) \to \mathrm{T}(A_{\{y\}})$ (see Theorem 4.1 (3) of [14] and [11, Proposition 16], respectively—a sketched proof of the latter result can be found in [12]). It follows that projections separate traces in $A_{\{y\}}$. Using the fact that B_y is unital, simple, and absorbs $\mathfrak{U}_l$ tensorially, we may appeal to [17] to conclude that B_y has real rank

zero. We have now collected the hypotheses of [24, Theorem 4.1]. We conclude that B_y has tracial rank zero, as desired.   ▌

This settles the first step of 4.2 above. Let us now take care of the second step.

4.4 THEOREM: [21] *Let X be an infinite compact metric space, and $\alpha : X \to X$ a minimal homeomorphism. Let $y \in X$ and set $A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{y\}))$. Let l be a prime, and suppose that $\mathfrak{U}_l \otimes A_{\{y\}}$ is $TA\mathcal{S}$, where $\mathcal{S}$ is some class of unital semiprojective C^*-algebras which is closed under taking quotients. Then, $\mathfrak{U}_l \otimes (\mathcal{C}(X) \rtimes_\alpha \mathbb{Z})$ is $TA\mathcal{S}$ as well.*

When we take $\mathcal{S}$ to be the class of finite dimensional C^*-algebras, the theorem indeed settles the second step of 4.2 above. Since we will only need this special case, for the convenience of the reader we briefly outline its proof in the sequel (cf. Proposition 4.7 below). We refer to [21] for full details, and for a discussion of the more general Theorem 4.4 and its implications for the classification program.

4.5 The key step in the proof of Theorem 4.4 is the following generalisation of [10, Lemma 4.2], which will be derived in [21]; here we only need it in the case of real rank zero.

LEMMA: *Let X be an infinite compact metric space, and $\alpha : X \to X$ a minimal homeomorphism. Let $y \in X$ and set $A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X\setminus\{y\}))$. Let l be a prime, and set $B_y = \mathfrak{U}_l \otimes A_{\{y\}}$. It follows that for any $\epsilon > 0$ and any finite subset $\mathcal{F}$ of $B := \mathfrak{U}_l \otimes (\mathcal{C}(X) \rtimes_\alpha \mathbb{Z})$, there is a projection $p \in B_y$ such that the following statements hold:*

   (i) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$;
   (ii) $\mathrm{dist}(pap, pB_yp) < \epsilon$ *for all* $a \in \mathcal{F}$;
   (iii) $\tau(1 - p) < \epsilon$ *for all* $\tau \in \mathrm{T}(B)$.

The proof is a modification of that of [10, Lemma 4.2]. In fact, when we replace $\mathfrak{U}_l$ by $\mathbb{C}$ in the above (i.e., we take $l = 1$), and assume in addition that $A_{\{y\}}$ has real rank zero and stable rank one, the lemma is exactly [10, Lemma 4.2]. Now if $\mathfrak{U}_l$ is nontrivial, and if B_y has real rank zero, then the proof of [10, Lemma 4.2] carries over almost verbatim, at least in the case where $\mathcal{F}$ is a subset of $\mathbf{1}_{\mathfrak{U}_l} \otimes A_{\{y\}}$. The case of general $\mathcal{F}$ can be reduced to the situation where $\mathcal{F}$ is in $\mathbf{1}_{\mathfrak{U}_l} \otimes \mathfrak{U}_l \otimes \mathbf{1}_{A_{\{y\}}} \cup \mathbf{1}_{\mathfrak{U}_l} \otimes \mathbf{1}_{\mathfrak{U}_l} \otimes A_{\{y\}}$ (using the fact that $\mathfrak{U}_l \cong \mathfrak{U}_l \otimes \mathfrak{U}_l$); the proof in this situation is essentially the same as in the first case.

4.6 Next, we recall another result from [10].

LEMMA: [10, Lemma 4.4] *Let C be a simple unital C^*-algebra. Suppose that for every finite subset $\mathcal{F} \subseteq C$, every $\epsilon > 0$, and every nonzero positive element $b \in C$, there exists a projection $p \in C$ and a simple unital subalgebra $D \subseteq pCp$ with tracial rank zero such that:*

   (i) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$;
   (ii) $\mathrm{dist}(pap, D) < \epsilon$ *for all* $a \in \mathcal{F}$;
   (iii) p *is Murray-von Neumann equivalent to a projection in* $\overline{bCb}$.

It follows that A has tracial rank zero.

4.7 Using essentially the same technique as in [10, Theorem 4.5], we may now use the two preceding lemmas to prove Theorem 4.4, at least in the case where $\mathcal{S}$ is the class of finite-dimensional C*-algebras.

PROPOSITION: *Let X be a compact infinite metric space of finite covering dimension, and $\alpha : X \to X$ a minimal homeomorphism. Suppose further that, for some prime l and $y \in X$, the C*-algebra $B_y := \mathfrak{U}_l \otimes A_{\{y\}}$ has tracial rank zero. It follows that the C*-algebra $B := \mathfrak{U}_l \otimes (\mathcal{C}(X) \rtimes_\alpha \mathbb{Z})$ has tracial rank zero.*

PROOF: Since B_y has tracial rank zero, so too does pB_yp for any projection $p \in B_y$. Using this fact and Lemma 4.5 we may apply [10, Lemma 4.4] (stated as Lemma 4.6 above) with C replaced by $B := \mathfrak{U}_l \otimes (\mathrm{C}(X) \rtimes_\alpha \mathbb{Z})$ and D replaced by our pB_yp, where $p \in B_y$ is the projection provided by the conclusion of Lemma 4.5. To conclude that B has tracial rank zero, we need to show that the projection p in the conclusion of Lemma 4.5 is Murray-von Neumann equivalent to a projection in $\overline{bBb}$ (we assume that ϵ, $\mathcal{F}$, and b as in the hypotheses of Lemma 4.6 are given). By shrinking ϵ, however, the existence of this projection follows from the second part of [13, Proposition 3.8]—this proposition applies since B is unital, simple, and $\mathcal{Z}$-stable. We conclude, finally, that B has tracial rank zero, as desired. ∎

We have thus completed the two steps of 4.2, hence the proof of Theorem 0.1.

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