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Recasting the Elliott conjecture

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Abstract Let A be a simple, unital, finite, and exact C*-algebra which absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially. We prove that the Cuntz semigroup of A admits a complete order embedding into an ordered semigroup which is obtained from the Elliott invariant in a functorial manner. We conjecture that this embedding is an isomorphism, and prove the conjecture in several cases. In these same cases—$\mathcal{Z}$-stable algebras all—we prove that the Elliott conjecture in its strongest form is equivalent to a conjecture which appears much weaker. Outside the class of $\mathcal{Z}$-stable C*-algebras, this weaker conjecture has no known counterexamples, and it is plausible that none exist. Thus, we reconcile the still intact principle of Elliott's classification conjecture—that K-theoretic invariants will classify separable and nuclear C*-algebras—with the recent appearance of counterexamples to its strongest concrete form.

1 Introduction

The Elliott conjecture for C*-algebras operates on two levels: on the one hand, it is a meta-conjecture asserting that separable and nuclear C*-algebras will be

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classified up to $*$-isomorphism by K-theoretic invariants; on the other, it is a collection of concrete classification conjectures, where the K-theoretic invariants in question are specified and depend on the class of algebras being considered. In the case of stable Kirchberg algebras (simple, nuclear, purely infinite, and satisfying the Universal Coefficients Theorem), the correct invariant is the graded Abelian group $K_0 \oplus K_1$ [20,28]. For non-simple algebras of real rank zero, K-theory with coefficients seems to suffice [6,7]. For a unital, separable, and nuclear C*-algebra A, the invariant

$$I(A) := \big((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A\big)$$

—topological K-theory, the (possibly empty) Choquet simplex $T(A)$ of tracial states, and the pairing $r_A : T(A) \times K_0(A) \to \mathbb{R}$ given by evaluating a trace at a K_0-class—is known as the Elliott invariant, and has been very successful in confirming Elliott's conjecture for simple algebras.

In its most general form, the Elliott conjecture may be stated as follows:

1.1 (Elliott, c. 1989). There is a K-theoretic functor F from the category of separable and nuclear C*-algebras such that if A and B are separable and nuclear, and there is an isomorphism

$$\phi : F(A) \to F(B),$$

then there is a $*$-isomorphism

$$\Phi : A \to B$$

such that $F(\Phi) = \phi$.

We will let (EC) denote the conjecture above with the Elliott invariant $I(\bullet)$ substituted for $F(\bullet)$, and with the class of algebras under consideration restricted to those which are simple and unital. (EC) has been shown to hold in many situations. An exhaustive list of these results would be impossibly long, but [8–10,12,20,22] are among the most important works. We refer the reader to Rørdam's book [30] for a comprehensive overview of Elliott's classification programme.

Recent examples due first to Rørdam and later the second named author have shown the currently proposed invariants (i.e., the proposed values of F in Conjecture 1.1) to be insufficient for the classification of all simple, separable, and nuclear C*-algebras [31,33,34]. In particular, (EC) does not hold. There are two options: enlarge the proposed invariants, or restrict the class of algebras considered.

The Cuntz semigroup of a C*-algebra A is a positively ordered Abelian semigroup whose elements are equivalence classes of positive elements in matrix algebras over A (see Sect. 2 for details). Let $W(A)$ denote this semigroup, and let $\langle a\rangle$ denote the equivalence class of a positive element $a \in M_n(A)$. The semigroup $W(A)$ may be thought of as a generalisation of the semigroup $V(A)$ of

Murray–von Neumann equivalence classes of projections in matrices over A, provided that A is stably finite. Theorem 1 of [34] states that there exist simple, separable, nuclear, and non-isomorphic C*-algebras which agree on each continuous and homotopy invariant functor from the category of C*-algebras, and which furthermore have the same simplex of tracial states. These algebras are distinguished by their Cuntz semigroups, whence this invariant is extremely sensitive. (Indeed, it is already unmanageably large for commutative C*-algebras with contractible spectrum—see [34, Lemma 5.1].) It thus suggests itself as the minimum quantity by which the Elliott invariant $I(\bullet)$ ought to be enlarged. The sequel will be concerned in large part with the relationship between (EC) and the following statement:

1.2 (WEC). Let A and B be simple, separable, unital, and nuclear C*-algebras. If there is an isomorphism

$$\phi\colon\ (W(A), \langle 1_A\rangle, I(A)) \to (W(B), \langle 1_B\rangle, I(B))\,,$$

then there is a $*$-isomorphism $\Phi\colon A \to B$ which induces ϕ.

There are no known counterexamples to the conjecture (WEC) among stably finite algebras, and perhaps none exist. But asking for the Cuntz semigroup as part of the invariant seems strong indeed, given its sensitivity and the fact that (EC) alone is so often true. The theme of the sequel is that (WEC) and (EC) are reconciled upon restriction to the largest class of C*-algebras for which (EC) may be expected to hold. (WEC) may thus be viewed as the appropriate specification of the Elliott conjecture for simple, separable, unital, nuclear, and stably finite C*-algebras. (We have, for the time being, glossed over what exactly is meant by isomorphism at the level of invariants in both (EC) and (WEC), so as not to burden this introduction with technicalities. The appropriate notions of isomorphism will be introduced in Sect. 4.)

It is generally agreed that the largest restricted class of algebras for which (EC) can hold consists of those algebras which absorb the Jiang–Su algebra $\mathcal{Z}$ tensorially [19]. Indeed, this fact is obvious if one considers only algebras with weakly unperforated ordered K_0-groups (a condition which holds in every confirmation of (EC))—by Theorem 1 of [13], the tensor product of such an algebra, say A, with $\mathcal{Z}$ has the same Elliott invariant as A, and so (EC) predicts that $A \cong A \otimes \mathcal{Z}$. If A is any C*-algebra and the minimal tensor product $A \otimes \mathcal{Z}$ is isomorphic to A, then we say that A is *$\mathcal{Z}$-stable*. Our first main result is:

Theorem 1.3 *Upon restriction to $\mathcal{Z}$-stable C*-algebras, (EC) implies (WEC).*

Notice that this theorem does not follow from the mere fact that the invariant considered in (WEC) is finer that the Elliott invariant. This is due to the functorial nature of Elliott-type conjectures: an isomorphism at the level of the invariant must lift to an isomorphism at the level of C*-algebras which, moreover, induces the original isomorphism of invariants.

More surprising, perhaps, is this:

Theorem 1.4 *Let $\mathcal{C}$ denote the class of all simple, unital, separable, nuclear, and $\mathcal{Z}$-stable C*-algebras A which are either*

(i) *of real rank zero, or*
(ii) *have finitely many pure tracial states.*

Then, (EC) and (WEC) are equivalent in $\mathcal{C}$. Moreover, there is a functor G from the category of Elliott invariants to the category of Elliott invariants augmented by the Cuntz semigroup such that

$$G(I(A)) = (W(A), \langle 1_A \rangle, I(A)).$$

In proving Theorem 1.4 we shall see that an algebra $A \in \mathcal{C}$ has, up to Cuntz equivalence, relatively few positive elements. This contrasts sharply with the counterexample to (EC) in [34]. Significant is the fact that A need not be of real rank zero; it may be projectionless but for zero and the unit. Most progress on (EC) from a general point of view has so far required the real rank zero assumption. We also outline a proof that Theorem 1.4 holds for Goodearl algebras, so that conditions (i) and (ii) of the theorem are, in principle, removeable. (Indeed, we conjecture as much.) The proof of Theorem 1.4 gives the first calculations of Cuntz semigroups for C*-algebras without the real rank zero property, and even in the real rank zero case generalises considerably the earlier results of Blackadar and Handelman [3].

The paper is organised as follows: in Sect. 2 we recall the definition of the Cuntz semigroup, and establish several results about its order structure; in Sect. 3 we compute $W(\mathcal{Z})$, and examine the basic structure of $W(A \otimes \mathcal{Z})$; Sect. 4 contains an embedding theorem which establishes Theorem 1.3; Sect. 5 contains a calculation of the Grothendieck enveloping group of the Cuntz semigroup for finite $\mathcal{Z}$-stable algebras; Sects. 6 and 7 are devoted to proving Theorem 1.4 in cases (i) and (ii), respectively; in Sect. 8 we sketch a proof of Theorem 1.4 for Goodearl algebras; Sect. 9 raises some questions for future research.

2 The Cuntz semigroup and comparison

Cuntz introduced in [5] a notion of comparison between positive elements in a C*-algebra that extends the usual (Murray–von Neumann) comparison for projections. This allowed him to prove the existence of dimension functions in stably finite simple C*-algebras. (The assumption of simplicity was subsequently removed by Handelman [18].)

Explicitly, if a and b are positive elements in a C*-algebra A, then we write $a \precsim b$ provided there is a sequence of elements (x_n) in A such that $a = \lim_{n\to\infty} x_n b x_n^*$. This relation can be extended to the (local) C*-algebra $M_\infty(A)$ defined as the inductive limit of $M_n(A)$ via the inclusion mappings $M_n(A) \hookrightarrow M_{n+1}(A)$ given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. Let $\mathrm{M}_\infty(A)_+$ denote the set of positive elements in $M_\infty(A)$. For elements a, b in $M_\infty(A)_+$, we write $a \precsim b$ provided that $a \precsim b$ in $M_n(A)$ for some n such that $a, b \in M_n(A)$. (If we view a and b in two different sized matrices over A, the above is equivalent to having $a = \lim_{n\to\infty} x_n b x_n^*$ where the x_n are suitable rectangular matrices.) If both $a \precsim b$ and $b \precsim a$, we will write $a \sim b$ and call a and b *Cuntz equivalent*. We shall denote the equivalence class of an element a in $M_\infty(A)_+$ by $\langle a\rangle$, and we will in this paper denote the set of all such equivalence classes by $W(A)$ (although this notation is not uniform in the literature). For $a, b \in M_\infty(A)_+$ we write $a \oplus b$ for the element $\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right) \in M_\infty(A)_+$. If $\langle a\rangle, \langle b\rangle \in W(A)$, we define $\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$. It is easy to verify that this operation is does not depend on the representatives chosen and that $W(A)$ becomes an Abelian semigroup with identity element $\langle 0\rangle$ (and thus an Abelian monoid). We shall refer to $W(A)$ as the *Cuntz semigroup of A*. All semigroups in this paper will be Abelian and assumed to have an identity element, which we shall denote by 0.

Recall that projections $p, q \in M_\infty(A)$ are *Murray–von Neumann equivalent* ($p \sim q$) if there is an element x in $M_\infty(A)$ such that $p = xx^*$ and $q = x^*x$; p is *subequivalent* to q (in symbols $p \precsim q$) if there is a projection $q' \in M_\infty(A)$ such that $p \sim q'$ and $q' \leq q$. The notions of Murray–von Neumann equivalence and Cuntz equivalence coincide for the set of projections in matrices over a stably finite C*-algebra, but do not coincide in general. Let $[p]$ denote the Murray–von Neumann equivalence class of p. The set of all such equivalence classes is denoted $V(A)$, and is also an Abelian semigroup (with identity element $[0]$) under the operation $[p]+[q] = [p \oplus q]$. There is a natural semigroup morphism $\varphi\colon V(A) \to W(A)$, given by $[p] \mapsto \langle p\rangle$, which is injective if A is stably finite. In this case, we identify $V(A)$ with its image under φ.

Definition 2.1 Let A be a C*-algebra, and let $W(A)_+$ denote the subset of $W(A)$ consisting of classes which are not the classes of projections. If $a \in A_+$ and $\langle a\rangle \in W(A)_+$, then we will say that a is purely positive and denote the set of such elements by A_{++}.

One of the advantages of the relation $\precsim$ is that it allows the decomposition of elements up to arbitrary approximations. If $\epsilon > 0$ and $a \in A_+$, then $(a - \epsilon)_+$ will denote the positive part of $a - \epsilon \cdot 1$, that is, $(a-\epsilon)_+ = f(a)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(t) = \max\{t-\epsilon, 0\}$. It is proved in [29, Proposition 2.4] (see also [21, Proposition 2.6]) that $a \precsim b$ if and only if for any $\epsilon > 0$, there exists $\delta > 0$ and x in A such that $(a - \epsilon)_+ = x(b - \delta)_+ x^*$. (This is in turn equivalent to the statement that, for any $\epsilon > 0$, there is $\delta > 0$ such that $(a - \epsilon)_+ \precsim (b - \delta)_+$.)

The next proposition shows that despite the typically non-algebraic ordering on the Cuntz semigroup, one can always complement projections.

Proposition 2.2 *Let A be a C*-algebra. Let $a, p \in M_\infty(A)_+$ be such that p is a projection and $p \precsim a$. Then, there exists $b \in M_\infty(A)_+$ such that $p \oplus b \sim a$.*

Proof By passing to a suitable matrix over A, we may assume that actually p, $a \in A$. Let $0 < \epsilon < 1$. Since $p \precsim a$, we have that $p \sim (p-\epsilon)_+ = xax^*$, for some x in pA. Set $p' = a^{\frac{1}{2}} x^* x a^{\frac{1}{2}}$. Then p' is a projection equivalent to p and $p' \leq \|x\|^2 a$, which is Cuntz equivalent to a. Therefore we may assume at the outset that $p \leq a$.

We claim now that $p \oplus (1-p)a(1-p) \sim a$. By [21, Lemma 2.8], we always have that $a \precsim pap \oplus (1-p)a(1-p)$. Since $pap \leq \|a\|^2 p \sim p$, we obtain that $a \precsim p \oplus (1-p)a(1-p)$. To establish the converse subequivalence, it will suffice to show that both p and $(1-p)a(1-p)$ belong to the hereditary algebra A_a generated by a, because then $p + (1-p)a(1-p) \in A_a$. From this it follows that $p + (1-p)a(1-p) \precsim a$.

By our assumption we have that $p \leq a$ and thus $p \in A_a$. Also, $(1-p)a^{\frac{1}{2}} = a^{\frac{1}{2}} - pa^{\frac{1}{2}} \in A_a$, whence $(1-p)a(1-p) \in A_a$. □

Let M be a preordered Abelian semigroup, with order relation denoted by $\leq$. Recall that a non-zero element u in M is said to be an *order-unit* provided that for any x in M there is a natural number n such that $x \leq nu$. A *state* on a preordered monoid M with order-unit u is an order preserving monoid morphism $s\colon M \to \mathbb{R}$ such that $s(u) = 1$. We denote the (convex) set of states by $\mathrm{S}(M, u)$. In the case of a unital C*-algebra A, the set of states on the Cuntz monoid $W(A)$ is referred as to the *dimension functions* on A and denoted by $\mathrm{DF}(A)$ (see also [3,25,29]).

A dimension function s is lower semicontinuous if $s(\langle a\rangle) \leq \liminf_{n\to\infty} s(\langle a_n\rangle)$ whenever $a_n \to a$ in norm. The set of all lower semicontinous dimension functions on A is denoted by $\mathrm{LDF}(A)$. Note that any dimension function s induces a function $d_s\colon M_\infty(A) \to \mathbb{R}$ given by $d_s(a) = s\langle a^*a\rangle$. With this notation, lower semicontinuity of s as defined above is equivalent to lower semicontinuity of the function d_s.

We shall denote by $\mathrm{T}(A)$ the simplex of normalised traces defined on a unital C*-algebra A, and by $\mathrm{QT}(A)$ the simplex of quasitraces. (We will work mostly with simple unital C*-algebras in the sequel, and so take the term "quasitrace" to mean a normalised 2-quasitrace—see [3].) We have $\mathrm{T}(A) \subseteq \mathrm{QT}(A)$, and equality holds if A is exact and unital by the main theorem of [17]. Any quasitrace τ defines a lower semicontinuous dimension function

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}),$$

provided that the domain of d_τ is restricted to positive elements. In fact, it was proved in [3, Theorem II.2.2] that if $d \in \mathrm{LDF}(A)$, then there is a unique quasitrace τ such that $d = d_\tau$. It is clear that if $a \precsim b$, then for any dimension function d we have $d(a) \leq d(b)$.

The next definition is not new (see [2,25]), but bears repeating.

Definition 2.3 Let A be a unital C*-algebra with $a, b \in A_+\backslash\{0\}$. We say that A has the Fundamental Comparability Property for Positive Elements, denoted by (FCQ+), if $a \precsim b$ whenever $d_\tau(a) < d_\tau(b)$ for every $\tau \in \mathrm{QT}(A)$.

Villadsen gave the first example of a simple C*-algebra for which (FCQ+) fails [37]. In his example the positive elements a and b are projections. (FCQ+)

may hold for all pairs of projections, yet fail in general [34]. The abbreviation (FCQ+) derives from Blackadar's Fundamental Comparability Question, which asks if (FCQ+) holds whenever a and b are projections. In the literature, however, a C*-algebra A with (FCQ+) is usually said to have *strict comparison of positive elements* or simply *strict comparison*. The latter terminology will be employed in the sequel.

Lemma 2.4 *Let A be a unital C*-algebra with $a \in A_+$. For any faithful quasitrace τ and $\epsilon, \eta, \delta \in \sigma(a)$ with $\epsilon < \eta < \delta$ we have $d_\tau((a-\delta)_+) < d_\tau((a-\epsilon)_+)$.*

Proof Since $(a-\epsilon)_+$ and $(a-\delta)_+$ belong to the C*-algebra $C^*(a)$ generated by a, we may assume that $A = C^*(a)$. Then τ corresponds to a probability measure μ_τ on $\sigma(a)$ which is nonzero on every open set. By [3, Proposition I.2.1] we have $d_\tau(b) = \mu_\tau(\mathrm{Coz}(b))$, where $\mathrm{Coz}(b)$ is the cozero set of a nonnegative function b in $C^*(a)$.

Put $U_\epsilon = \{(\epsilon, \infty] \cap \sigma(a)\}$; define U_δ similarly. Let V be an open subset of $\sigma(a)$ containing η and such that $V \subseteq U_\epsilon \cap U_\delta^c$. Let b be a nonnegative function on $\sigma(a)$ such that $\mathrm{Coz}(b) = V$ and $b \leq (a-\epsilon)_+$. Now b is orthogonal to $(a-\delta)_+$ and $(a-\delta)_+ + b \leq (a-\epsilon)_+$, so

$$d_\tau((a-\delta)_+) + d_\tau(b) \leq d_\tau((a-\epsilon)_+);$$

$d_\tau(b) = \mu_\tau(V)$ is nonzero, and the lemma follows. □

Remark 2.5 We will occasionally refer the the spectrum $\sigma(a)$ of a positive element $a \in \mathrm{M}_\infty(A)$. Since a may be viewed as an element of arbitrarily large matrix algebras over A, we always assume that $0 \in \sigma(a)$ for consistency.

Proposition 2.6 *Let A be a simple C*-algebra with strict comparison of positive elements. Let $a \in A_{++}$ and $b \in A_+$ satisfy $d_\tau(a) \leq d_\tau(b)$ for every $\tau \in \mathrm{QT}(A)$. Then, $a \precsim b$.*

Proof If A has no quasitrace, then strict comparison of positive elements reduces to the condition that for any nonzero positive elements $a, b \in A$, there is a sequence x_j in A such that $x_j b x_j^* \to a$ as $j \to \infty$. Thus, $a \precsim b$.

Suppose that $\mathrm{QT}(A)$ is nonempty. Each quasitrace is faithful since A is simple. Since $a \in A_{++}$, we have that $a \neq 0$ and $0 \in \sigma(a)$. Then, there is a strictly decreasing sequence ϵ_n of positive reals in $\sigma(a)$ converging to zero. We also know by [2, Sect. 6] (see also [21, Proposition 2.6]) that the set $\{x \in A_+ \mid x \precsim b\}$ is closed, and since $(a-\epsilon_n)_+ \to a$ in norm it suffices to prove that $(a-\epsilon_n)_+ \precsim b$ for every $n \in \mathbb{N}$.

Let $\tau \in \mathrm{QT}(A)$ be given, and apply Lemma 2.4 with $\epsilon = 0$, $\eta = \epsilon_{n+1}$, and $\delta = \epsilon_n$ to see that

$$d_\tau((a-\epsilon_n)_+) < d_\tau(a) \leq d_\tau(b).$$

Using strict comparison we conclude that $(a-\epsilon_n)_+ \precsim b$ for all n, as desired. □

Proposition 2.7 *Let A be as in Proposition 2.6. Let p be a projection in A, and let $a \in A_{++}$. Then, $p \precsim a$ if and only if $d_\tau(p) < d_\tau(a)$ for each $\tau \in \mathrm{QT}(A)$.*

Proof If A has no quasitrace, then it is purely infinite and $p \precsim a$ [21].

Assume that $\mathrm{QT}(A)$ is nonempty. The reverse implication follows from strict comparison. We prove the contrapositive of the forward implication. Suppose that $d_\tau(a) \leq d_\tau(p)$ for some $\tau \in \mathrm{QT}(A)$, and let $1 > \epsilon > 0$ be given. By [29, Proposition 2.4] there exists a $\delta > 0$ such that

$$(p-\epsilon)_+ \precsim (a-\delta)_+.$$

This implies that

$$d_\tau((p-\epsilon)_+) \leq d_\tau((a-\delta)_+).$$

But p is a projection, so the functional calculus implies that

$$d_\tau((p-\epsilon)_+) = d_\tau(p).$$

Now

$$d_\tau((p-\epsilon)_+) \leq d_\tau((a-\delta)_+) < d_\tau(a) \leq d_\tau(p) = d_\tau((p-\epsilon)_+),$$

a contradiction. □

The hypotheses of Propositions 2.6 and 2.7 are satisfied whenever A is simple, unital, and $W(A)$ satisfies the technical condition of being almost unperforated (see [32]). In particular, A could be a simple, unital and finite C*-algebra absorbing the Jiang–Su algebra $\mathcal{Z}$ tensorially [32, Corollary 4.6].

Proposition 2.8 *Let A be a simple, unital, and stably finite C*-algebra, and let $a \in \mathrm{M}_\infty(A)_+$. Then, $\langle a\rangle = \langle p\rangle$ for a projection p in $\mathrm{M}_\infty(A)_+$ if and only if $0 \notin \sigma(a)$ or 0 is an isolated point of $\sigma(a)$.*

Proof If $0 \notin \sigma(a)$ or 0 is an isolated point of $\sigma(a)$, then $\langle a\rangle = \langle p\rangle$ for a projection p in $\mathrm{M}_\infty(A)_+$ by a straightforward functional calculus argument.

Now suppose that $\langle a\rangle = \langle p\rangle$ for a projection p in $\mathrm{M}_\infty(A)_+$ and 0 is an accumulation point of $\sigma(a)$. Choose $\epsilon \in [0,1) \cap \sigma(a)$ and a (necessarily faithful) quasitrace $\tau \in \mathrm{QT}(A)$. Using [29, Proposition 2.4], there is $0 < \delta \in \sigma(a)$ such that $(p-\epsilon)_+ \precsim (a-\delta)_+$. If $\delta > \epsilon$, then

$$d_\tau(a) = d_\tau(p) = d_\tau((p-\epsilon_+) \leq d_\tau((a-\delta)_+) \leq d_\tau((a-\epsilon)_+) < d_\tau(a)$$

by Lemma 2.4; this is impossible. Thus $\delta \leq \epsilon$, and by assumption we may find $\delta' \in \sigma(a)$ such that $\delta' < \delta$. A second application of Lemma 2.4 implies that

$$d_\tau(a) = d_\tau(p) \leq d_\tau((a-\delta)_+) \leq d_\tau((a-\delta')_+) < d_\tau(a);$$

this, too, is impossible. □

If one replaces the assumptions of simplicity and being stably finite with stable rank one, then Proposition 2.8 is due to Perera [25, Proposition 3.12].

Corollary 2.9 *Let A be a unital C*-algebra which is either simple and stably finite or of stable rank one. Then*:

(i) $W(A)_+$ *is a semigroup, and is absorbing in the sense that if one has* $a \in W(A)$ *and* $b \in W(A)_+$, *then* $a + b \in W(A)_+$;

(ii)
$$V(A) = \{x \in W(A) \mid \text{ if } x \leq y \text{ for } y \in W(A), \text{ then } x + z = y \text{ for some } z \in W(A)\}.$$

Proof For (i), take $\langle a\rangle, \langle b\rangle \in W(A)_+$ and notice that the spectrum of $a \oplus b$ contains the union of the spectra of a and b. Apply Proposition 2.8.

For (ii), set $X = \{x \in W(A) \mid \text{ if } x \leq y \text{ for } y \in W(A), \text{then } x + z = y \text{ for some } z \in W(A)\}$. By Proposition 2.2, we already know that $V(A) \subseteq X$.

Conversely, if $\langle x\rangle \in X$, then we may find a projection p (in $M_\infty(A)$) such that $\langle x\rangle \leq \langle p\rangle$. But then there is z in $M_\infty(A)$ for which $x \oplus z \sim p$. Since 0 is an isolated point in $\sigma(p)$, the same will be true of $\sigma(x)$. Invoking Proposition 2.8 or [25, Proposition 3.12] as appropriate, we find a projection q such that $q \sim x$, and so $\langle x\rangle \in V(A)$. □

The last proposition of this section, though straightforward, will be quite important in the sequel.

Proposition 2.10 *Let A be a stably finite unital C*-algebra, and let $a \in A_+$. Then, the map $\tau \mapsto d_\tau(a)$ is a lower semicontinuous bounded function on* $\mathrm{T}(A)$.

Proof Since $\langle \lambda a\rangle = \langle a\rangle$ for every $\lambda \in \mathbb{R}^+\backslash\{0\}$, we may assume that $||a|| \leq 1$. Then, $f_n(\tau) := \tau(a^{1/n})$ is an increasing sequence of continuous functions on $\mathrm{T}(A)$ with pointwise limit $f(\tau) := d_\tau(a)$. □

3 $\mathcal{Z}$-stable C*-algebras

In this section we give a precise description of $W(\mathcal{Z})$ (Theorem 3.1 below), and establish the important fact that $W(\bullet)_+$ is a $\mathbb{R}^+$-cone for certain finite and $\mathcal{Z}$-stable C*-algebras. In the study of the Cuntz semigroup for simple, unital, and $\mathcal{Z}$-stable C*-algebras, the finite case is the only interesting one. Indeed, a simple, unital, and $\mathcal{Z}$-stable C*-algebra A either has stable rank one or is purely infinite (see [13, Theorem 3] and also [32, Corollary 5.1 and Theorem 6.7]). If A is purely infinite, then $a \precsim b$ for all non-zero positive elements (see [23]). It follows that $W(A) = \{0, \langle 1\rangle\}$ ($\langle 1\rangle + \langle 1\rangle = \langle 1\rangle$), and that the Grothendieck group $\mathrm{K}_0^*(A)$ of $W(A)$ is zero.

We begin with some notation. For a compact convex set K, denote by $\mathrm{Aff}(K)^+$ the semigroup of all positive, affine, continuous, and real-valued functions on

K; $\mathrm{LAff}(K)^+ \subseteq \mathrm{Aff}(K)^+$ is the subsemigroup of lower semicontinuous functions, and $\mathrm{LAff}_b(K)^+ \subseteq \mathrm{LAff}(K)^+$ is the subsemigroup consisting of those functions which are bounded above. The use of an additional "+" superscript (e.g., $\mathrm{Aff}(K)^{++}$) indicates that we are considering only strictly positive functions together with the zero function. Unless otherwise noted, the order on these semigroups will be pointwise. $\mathrm{Aff}(K)^+$ is algebraically ordered with this ordering, but $\mathrm{LAff}(K)^+$, in general, is not (unless K is, for example, finite dimensional).

Given two partially ordered semigroups M and N, a homomorphism $\varphi\colon M \to N$ is said to be an *order-embedding* provided that $\varphi(x) \leq \varphi(y)$ if and only if $x \leq y$. A surjective order-embedding will be called an *order-isomorphism*.

Let $\leq_{\mathbb{R}}$ denote the usual order on the real numbers. We equip the disjoint union $\mathbb{Z}^+ \sqcup \mathbb{R}^{++}$ with a semigroup structure by using the usual addition inside the components $\mathbb{Z}^+$ and $\mathbb{R}^{++}$ and declaring that $x + y \in \mathbb{R}^{++}$ whenever $x \in \mathbb{Z}^+$ and $y \in \mathbb{R}^{++}$. Define an order $\leq_{\mathcal{Z}}$ on this semigroup by using the usual order inside the components $\mathbb{Z}^+$ and $\mathbb{R}^{++}$, and the following order for comparing $x \in \mathbb{Z}^+$ and $y \in \mathbb{R}^{++}$: $x \leq_{\mathcal{Z}} y$ iff $x <_{\mathbb{R}} y$, while $x \geq_{\mathcal{Z}} y$ iff $x \geq_{\mathbb{R}} y$. With this ordering, $1_{\mathbb{Z}^+}$ is an order-unit.

Theorem 3.1 *The ordered semigroup* $(W(\mathcal{Z}), \langle 1_{\mathcal{Z}} \rangle)$ *is order-isomorphic* (*as an ordered monoid with order-unit*) *to*

$$(\mathbb{Z}^+ \sqcup \mathbb{R}^{++}, 1_{\mathbb{Z}^+}, \leq_{\mathcal{Z}}).$$

Proof As observed in Corollary 2.9, the Cuntz semigroup of a C*-algebra A of stable rank one is always the disjoint union of the monoid $V(A)$ and $W(A)_+$. Since $\mathcal{Z}$ is unital, projectionless, and of stable rank one we have $V(\mathcal{Z}) \cong \mathbb{Z}^+$. By Proposition 2.6 there is an order-embedding

$$\iota\colon W(\mathcal{Z})_+ \to \mathbb{R}^{++}$$

given by

$$\iota(\langle a \rangle) = d_{\tau_{\mathcal{Z}}}(a),$$

where $\tau_{\mathcal{Z}}$ is the unique normalised trace on $\mathcal{Z}$. By [32, Theorem 2.1] there is a unital embedding of $\mathrm{C}([0,1])$ into $\mathcal{Z}$ such that $\tau_{\mathcal{Z}}$ is implemented by the uniform distribution on $[0,1]$. Given $\lambda \in (0,1]$, let $z_\lambda \in \mathrm{C}([0,1])$ be a positive function with support $(0,\lambda)$. It follows that $d_{\tau_{\mathcal{Z}}}(z_\lambda) = \lambda$, whence ι is surjective. We therefore have a bijection

$$\varphi\colon W(\mathcal{Z}) = V(\mathcal{Z}) \sqcup W(\mathcal{Z})_+ \to \mathbb{Z}^+ \sqcup \mathbb{R}^{++}.$$

That φ is an order-isomorphism follows from the fact that $\mathcal{Z}$ has strict comparison of positive elements [32, Corollary 4.6] and Propositions 2.7 and 2.6. □

Notation 3.2 *For each* $\lambda \in (0,1]$ *we will use* z_λ *denote any element in* $\mathcal{Z}_{++}$ *such that* $d_{\tau_{\mathcal{Z}}}(z_\lambda) = \lambda$.

Proposition 3.3 *Let A be a* C^**-algebra of stable rank one for which every trace is faithful. Then, the map*

$$\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$$

given by $\iota(\langle a\rangle)(\tau) = d_\tau(a)$ *is a homomorphism. If A has strict comparison of positive elements, then* ι *is an order embedding.*

Proof The requirement that every trace on A be faithful guarantees that $\iota(\langle a\rangle)$ is strictly positive. A has stable rank one, so $W(A)_+$ is a semigroup by Proposition 2.8 and ι is a homomorphism.

If A has strict comparison of positive elements, then ι is an order embedding by Proposition 2.6. □

Lemma 3.4 *Let A be a unital and* $\mathcal{Z}$*-stable* C^**-algebra, with* $a \in A_+$*. Then, a is Cuntz equivalent to a positive element of the form* $b \otimes \mathbf{1}_{\mathcal{Z}} \in A \otimes \mathcal{Z} \cong A$.

Proof Let $\psi : \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$ be a $*$-isomorphism, and put $\phi = (id_{\mathcal{Z}} \otimes \mathbf{1}_{\mathcal{Z}}) \circ \psi$. By [35, Corollary 1.12], ϕ is approximately inner, and therefore so also is

$$id_A \otimes \phi : A \otimes \mathcal{Z}^{\otimes 2} \to A \otimes \mathcal{Z} \otimes \mathbf{1}_{\mathcal{Z}}.$$

In particular, there is a sequence of unitaries u_n in $A \cong A \otimes \mathcal{Z}^{\otimes 2}$ such that

$$||u_n a u_n^* - \phi(a)|| \overset{n\to\infty}{\longrightarrow} 0.$$

Approximate unitary equivalence preserves Cuntz equivalence classes, whence $\langle a\rangle = \langle \phi(a)\rangle$. The image of $\phi(a)$ is, by construction, of the form $b \otimes \mathbf{1}_{\mathcal{Z}}$ for some $b \in A \otimes \mathcal{Z} \cong A$. □

Lemma 3.5 *Let A be a unital, stably finite, and* $\mathcal{Z}$*-stable* C^**-algebra. Suppose that* $f \in \mathrm{LAff}(\mathrm{T}(A))^{++}$ *is equal to* $d_\tau(a)$ *for some* $a \in \mathrm{M}_\infty(A)_+$*. Then, the image of* $a \otimes z_\lambda$ *in* $\mathrm{LAff}(\mathrm{T}(A \otimes \mathcal{Z}))^{++}$ *is* $\lambda\tilde{f}$*, where* $\tilde{f} = d_\tau(a \otimes \mathbf{1}_{\mathcal{Z}}$.

Proof For any $\tau \in \mathrm{T}(A)$ one has

$$\begin{aligned} d_\tau(a \otimes z_\lambda) &= \lim_{n\to\infty} \tau\left((a \otimes z_\lambda)^{1/n}\right) \\ &= \lim_{n\to\infty} \tau(a^{1/n})\tau_{\mathcal{Z}}(z_\lambda^{1/n}) \\ &= d_\tau(a) d_{\tau_{\mathcal{Z}}}(z_\lambda) \\ &= \lambda d_\tau(a). \end{aligned}$$

□

Corollary 3.6 *Let A be as in Lemma 3.5. Then, the image of $\mathrm{M}_\infty(A)_+$ under the map ι of Proposition 3.3 is a cone over $\mathbb{R}^+$*

Proof It will be enough to prove that if $\lambda \in \mathbb{R}^+$ and $a \in A_+$, then there exists $b \in A_+$ with $d_\tau(b) = \lambda d_\tau(a)$. Identify A with $A \otimes \mathcal{Z}$, and use Lemma 3.4 to find $b \in A_+$ such that $b \otimes \mathbf{1}_\mathcal{Z}$ and a are Cuntz equivalent. It follows that $d_\tau(b \otimes \mathbf{1}_\mathcal{Z}) = d_\tau(a)$ for each $\tau \in \mathrm{T}(A)$. Now $d_\tau(b \otimes z_\lambda) = \lambda d_\tau(a)$ by Lemma 3.5. □

Summarising, we have:

Corollary 3.7 *Let A be a simple, unital, exact, finite, and $\mathcal{Z}$-stable C^*-algebra. Then, the map ι of Proposition 3.3 is an order embedding, and $W(A)_+$ is a $\mathbb{R}^+$-cone.*

Note that exactness is required above in order to identify the image of ι with a collection of functions on $\mathrm{T}(A)$ as opposed to $\mathrm{QT}(A)$.

We close this section with an aside on some algebras of particular interest in Elliott's classification programme. Recall that a C^*-algebra is said to have property (SP) if every hereditary subalgebra contains a non-zero projection. With Theorem 3.1 in hand, we can prove the following proposition:

Proposition 3.8 *Let A be a simple, unital, exact, finite, and $\mathcal{Z}$-stable C^*-algebra. Then A has property (SP) if and only if for every $\epsilon > 0$ there exists a non-zero projection $p \in A$ such that $d_\tau(p) = \tau(p) < \epsilon$ for every trace on A.*

In particular, a projection p is Murray–von Neumann equivalent to a projection q in a hereditary subalgebra $\overline{aAa}$ whenever $\tau \mapsto d_\tau(p)$ if uniformly sufficiently small.

Proof For the forward implication, write $A \cong A \otimes \mathcal{Z}$, and notice that $d_\tau(1_A \otimes z_\lambda) = \lambda$, for all $\tau \in \mathrm{T}(A)$. Since A has property (SP), the algebra $\overline{(1_A \otimes z_\lambda)A(1_A \otimes z_\lambda)}$ contains a projection p, whence $p \precsim 1_A \otimes z_\lambda$. Setting $\lambda = \epsilon/2$, we have that

$$\tau(p) \leq d_\tau(1_A \otimes z_{\epsilon/2}) < \epsilon, \quad \text{for all } \tau \in \mathrm{T}(A).$$

For the reverse implication, let $a \in A_+$ be given. The compactness of $\mathrm{T}(A)$ and the lower semicontinuity of the function $f_a \colon \mathrm{T}(A) \to \mathbb{R}^{++}$ given by $f_a(\tau) = d_\tau(a)$ (that follows from Proposition 2.10) imply that there exists $\epsilon > 0$ such that $d_\tau(a) > \epsilon$, for every τ in $\mathrm{T}(A)$. Choose a non-zero projection p in A such that $d_\tau(p) = \tau(p) < \epsilon$ for every trace on A. The hypotheses on A guarantee strict comparison for positive elements (cf. [32, Corollary 4.6]), so that $p \precsim a$ inside $W(A)$. Following the proof of Proposition 2.2, we see that there is a projection $q \in \overline{aAa}$ which is Murray–von Neumann equivalent to p. □

Let $\mathcal{B}$ be a class of unital C^*-algebras. Recall that a unital C^*-algebra A is said to be *tracially approximately* $\mathcal{B}$ (TA$\mathcal{B}$) if for any $\epsilon > 0$, finite set $F \subset A$, and $a \in A_+$ there exists a C*-subalgebra C of A such that $C \in \mathcal{B}$, $\mathbf{1}_C \neq 0$, and

1. $[f, \mathbf{1}_C] < \epsilon$, for all f in F;
2. $\mathrm{dist}(\mathbf{1}_C f \mathbf{1}_C, C) < \epsilon$, for all f in F;
3. $\mathbf{1}_A - \mathbf{1}_C$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

One may wonder why the term "tracially" is used in the description of such algebras, given that no reference to traces is made in their definition. The reason is that condition (3) above can sometimes be replaced by the condition

$$(3)' \quad \tau(\mathbf{1}_C) > 1 - \epsilon, \text{ for all } \tau \text{ in } \mathrm{T}(A),$$

provided that the class of TA$\mathcal{B}$ algebras is sufficiently well behaved.

TA$\mathcal{B}$ algebras are used mainly in Elliott's classification program. In this setting, it is necessary to assume exactness, and the largest class for which classification can be hoped for consists of $\mathcal{Z}$-stable algebras. Since, in the simple case, the program is more or less complete for purely infinite algebras, we may also assume finiteness. Taken together, these conditions constitute the hypotheses of Proposition 3.8, and the proof of the proposition then shows that conditions (3) and (3)$'$ above are equivalent. Thus, in most situations where TA$\mathcal{B}$ algebras might be useful, there is no ambiguity in their definition.

4 An embedding theorem

In order to make sense of (EC) and (WEC), we must define the categories in which the relevant invariants sit.

Let $\mathcal{I}$ denote the category whose objects are four-tuples

$$\left((G_0, G_0^+, u), G_1, X, r\right),$$

where (G_0, G_0^+, u) is a simple partially ordered Abelian group with distinguished order-unit u and state space $S(G_0, u)$, G_1 is a countable Abelian group, X is a metrizable Choquet simplex, and $r\colon X \to S(G_0, u)$ is an affine map. A morphism

$$\Theta\colon \left((G_0, G_0^+, u), G_1, X, r\right) \to \left((H_0, H_0^+, v), H_1, Y, s\right)$$

in $\mathcal{I}$ is a three-tuple

$$\Theta = (\theta_0, \theta_1, \gamma)$$

where

$$\theta_0\colon (G_0, G_0^+, u) \to (H_0, H_0^+, v)$$

is an order-unit-preserving positive homomorphism,

$$\theta_1\colon G_1 \to H_1$$

is any homomorphism, and

$$\gamma\colon Y \to X$$

is a continuous affine map that makes the diagram below commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\gamma} & X \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} \\ S(H_0, v) & \xrightarrow{\theta_0^*} & S(G_0, u). \end{array}$$

For a simple unital C^*-algebra A the Elliott invariant $I(A)$ is an element of $\mathcal{I}$, where $(G_0, G_0^+, u) = (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$, $G_1 = \mathrm{K}_1(A)$, $X = \mathrm{T}(A)$, and r_A is given by evaluating a given trace at a K_0-class. Given a class $\mathcal{C}$ of simple unital C*-algebras, let $\mathcal{I}(\mathcal{C})$ denote the subcategory of $\mathcal{I}$ whose objects can be realised as the Elliott invariant of a member of $\mathcal{C}$, and whose morphisms are all admissible maps between the now specified objects.

The definition of $\mathcal{I}$ removes an ambiguity from the statement of (EC), namely, what is meant by an isomorphism of Elliott invariants. We now do the same for (WEC). Let $\mathcal{W}$ be the category whose objects are ordered pairs

$$((W(A), \langle 1_A \rangle), I(A)),$$

where A is a simple, unital, exact, and stably finite C*-algebra, $(W(A), \langle 1_A \rangle)$ is the Cuntz semigroup of A together with the distinguished order-unit $\langle 1_A \rangle$, and $I(A)$ is the Elliott invariant of A. A morphism

$$\Psi\colon\ ((W(A), \langle 1_A \rangle), I(A)) \to ((W(B), \langle 1_B \rangle), I(B))$$

in $\mathcal{W}$ is an ordered pair

$$\Psi = (\Lambda, \Theta),$$

where $\Theta = (\theta_0, \theta_1, \gamma)$ is a morphism in $\mathcal{I}$ and $\Lambda\colon (W(A), \langle 1_A \rangle) \to (W(B), \langle 1_B \rangle)$ is an order- and order-unit-preserving semigroup homomorphism satisfying two compatibility conditions: first,

$$\begin{array}{ccc} (V(A), \langle 1_A \rangle) & \xrightarrow{\Lambda|_{V(A)}} & (V(B), \langle 1_B \rangle) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ (\mathrm{K}_0(A), [1_A]) & \xrightarrow{\theta_0} & (\mathrm{K}_0(B), [1_B]), \end{array}$$

where ρ is the usual Grothendieck map from $V(\bullet)$ to $\mathrm{K}_0(\bullet)$ (recall that there is an order-unit-preserving order-embedding of $(V(A), \langle 1_A \rangle)$ into $(W(A), \langle 1_A \rangle)$, and that Cuntz equivalence of projections agrees with Murray–von Neumann equivalence in stably finite algebras); second,

$$\begin{array}{ccc} \mathrm{LDF}(B) & \xrightarrow{\Lambda^*} & \mathrm{LDF}(A) \\ \downarrow \eta & & \downarrow \eta \\ \mathrm{T}(B) & \xrightarrow{\gamma} & \mathrm{T}(A), \end{array}$$

where η is the affine bijection between $\mathrm{LDF}(\bullet)$ and $\mathrm{T}(\bullet)$ given by $\eta(d_\tau) = \tau$ (see [3, Theorem II.2.2]). These compatibility are automatically satisfied if Ψ is induced by a $*$-homomorphism $\psi : A \to B$.

Recall that we have previously defined, for a C*-algebra with stable rank one, a semigroup homomorphism

$$\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$$

by

$$\iota(\langle a\rangle)(\tau) = d_\tau(a), \text{ for all } \tau \in \mathrm{T}(A).$$

In the following definition we generalise the semigroup and order structure on $\mathbb{Z}^+ \sqcup \mathbb{R}^{++}$ considered in Theorem 3.1. Semigroups of this type have been considered previously in the study of multiplier algebras (see [26]).

Definition 4.1 Let A be a unital C*-algebra. Define a semigroup structure on the set

$$\widetilde{W}(A) := V(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++}$$

by extending the natural semigroup operations and setting $[p] + f = \widehat{p} + f$, where $\widehat{p}(\tau) = \tau(p)$. Define an order $\leq$ on $\widetilde{W}(A)$ such that:

(i) $\leq$ agrees with the usual order on $V(A)$;
(ii) $f \leq g$ for f, g in $\mathrm{LAff}(\mathrm{T}(A))^{++}$ if and only if

$$f(\tau) \leq_{\mathbb{R}} g(\tau) \quad \text{for all } \tau \in \mathrm{T}(A);$$

(iii) $f \leq [p]$ for $[p] \in V(A)$ and f in $\mathrm{LAff}(\mathrm{T}(A))^{++}$ if and only if

$$f(\tau) \leq_{\mathbb{R}} \tau(p) \quad \text{for all } \tau \in \mathrm{T}(A);$$

(iv) $[p] \leq f$ for $f, [p]$ as in (iii) whenever

$$\tau(p) <_{\mathbb{R}} f(\tau) \quad \text{for all } \tau \in \mathrm{T}(A).$$

Let $\widetilde{\mathcal{W}}$ be the category whose objects are of the form $(\widetilde{W}(A), [1_A])$ for some exact, unital, and stable rank one C*-algebra A, and whose morphisms are positive order-unit-preserving homomorphisms

$$\Gamma\colon (\widetilde{W}(A), [1_A]) \to (\widetilde{W}(B), [1_B])$$

such that

$$\Gamma(V(A)) \subseteq V(B)$$

and

$$\Gamma|_{\mathrm{LAff}_b(\mathrm{T}(A))^{++}}\colon \mathrm{LAff}_b(\mathrm{T}(A))^{++} \to \mathrm{LAff}_b(\mathrm{T}(B))^{++}$$

is induced by a continuous affine map from $\mathrm{T}(B)$ to $\mathrm{T}(A)$.

For the next definition, we remind the reader that $V(A) \cong \mathrm{K}_0(A)^+$ for a C*-algebra of stable rank one.

Definition 4.2 Let $\mathcal{C}$ denote the class of simple, unital, exact, and stable rank one C*-algebras. Let

$$F\colon \mathbf{Obj}(\mathcal{I}(\mathcal{C})) \to \mathbf{Obj}(\widetilde{\mathcal{W}})$$

be given by

$$F\left((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r_A\right) = (\widetilde{W}(A), [1_A]).$$

Define

$$F\colon \mathbf{Mor}(\mathcal{I}(\mathcal{C})) \to \mathbf{Mor}(\widetilde{\mathcal{W}})$$

by sending $\Theta = (\theta_0, \theta_1, \gamma)$ to the morphism

$$\Gamma\colon (\widetilde{W}(A), [1_A]) \to (\widetilde{W}(B), [1_B])$$

given by θ_0 on $\mathrm{K}_0(A)^+ = V(A)$ and induced by γ on $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$.

The next proposition holds by definition.

Proposition 4.3 *With $\mathcal{C}$ as in Definition 4.2, the map $F\colon \mathcal{I}(\mathcal{C}) \to \widetilde{\mathcal{W}}$ is a functor.*

For the theorem below, we remind the reader that the definition of the map ι is contained in Proposition 3.3.

Theorem 4.4 *Let A be a simple, unital, and exact C*-algebra having stable rank one and strict comparison of positive elements. Then, there is an order embedding*

$$\phi\colon W(A) \to \widetilde{W}(A)$$

such that $\phi|_{V(A)} = \mathrm{id}_{V(A)}$ and $\phi|_{W(A)_+} = \iota$.

Proof The map ϕ is well-defined, so it will suffice to prove that it is an order embedding. We verify conditions (i)–(iv) from Definition 4.1: the image of $\phi|_{V(A)}$ is $V(A)$, with the same order, so (i) is satisfied; (ii) and (iii) follow from Proposition 2.6; (iv) is Proposition 2.7. □

We are now ready to prove Theorem 1.3. In fact, we can prove a formally stronger result.

Theorem 4.5 (*EC*) *implies* (*WEC*) *for the class of simple, unital, separable, and nuclear C*-algebras with strict comparison of positive elements and* $\mathrm{sr} \in \{1, \infty\}$.

Proof Algebras in the class under consideration are either purely infinite or stably finite. The theorem is trivial for the subclass of purely infinite algebras, due to the degenerate nature of the Cuntz semigroup in this setting. The remaining case is that of stable rank one.

Let A and B be simple, separable, unital, nuclear, and stably finite C*-algebras with strict comparison of positive elements, and suppose that (EC) holds. Let there be given an isomorphism

$$\phi\colon\ (W(A), \langle 1_A \rangle, I(A)) \to (W(B), \langle 1_B \rangle, I(B))\,.$$

Then by restricting ϕ we have an isomorphism

$$\phi|_{I(A)}\colon I(A) \to I(B),$$

and we may conclude by (EC) that there is a $*$-isomorphism $\Phi\colon A \to B$ such that $I(\Phi) = \phi|_{I(A)}$. Φ is unital and so preserves the Cuntz class of the unit. The compatibility conditions imposed on ϕ (see the discussion preceding Definition 4.1) together with Theorem 4.4 ensure that $\phi|_{W(A)}$ is determined by $\phi|_{V(A)}$ and $\phi^\sharp : \mathrm{T}(B) \to \mathrm{T}(A)$. Thus, Φ induces ϕ, and (WEC) holds. □

Note that the semigroup homomorphism ϕ in Theorem 4.4 is an isomorphism if and only if ι is surjective.

Let (EC)$'$ and (WEC)$'$ denote the statements (EC) and (WEC), respectively, but expanded to apply to all simple, unital, exact, and stably finite C*-algebras. Collecting the results of this section we have:

Theorem 4.6 *Let $\mathcal{C}$ be a class of simple, unital, exact, finite, and $\mathcal{Z}$-stable C*-algebras. Suppose that ι is surjective for each member of $\mathcal{C}$. Then,* (*EC*)$'$ *and* (*WEC*)$'$ *are equivalent in $\mathcal{C}$. Moreover, there is a functor $G\colon \mathcal{I}(\mathcal{C}) \to \mathcal{W}$ such that*

$$G(I(A)) \stackrel{\mathrm{def}}{=} (F(I(A)), I(A)) = ((\widetilde{W}(A), [1_A]), I(A)) \cong ((W(A), \langle 1_A \rangle), I(A)).$$

Even in situations where (EC) holds, there is no inverse functor which reconstructs C*-algebras from Elliott invariants. (This is *not* the same as saying that one cannot reconstruct the algebra from the Elliott invariant at all—this is always possible when one has a range result for a class of algebras satisfying

(EC).) Contrast this with Theorem 4.6, where G reconstructs the finer invariant from the coarser one functorially.

We now see that (EC) and (WEC) are equivalent among simple, unital, separable, nuclear, finite, and $\mathcal{Z}$-stable C*-algebras whenever ι is surjective. (It is not clear whether the converse holds.) In Sects. 6, 7, and 8 we will prove that ι is surjective for algebras satisfying hypotheses (i), (ii), or (iii) of Theorem 1.4, respectively, thereby proving the theorem.

We note that if ι is surjective and A satisfies the hypotheses of Theorem 4.6, then the invariant

$$((W(A), \langle 1_A \rangle), I(A))$$

carries redundant information. A has stable rank one, so one may, by using Corollary 2.9, recover $V(A) \cong \mathrm{K}_0(A)^+$, and hence $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$, from $(W(A), \langle 1_A \rangle)$. The convex affine space $\mathrm{T}(A)$ is identified with $\mathrm{LDF}(A)$ (although we cannot, in general, recover the topology on $\mathrm{T}(A)$—see the discussion following Corollary I.2.2 of [3]). The pairing r_A can be recovered by applying the elements of $\mathrm{LDF}(A)$ to $V(A) \cong \mathrm{K}_0(A)^+$.

We close this section by observing that if ι is surjective, then the failure of the order on $W(\bullet)$ to be algebraic in general is easily explained.

Proposition 4.7 *Let A be an exact C*-algebra with strict comparison of positive elements. Suppose that ι is surjective and that each $\tau \in \mathrm{T}(A)$ is faithful. Let $a \precsim b$ in $\mathrm{M}_\infty(A)_{++}$. Then, there exists a positive element $c \in \mathrm{M}_\infty(A)_{++}$ such that $a \oplus c \sim b$ if and only if the difference*

$$d_\tau(b) - d_\tau(a) \colon \mathrm{T}(A) \to \mathbb{R}^+$$

is in $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$.

Proof If $b \sim a \oplus c$, then $d_\tau(b) - d_\tau(a) = d_\tau(c)$ and $d_\tau(c) \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ by Proposition 2.10.

Suppose that $f(\tau) := d_\tau(b) - d_\tau(a) \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$. Choose, by the surjectivity of ι, an element $c \in \mathrm{M}_\infty(A)_{++}$ for which $d_\tau(c) = f(\tau)$. Then $d_\tau(a \oplus c) = d_\tau(b)$, whence $a \oplus c \sim b$ by Proposition 2.6. □

5 The structure of K_0^*

The Grothendieck enveloping group of $W(A)$ is denoted $\mathrm{K}_0^*(A)$, and its structure has been previously analysed in [3,5,18,25]. Because $W(A)$ carries its own order coming from the Cuntz comparison relation, $\mathrm{K}_0^*(A)$ may be given two natural (partial) orderings. For an abelian semigroup M with a partial order $\leq$ that extends the algebraic order, we use $G(M)$ to denote its enveloping group. Write $\gamma \colon M \to G(M)$ for the natural Grothendieck map. We define the following cones:

$$G(M)^+ = \gamma(M),$$

and

$$G(M)^{++} = \{\gamma(x) - \gamma(y) \mid x, y \in M \text{ and } y \leq x\}.$$

Since M is partially ordered, so is $(G(M), G(M)^{++})$. Clearly, $G(M)^+ \subseteq G(M)^{++}$, and the inclusion may be strict. Therefore, $(G(M), G(M)^+)$ is also partially ordered. For the reader's convenience, we offer a short argument which shows the cone $G(M)^{++}$ to be strict (compare with [18] and [3]). Assume that $\gamma(x) - \gamma(y) \in G(M)^{++} \cap (-G(M)^{++})$. Then there are elements s, t, u, v in M such that

$$x + z \leq y + z, \quad t + v \leq s + v, \quad x + s + u = y + t + u,$$

so that $\gamma(y) - \gamma(x) = \gamma(s) - \gamma(t) \in G(M)^{++}$. Set $w = u + v + z + t$ and check that $x + w = y + w$, whence $\gamma(x) = \gamma(y)$.

Recall that a partially ordered Abelian group with order-unit (G, G^+, u) is *Archimedean* provided that $nx \leq y$ for $x, y \in G$ and for all natural numbers n only if $x = 0$ (see [14, p. 20]). This is equivalent (by [14, Theorem 4.14]) to saying that the order on G is determined by its states, i.e., $G^+ = \{x \in G \mid s(x) \geq 0$ for all $s \in \mathrm{S}(G, u)\}$. (Recall that a state s on (G, G^+, u) is a positive group homomorphism into $\mathbb{R}$ such that $s(u) = 1$ — s need not be order preserving, in contrast with a state on a positive ordered Abelian semigroup.) We say that (G, G^+) is *unperforated* if $nx \geq 0$ implies that $x \geq 0$ (see [14]). Archimedean directed groups are unperforated (cf. [14, Proposition 1.24]).

For an element a in $M_\infty(A)_+$, we shall denote by $[a]$ the class of $\langle a \rangle$ in $\mathrm{K}_0^*(A)$.

Lemma 5.1 *Let A be a simple C^*-algebra with strict comparison of positive elements. Suppose that* $\mathrm{M}_\infty(A)_{++} \neq \emptyset$. *Then*:

$$\mathrm{K}_0^*(A)^{++} = \{[a] - [b] \mid a, b \in M_\infty(A)_+ \text{ and } d_\tau(a) \geq d_\tau(b) \text{ for all } \tau \in \mathrm{QT}(A)\}.$$

Proof By the properties of dimension functions, it is clear that if $a, b \in M_\infty(A)_+$ and $b \precsim a$, we have $d_\tau(b) \leq d_\tau(a)$ for any $\tau \in \mathrm{QT}(A)$.

For the converse inclusion, let $[a] - [b] \in \mathrm{K}_0^*(A)$ be such that $d_\tau(b) \leq d_\tau(a)$ for each $\tau \in \mathrm{QT}(A)$. Then, for any $0 \neq c \in \mathrm{M}_\infty(A)_{++}$ we have $a \oplus c, b \oplus c \in \mathrm{M}_\infty(A)_{++}$ and

$$d_\tau(b \oplus c) \leq d_\tau(a \oplus c).$$

It follows from Proposition 2.6 that

$$b \oplus c \precsim a \oplus c,$$

and thus $[a] - [b] = [a \oplus c] - [b \oplus c] \in \mathrm{K}_0^*(A)^{++}$. □

Corollary 5.2 *Let A be a C^*-algebra satisfying the hypotheses of Lemma 5.1. Then $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++})$ is Archimedean, and in particular is unperforated.*

Proof The second conclusion follows from the first since, as observed above, archimedean groups are unperforated. (Notice that $\mathrm{K}_0^*(A)$ is directed since A is unital.)

We only need to show that if $[a]-[b] \in \mathrm{K}_0^*(A)$ is such that $s([a]-[b]) \geq 0$ for any state s on $\mathrm{K}_0^*(A)$ (i.e. $s([b]) \leq s([a])$), then $[a]-[b] \in \mathrm{K}_0^*(A)^{++}$. Recalling that the states on $\mathrm{K}_0^*(A)$ are precisely the dimension functions, we have that in particular $d_\tau(b) \leq d_\tau(a)$ for any quasitrace τ, hence we may use Lemma 5.1. □

We shall show below that $\mathrm{K}_0^*(A)$ is also unperforated when endowed with the ordering defined by taking as positive cone $\mathrm{K}_0^*(A)^+ = \gamma(W(A))$, that is, the image of $W(A)$ under the Grothendieck map.

A partially ordered semigroup $(M, \leq)$ is said to be *almost unperforated* if for all x, y in M and $n \in \mathbb{N}$ with $(n+1)x \leq ny$, one has that $x \leq y$. A simple partially ordered group (G, G^+) is *weakly unperforated* if $nx \in G^+ \setminus \{0\}$ implies that $x \in G^+ \setminus \{0\}$ [32, Lemma 3.4].

Proposition 5.3 *Let A be a simple, unital, exact, and finite C^*-algebra which absorbs the Jiang–Su algebra $\mathcal{Z}$ tensorially. Then, the partially ordered Abelian group $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^+)$ is weakly unperforated.*

Proof We have already noticed that A has strict comparison of positive elements, by Corollary 4.6 of [32]. The simplicity of A guarantees that each trace on A is faithful. Since $1_A \otimes z_1 \in A \otimes \mathcal{Z} \cong A$, we have that $\mathrm{M}_\infty(A)_{++} \neq \emptyset$. Thus, A satisfies the hypotheses of Lemma 5.1.

Given $[a] \in \mathrm{K}_0^*(A)^+$, for $a \in M_\infty(A)_+$, we may assume that $a \in \mathrm{M}_\infty(A)_{++}$. To see this, first identify A with $A \otimes \mathcal{Z}$, and replace a with a Cuntz equivalent element $b \otimes \mathbf{1}_{\mathcal{Z}}$ (see Lemma 3.4). Now for each $\tau \in \mathrm{T}(A)$ we have

$$d_\tau(a) = d_\tau(b \otimes \mathbf{1}_{\mathcal{Z}}) = d_\tau(b \otimes z_1)$$

(see Notation 3.2 and Lemma 3.5). Now $[a] = [b \otimes z_1]$ by Lemma 5.1 and the proof of the fact that $\mathrm{K}_0^*(A)^{++}$ is strict. We have $z_1 \in \mathcal{Z}_{++}$ by construction, and a straighforward functional calculus argument then shows that $b \otimes z_1 \in \mathrm{M}_\infty(A)_{++}$.

Suppose that $[a], [b] \in \mathrm{K}_0^*(A)^+$ are such that

$$(n+1)[a] \leq n[b], \quad \text{for some } n \in \mathbb{N}.$$

This means that there is $c \in M_\infty(A)_+$ such that $(n+1)[a] + [c] = n[b]$.

Assume that $a, b \in \mathrm{M}_\infty(A)_{++}$. By Lemma 5.1, we have $(n+1)d_\tau(a) + d_\tau(c) = nd_\tau(b)$, whence $d_\tau(a) + \frac{1}{n}d_\tau(a \oplus c) = d_\tau(b)$. Invoke Corollary 3.6 to find a (purely positive) element c' such that $\frac{1}{n}d_\tau(a \oplus c) = d_\tau(c')$. Now, Proposition 2.6 implies that $a \oplus c' \sim b$, whence $[a] + [c'] = [b]$. This shows that $\mathrm{K}_0^*(A)^+$ is almost unperforated. Apply Lemma 3.4 of [32] and the discussion thereafter to conclude that $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^+)$ is weakly unperforated. □

Note that if A is simple, then $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^+)$ is a simple group. This raises the question of whether $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++})$ will also be simple for a simple C*-algebra A. We give a criterion below to decide when a given (positive) element in $\mathrm{K}_0^*(A)^{++}$ is an order-unit. If $a \in M_\infty(A)_+$, write $n \cdot a$ to mean $a \oplus \cdots \oplus a$ (n times).

Proposition 5.4 *Let A be a unital, simple, stably finite, exact C*-algebra with strict comparison of positive elements. Suppose that $M_\infty(A)_{++}$ is non-empty. Then, an element $[a]-[b] \in \mathrm{K}_0^*(A)^{++}$ is an order-unit if and only if there is $\epsilon > 0$ such that $d_\tau(a) - d_\tau(b) > \epsilon$ for all traces τ.*

Proof If $[a] - [b]$ is an order-unit, then clearly $[a] \neq 0$. If $b = 0$ then

$$d_\tau(a) - d_\tau(b) = d_\tau(a) > 0$$

for each $\tau \in \mathrm{T}(A)$. The function $\tau \mapsto d_\tau(a)$ is lower semicontinuous on a compact set, and therefore achieves a minimum $\delta > 0$. Setting $\epsilon = \delta/2$ gives the desired conclusion.

Now suppose that $b \neq 0$. There is a natural number n such that $[a] \leq n[a] - n[b]$, hence we can find $c \in M_\infty(A)_+$ such that $a \oplus c \oplus n \cdot b \precsim n \cdot a \oplus c$. Therefore, for any $\tau \in \mathrm{T}(A)$, we have $d_\tau(a) + nd_\tau(b) \leq nd_\tau(a)$. Since $b \neq 0$ we conclude that

$$(n-1)(d_\tau(a) - d_\tau(b)) > d_\tau(b) > 0.$$

Using the same argument as in the $b = 0$ case, we conclude that there is some $\epsilon > 0$ such that $d_\tau(b) > (n-1)\epsilon$ for every $\tau \in \mathrm{T}(A)$. It follows that $d_\tau(a) - d_\tau(b) > \epsilon$, as desired.

Conversely, if $d_\tau(a) - d_\tau(b) > \epsilon$ for all τ, choose n such that $d_\tau(n \cdot a) - d_\tau(n \cdot b) = n(d_\tau(a) - d_\tau(b)) > 1 = d_\tau(1_A)$. Let $c \in M_\infty(A)_{++}$. Then

$$d_\tau(n \cdot a \oplus n \cdot c) - d_\tau(n \cdot b \oplus n \cdot c) > d_\tau(1_A),$$

whence $d_\tau(n \cdot a \oplus n \cdot c) > d_\tau(n \cdot b \oplus n \cdot c \oplus 1_A)$ for all τ. If follows now from Proposition 2.6 that $n \cdot b \oplus n \cdot c \oplus 1_A \precsim n \cdot a \oplus n \cdot c$. This implies that $n([a]-[b]) \geq [1_A]$, whence $[a] - [b]$ is an order-unit. □

Lemma 5.5 *Let A be a C*-algebra with stable rank one and such that the semigroup $W(A)_+$ of purely positive elements is non-empty. Then there exists an ordered group isomorphism*

$$\alpha \colon (\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}) \to (G(W(A)_+), G(W(A)_+)^+).$$

If, furthermore, A is simple and $\mathcal{Z}$-stable, then $\alpha([1_A]) = ([1 \otimes z_1])$.

Proof Recall from Sect. 2 that if A has stable rank one, then $W(A) = V(A) \sqcup W(A)_+$. Denote by $\gamma\colon W(A)_+ \to G(W(A)_+)$ the Grothendieck map, and choose any element $c \in W(A)_+$. Then, define

$$\alpha\colon W(A) \to G(W(A)_+)$$

by $\alpha(\langle a\rangle) = \gamma(\langle a\rangle)$ if $\langle a\rangle \in W(A)_+$, and by $\alpha(\langle p\rangle) = \gamma(\langle p\rangle + c) - \gamma(c)$ for any projection in $M_\infty(A)$.

Note that α is a well defined semigroup homomorphism. Indeed, since A has stable rank one, $\langle p\rangle + c \in W(A)_+$ whenever $c \in W(A)_+$ (Lemma 2.9), and if $c' \in W(A)_+$ is any other element, then one has that $\gamma(\langle p\rangle + c) - \gamma(c) = \gamma(\langle p\rangle + c') - \gamma(c')$.

In order to check that α is a homomorphism, let p, q and a be elements in $M_\infty(A)_+$ with p and q projections and a purely positive. Then,

$$\begin{aligned}\alpha(\langle p\rangle + \langle q\rangle) &= \gamma(\langle p \oplus q\rangle + 2c) - \gamma(2c)\\ &= \gamma(\langle p\rangle + c) - \gamma(c) + \gamma(\langle q\rangle + c) - \gamma(c)\\ &= \alpha(\langle p\rangle) + \alpha(\langle q\rangle).\end{aligned}$$

Also

$$\begin{aligned}\alpha(\langle p\rangle + \langle a\rangle) &= \gamma(\langle p \oplus a\rangle)\\ &= \gamma(\langle p \oplus a\rangle + c) - \gamma(c)\\ &= \gamma(\langle p\rangle + c) - \gamma(c) + \gamma(\langle a\rangle)\\ &= \alpha(\langle p\rangle) + \alpha(\langle a\rangle).\end{aligned}$$

By definition,

$$G(W(A)_+)^+ = \{[x] - [y] \mid x, y \in W(A)_+,\ y + r \le x + r \text{ for some } r \in W(A)_+\},$$

whence $\alpha(W(A)_+) = \gamma(W(A)_+) \subseteq G(W(A)_+)^+$ by construction. If $p \in \mathrm{M}_\infty(A)$ is a projection, then its image under alpha is $\gamma(\langle p\rangle + c) - \gamma(c)$. Since $c \le \langle p\rangle + c$, we conclude that $\alpha(\langle p\rangle) \in G(W(A)_+)^+$, too. Thus, $\alpha(W(A)) \subseteq G(W(A)_+)^+$, and so α extends to an ordered group homomorphism

$$\alpha\colon \mathrm{K}_0^*(A) = G(W(A)) \to G(W(A)_+),$$

given by the rule $\alpha([a] - [b]) = \alpha(\langle a\rangle) - \alpha(\langle b\rangle)$. Evidently, α is surjective and satisfies

$$\alpha(\mathrm{K}_0^*(A)^{++}) \subseteq G(W(A)_+)^+$$

To prove injectivity, assume that $\alpha(\langle a\rangle) = \alpha(\langle p\rangle)$ for $\langle a\rangle \in W(A)_+$ and p a projection. This means that $\gamma(\langle a\rangle) = \gamma(\langle p\rangle + c) - \gamma(c)$, and hence $\langle a\rangle + c + c' = \langle p\rangle + c + c'$ for some $c' \in W(A)$. Thus $[a] = [p]$ in $\mathrm{K}_0^*(A)$. If, for projections p

and q, we have that $\alpha(\langle p\rangle)=\alpha(\langle q\rangle)$, then $\gamma(\langle p\rangle+c)-\gamma(c)=\gamma(\langle q\rangle+c)-\gamma(c)$. It follows that $[p]=[q]$ in $\mathrm{K}_0^*(A)$.

Finally, if A is simple and $\mathcal{Z}$-stable, we may apply Proposition 2.6 to conclude that

$$(1_A\otimes 1_{\mathcal{Z}})\oplus(1_A\otimes z_1)\sim(1_A\otimes z_1)\oplus(1_A\otimes z_1).$$

Thus, there is an identification of A with $A\otimes\mathcal{Z}$ for which

$$\begin{aligned}\alpha([1_A])&=\gamma(\langle(1_A\otimes 1_{\mathcal{Z}})\oplus(1_A\otimes z_1)\rangle)-\gamma(\langle 1_A\otimes z_1\rangle)\\&=\gamma(\langle 1_A\otimes z_1\rangle)=\alpha([1_A\otimes z_1]).\end{aligned}$$

(Note that the 1_As on the far right and far left are, strictly speaking, not the same.) □

Corollary 5.6 *Let A be simple, unital, and exact C*-algebra having stable rank one and strict comparison of positive elements. Suppose further that $\mathrm{M}_\infty(A)\neq\emptyset$. Then, $\mathrm{K}_0^*(A)$ is the Grothendieck enveloping group of $\iota(W(A)_+)$, where ι is the map defined in Proposition 3.3.*

Proof Under the hypotheses, ι is an order-embedding (see Theorem 4.4). The result then follows from Lemma 5.5. □

Corollary 5.6 gives a version of Theorem III.3.2 of [3] for C*-algebras which may lack non-trivial projections.

We close this section summarizing our findings in the following:

Theorem 5.7 *Let A be a simple, unital, nuclear and finite C*-algebra which is $\mathcal{Z}$-stable. Then,*

(i) $(\mathrm{K}_0^*(A),\mathrm{K}_0^*(A)^{++})$ *is an Archimedean partially ordered Abelian group.*
(ii) $(\mathrm{K}_0^*(A),\mathrm{K}_0^*(A)^{+})$ *is a simple and weakly unperforated partially ordered Abelian group.*
(iii) $\mathrm{K}_0^*(A)=G(\iota(W(A)_+)$, *where* $\iota\colon W(A)_+\to\mathrm{LAff}_b(\mathrm{T}(A))^{++}$ *is defined as in 3.3.*

6 $\mathcal{Z}$-stable algebras with finitely many pure tracial states

In the final sections of the paper, we study the surjectivity of the order-embedding ι. In this section we study algebras which satisfy the hypotheses of Theorem 1.4 by way of having finitely many pure tracial states. We begin by establishing a closure property for the image of ι.

Lemma 6.1 *Let A be a simple, unital, exact, finite, and $\mathcal{Z}$-stable C*-algebra*:

$$A\overset{\phi}{\cong}A\otimes\mathcal{Z},$$

where ϕ is as in the proof of Lemma 3.4. Suppose that $a \in \mathrm{M}_\infty(A)_+$ is such that $d_\tau(a) \leq r$, for some $r \in \mathbb{R}^{++}$ and for all $\tau \in \mathrm{T}(A)$. Then, for any z in $\mathcal{Z}$ such that $z \sim z_r$, there exists $\tilde{a} \in \mathrm{M}_\infty(A)_+$ such that

$$a \sim \tilde{a} \leq (1_A \oplus 1_A) \otimes z \in \mathrm{M}_\infty(A \otimes \mathcal{Z})_+ \stackrel{\phi}{\equiv} \mathrm{M}_\infty(A)_+.$$

Proof We assume throughout the proof that whenever elementary tensors in $A \otimes \mathcal{Z}$ are mentioned, they are being identified with elements of A via ϕ.

Suppose first that $a \sim p$ for some projection $p \in \mathrm{M}_\infty(A)$. Since

$$d_\tau(a) \leq r < 2r = d_\tau((1_A \oplus 1_A) \otimes z), \quad \text{for all } \tau \in \mathrm{T}(A),$$

we have that $a \sim p \precsim (1_A \oplus 1_A) \otimes z$ by Proposition 2.7. Applying [29, Proposition 2.4] we may find $x \in \mathrm{M}_\infty(A)$ such that

$$x^* ((1_A \oplus 1_A) \otimes z) x = (p - \epsilon)_+ \sim p \sim a,$$

so that $\tilde{a} := (1_A \oplus 1_A)xx^*(1_A \oplus 1_A)$ has the desired properties.

Now assume that $a \in \mathrm{M}_\infty(A)_{++}$. Use Lemma 3.4 to find representative $a' \otimes 1_{\mathcal{Z}} \in A \otimes \mathcal{Z}$ of $\langle a \rangle$. Put $b := a' \otimes z_{1/r} \in \mathrm{M}_\infty(A \otimes \mathcal{Z})_+$, so that $d_\tau(b) \leq 1$. We now identify A with $A \otimes \mathcal{Z}$ via ϕ. Our hypotheses ensure that A has strict comparison of positive elements (Corollary 4.6 of [32]), whence $b \precsim 1_A$ by Proposition 2.6. We apply [29, Proposition 2.4] to $b + \epsilon \cdot 1_A \precsim b \oplus \epsilon \precsim 1_A \oplus 1_A$, and obtain $x \in \mathrm{M}_\infty(A)_+$ such that

$$x^*(1_A \oplus 1_A)x = (b + \epsilon - \epsilon)_+ = b.$$

It follows that

$$b \sim \tilde{b} := (1_A \oplus 1_A)xx^*(1_A \oplus 1_A) \leq \|x\|^2 1_A \oplus 1_A.$$

Now $(1/\|x\|^2)\tilde{b} \sim \tilde{b}$ — Cuntz equivalence is robust under multiplication by elements of $\mathbb{R}^{++}$—and so

$$b \sim (1/\|x\|^2)\tilde{b} \leq 1_A \oplus 1_A.$$

It follows that

$$(1/\|x\|^2)(\tilde{b} \otimes z) \leq (1_A \oplus 1_A) \otimes z,$$

and that

$$(1/\|x\|^2)(\tilde{b} \otimes z) \sim b \otimes z = (a' \otimes z_{1/r}) \otimes z$$

[32, Lemma 4.1]. Put $\tilde{a} := (1/\|x\|^2)(\tilde{b} \otimes z)$. The last equation shows that $d_\tau(\tilde{a}) = d_\tau(a)$, whence $a \sim \tilde{a}$ by Proposition 2.6. □

Proposition 6.2 *Let A be a simple, unital, exact, and finite C^*-algebra absorbing the Jiang–Su algebra $\mathcal{Z}$ tensorially. Let there be given a sequence $(a_i)_{i=1}^{\infty} \subseteq \mathrm{M}_{\infty}(A)_+$, and put*

$$h_i(\tau) := d_\tau(a_i); \quad g_i := \sum_{j=1}^{i} h_j.$$

If

$$\lim_{i\to\infty} g_i = g; \quad \sum_{i=1}^{\infty} \|h_i\| < \infty,$$

then there exists $a \in \mathrm{M}_{\infty}(A)_{++}$ such that $d_\tau(a) = g(\tau)$, for all $\tau \in \mathrm{T}(A)$.

Proof We may assume that $a_i \in \mathrm{M}_{\infty}(A)_{++}$, since $d_\tau(a_i) = d_\tau(a_i \otimes z_1)$, for all $\tau \in \mathrm{T}(A)$. We may also assume that $\sum_{i=1}^{\infty} \|h_i\| < 1$ by scaling the a_i (using Corollary 3.6).

Using the embedding of $C[0,1]$ into $\mathcal{Z}$ as in Theorem 3.1 we may choose, for each $i \in \mathbb{N}$, a representative y_i of $\langle z_{\|h_i\|}\rangle$ inside $\mathcal{Z}$ such that $y_i y_j = y_j y_i = 0$ for all $i \neq j$. By Lemma 6.1, a_i is equivalent to $\tilde{a}_i \leq (1_A \oplus 1_A) \otimes y_i$. It follows that the $\tilde{a}_i$s are pairwise orthogonal, and that $d_\tau(\tilde{a}_i) = h_i$. Put

$$a := \sum_{i=1}^{\infty} \frac{1}{2^i} \tilde{a}_i \in \mathrm{M}_2(A \otimes \mathcal{Z}).$$

Then, $d_\tau(a) = g(\tau)$, as desired. □

Let A be a C*-algebra with finitely many pure tracial states. In this situation we make the identifications

$$\mathrm{LAff}_b(\mathrm{T}(A))^{++} \equiv \mathrm{Aff}(\mathrm{T}(A))^{++} \equiv \{(\lambda_1, \ldots, \lambda_n) | \lambda_i \in \mathbb{R}^{++}\},$$

where n is the number of pure tracial states on A. Now suppose further that A is simple, unital, exact, finite, and $\mathcal{Z}$-stable. Since $\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ is an order-embedding, we know (using [4, Theorem 2.6]) that $S((\mathbb{R}^{++})^n, 1)$ maps surjectively onto $S(W(A)_+, \langle 1 \otimes z_1\rangle)$, which by Lemma 5.5 agrees with

$$S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}, [1_A]) \stackrel{\text{def}}{=} \mathrm{DF}(A).$$

Remark 6.3 The definition of the term "state" is different for partially ordered Abelian semigroups and partially ordered Abelian groups. For semigroups a state must be order preserving, while for groups it is required to be positive. Both definitions require the state to be a linear map into $\mathbb{R}$ taking the order unit to 1. With this in mind it is easy to check that the states on $(W(A), \langle 1_A\rangle)$ coincide with the states on $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}, [1_A])$.

Now, if τ is an extremal trace, then the corresponding lower semicontinuous function d_τ is an extreme point in $\mathrm{DF}(A)$. This follows from the fact that $\mathrm{LDF}(A)$ is a face of $\mathrm{DF}(A)$ [3, Proposition II.4.6] and the fact that $\tau \mapsto d_\tau$ is an affine bijection from $\mathrm{T}(A)$ onto $\mathrm{LDF}(A)$. In our case of interest, where we have exactly n extreme traces, we find counting dimensions that $S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}, [1_A]) \cong \mathbb{R}^n$. It follows from Corollary 5.2 and [14, Theorem 4.14] that $\mathrm{K}_0^*(A) \cong \mathbb{R}^n$ in this case.

Next, from the obvious containment

$$S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}, [1_A]) \subseteq S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{+}, [1_A])$$

and the fact that $\mathrm{K}_0^*(A) \cong \mathbb{R}^n$, we see that in fact we have equality.

We shall need the following result (see, e.g. [14, Theorem 7.9]):

Theorem 6.4 *Let* (G, u) *be an unperforated partially ordered Abelian group with order-unit, and let*

$$\psi\colon G \to \mathrm{Aff}(S(G, u))$$

be the natural map (*given by evaluation*). *Then, the set*

$$\{\psi(x)/2^n \mid x \in G^+,\ n \in \mathbb{N}\}$$

is dense in $\mathrm{Aff}(S(G, u))^+$.

Inspection of the proof reveals that the same result will hold under the assumption that G is simple and weakly unperforated, which is what we shall use below.

Theorem 6.5 *Let A be an exact, simple, and unital C^*-algebra absorbing the Jiang–Su algebra $\mathcal{Z}$ tensorially. Suppose that A has n pure tracial states. Then,* $\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ *is surjective.*

Proof From the comments preceding Theorem 6.4, it follows that the state space of the group $\mathrm{K}_0^*(A)$ is $\mathbb{R}^n$, no matter which ordering we consider on it (either $\mathrm{K}_0^*(A)^+$ or $\mathrm{K}_0^*(A)^{++}$). Therefore,

$$\mathrm{Aff}(S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{+}, [1_A])) = \mathrm{Aff}(S(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{++}, [1_A])) = \mathrm{LAff}_b(\mathrm{T}(A)).$$

We also know from Proposition 5.3 that $(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^{+}, [1_A])$ is a weakly unperforated partially ordered simple abelian group. Our considerations above together with Theorem 6.4 imply that

$$\{\iota(a)/2^n \mid a \in \mathrm{M}_\infty(A)_{++},\ n \in \mathbb{N}\}$$

is dense in $\mathrm{LAff}_b(\mathrm{T}(A))$. But $\iota(a)/2^n = \iota(a \otimes z_{1/2^n})$ by Corollary 3.6, so

$$\{\iota(a)/2^n \mid a \in \mathrm{M}_\infty(A)_{++},\ n \in \mathbb{N}\} = \{\iota(a) \mid a \in \mathrm{M}_\infty(A)_{++}\}.$$

In other words, the image of ι in $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$ is dense.

Let $f \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ be given. A moment's reflection shows that one may choose a sequence $(h_i)_{i=1}^\infty \subseteq \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ with the following properties:

(i) $\lim_{i\to\infty} f_i = f$, where $f_i = \sum_{j=1}^{i} h_j$;
(ii) $\sum_{i=1}^\infty \|h_i\| < \infty$;
(iii) $h_i(\tau) = d_\tau(a_i)$ for some $a_i \in \mathrm{M}_\infty(A)_{++}$.

We may apply Proposition 6.2 to find $a \in \mathrm{M}_\infty(A)_{++}$ such that $d_\tau(a) = f(\tau)$, for all $\tau \in \mathrm{T}(A)$, whence ι is surjective, as desired. □

7 Real rank zero

In this section we show that our map ι is surjective whenever A is a $\mathcal{Z}$-stable, simple, exact C*-algebra with real rank zero and stable rank one. In fact, we can prove a more general result, namely that for such an A (not necessarily simple) $\mathrm{K}_0^*(A)$ is order-isomorphic to the group of differences of lower semicontinous, affine, real-valued and bounded functions defined on $\mathrm{T}(A)$, equipped with the pointwise ordering. Some of our arguments, namely the first part of Theorem 7.3 below, can be traced back to the ones in [3], and we include them for the convenience of the reader.

It should be no surprise, however, that the (WEC) implies the (EC) for this class. This can be justified by recalling that the Cuntz semigroup $W(A)$ is completely determined by $V(A)$ whenever A is σ-unital, has real rank zero and stable rank one. More concretely, one can obtain for such an A an order-isomorphism between $W(A)$ and the monoid of the so-called countably generated intervals in $V(A)$ that are bounded by the generating interval $D(A)$ (see [25] for a full account).

Given a positively ordered abelian semigroup with order-unit $(M, \leq, u)$, consider the natural representation map $\phi_u\colon M \to \mathrm{Aff}(S(M,u))^+$. It is said that M satisfies *condition (D)* provided that $\phi_u(M)$ is dense. A unital C*-algebra A satisfies condition (D) provided that the positive cone $\mathrm{K}_0(A)^+$ of its Grothendieck group satisfies condition (D). It was shown in [24] that any unital C*-algebra A with real rank zero satisfies condition (D) if and only if A has no finite dimensional representations.

Lemma 7.1 *Let A be a $\mathcal{Z}$-stable unital C*-algebra with stable rank one. Then $s(x) > 0$ for all states s on $S(\mathrm{K}_0(A), [1_A])$ if and only if x is an order-unit in $\mathrm{K}_0(A)$.*

Proof Since A has stable rank one, we have $\mathrm{K}_0(A)^+ = V(A)$. We also know from [32, Corollary 4.8] that $V(A)$ is almost unperforated. Assume that $s(x) > 0$ for all states s. It then follows from [14, Theorem 4.12] that mx is an order-unit for some natural number m. Write $x = a - b$ where $a, b \in V(A)$. We know that there is l in $\mathbb{N}$ such that $b \leq lm(a-b)$, and hence $(lm+1)b \leq lma$. Therefore $b \leq a$, and so $x > 0$. Thus x is an order-unit. □

If f, g are real-valued functions defined on a set X, write $f \gg g$ (or $f \ll g$) to mean that $f(x) > g(x)$ (or $f(x) < g(x)$) for every x in X.

Lemma 7.2 *Let A be a $\mathcal{Z}$-stable unital C*-algebra with real rank zero and stable rank one. Then A contains a sequence of orthogonal projections (p_n) such that $s([p_n]) > 0$ for all states $s \in S(V(A), [1_A])$. (Equivalently, $\tau(p_n) > 0$ for all quasitraces on A.)*

Proof (Outline.) Note first that $A \cong A \otimes \mathcal{Z}$ satisfies condition (D), because $\mathcal{Z}$ is simple and infinite dimensional. Denote by $u = [1_A] \in V(A)$ and by

$$\phi_u \colon V(A) \to \mathrm{Aff}(S(V(A), u)) = \mathrm{Aff}(S(\mathrm{K}_0(A), u))$$

the natural representation map, given by evaluation.

Using condition (D) we may then find a projection p_1 such that $0 \ll \phi_u([p_1]) \ll 1$. Thus, by compactness of the state space of $V(A)$ and condition (D) again, there is a projection p_2' satisfying $0 \ll \phi_u([p_2']) \ll \phi_u([1-p_1])$. Lemma 7.1 implies that $p_2' \sim p_2 \leq 1 - p_1$ for some projection p_2. Continuing in this way we find our sequence of projections (p_n).

The equivalent statement follows readily from the fact that the map $\mathrm{QT}(A) \to S(V(A), [1_A])$, given by evaluation, is an affine homeomorphism (see [3, Theorem III.1.3]). □

We remark that Lemma 7.2 also holds trading $\mathcal{Z}$-stability and stable rank one by weak divisibility. This latter property was introduced in [27]: a C*-algebra A is *weakly divisible* if for any element x in $V(A)$, we may find a natural number n and elements y and z in $V(A)$ such that $x = ny + (n+1)z$. Weak divisibility is always guaranteed for simple (non-type I) C*-algebras of real rank zero, and holds quite widely in the non-simple case (see [27, Theorem 5.8]). Basically, what we need to use to establish 7.2 in this setting is that for a non-zero x in $V(A)$, there is n and a non-zero y in $V(A)$ such that $ny \leq x \leq (n+1)y$.

Theorem 7.3 (cf. [3, Theorem III.3.2 and Corollary III.3.3]) *Let A be a $\mathcal{Z}$-stable, exact, separable and unital C*-algebra with real rank zero and stable rank one. Then $\mathrm{K}_0^*(A)$ is order-isomorphic to $G(\mathrm{LAff}_b(\mathrm{T}(A))$, equipped with the pointwise ordering.*

Proof Define $\iota \colon \mathrm{K}_0^*(A) \to G(\mathrm{LAff}_b(\mathrm{T}(A)))$ by $\iota([a])(\tau) = d_\tau(a)$. Note first that, for a positive element a, if (p_n) is an (increasing) approximate unit consisting of projections for the hereditary algebra generated by a, we have that $\iota([a])(\tau) = \sup_n \tau(p_n)$.

In order to get an order-isomorphism onto the image, we have to show that $[a] \leq [b]$ in $\mathrm{K}_0^*(A)$ whenever $\iota([a]) \leq \iota([b])$. Let (p_n) be the sequence of orthogonal projections constructed in Lemma 7.2, and let $c = \sum_{n=1}^{\infty} \frac{1}{2^n} r_n \in A_+$, where

$r_n = \sum_{i=1}^{n} p_i$. Let (e_n) and (f_n) be approximate units consisting of projections for the hereditary algebras generated by a and b, respectively. We then have that $(e_n \oplus r_n)$ (respectively, $(f_n \oplus r_n)$) is an (increasing) approximate unit consisting of projections for $a \oplus c$ (respectively, for $b \oplus c$). Note that $\iota([a \oplus c]) \leq \iota([b \oplus c])$. By construction of the sequence (r_n) and Lemma 7.2, the sequence $\tau(e_n \oplus r_n)$ is strictly increasing. Using compactness of the state space of $V(A)$, we find that for all n, there is m such that $\tau(e_n \oplus r_n) < \tau(f_m \oplus r_m)$ for all τ. It follows again from Lemma 7.2 that for all n, there is m such that $e_n \oplus r_n \precsim f_m \oplus r_m$. But this implies that $a \oplus c \precsim b \oplus c$ (see [25, Proposition 2.3] and also [3, Corollary III.3.8]).

We now prove that ι is surjective. Let $f \in \mathrm{LAff}_b(\mathrm{T}(A))$, which is bounded below by some constant k. Writing $h = f - k + 1$, we may assume that actually $f \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$. Since A is separable, we have that $\mathrm{T}(A)$ is metrizable, hence we may write f as a pointwise supremum of an increasing sequence (f_n) of functions in $\mathrm{Aff}(\mathrm{T}(A))^{++}$. Choose n_0 such that $f_n - \frac{1}{2^n} \gg 0$ whenever $n \geq n_0$. Write $u = [1_A] \in V(A)$ and denote as before ϕ_u the natural representation map.

Using condition (D) we may find projections p_n in $M_\infty(A)$ such that $f_n - \frac{1}{2^n} \ll \phi_u([p_n]) \ll f_n - \frac{1}{2^{n+1}}$ for all $n \geq n_0$, where $u = [1_A] \in V(A)$. Since $\phi_u([p_n]) \ll \phi_u([p_{n+1}])$ we get from Lemma 7.1 that $[p_n] \leq [p_{n+1}]$ in $V(A)$. Since f is also bounded, a second use of Lemma 7.1 shows that p_n all belong to $M_t(A)$ for some t. Using that A has stable rank one (whence projections cancel from direct sums) we may arrange that the sequence (p_n) is indeed increasing in the order of A.

It is clear that f, being the pointwise supremum of the f_n's, will satisfy that $f = \sup \phi_u([p_n])$. We know from [3, Theorem III.1.3] that the natural mapping $\mathrm{T}(A) \to S(\mathrm{K}_0(A), [1_A])$ is a homeomorphism.

If we then let $x = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n$, we find that $x \otimes z_1$ is a purely positive element in $M_t(A)$ such that $d_\tau(x \otimes z_1) = d_\tau(x) = \sup_n d_\tau(p_n) = \sup \tau(p_n) = \phi_u([p_n])(\tau) = f(\tau)$ for every $\tau \in \mathrm{T}(A)$. □

The argument of surjectivity in the proof of Theorem 7.3, allows us to state the following:

Corollary 7.4 *Let A be an exact, simple, and unital C*-algebra absorbing the Jiang–Su algebra $\mathcal{Z}$ tensorially. Suppose that A has real rank zero and stable rank one. Then, $\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ is surjective.*

8 Goodearl algebras

In this section we prove that ι is surjective for algebras we term *degenerate Goodearl algebras*, and outline a proof of the same fact for the simple Goodearl algebras studied in [15]. We do so to support the conjecture that ι is always surjective for unital and stably finite C*-algebras without nonzero finite-dimensional representations. In other words, hypotheses (i) and (ii) of Theorem 1.4

should be removeable. (Note that for a non-simple algebra, the image of ι will not always consist of strictly positive functions.) Our reasons for providing a sketch in lieu of a full proof in the simple case are twofold: first, the main ideas and technical details for a full proof are contained already in the argument for the degenerate case; second, simple Goodearl algebras are known to satisfy the Elliott conjecture, and so one gains little new insight into their structure by computing their Cuntz semigroups.

Let X and Y be compact Hausdorff spaces. A $*$-homomorphism

$$\phi\colon \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$$

is called *diagonal* if

$$\phi(f)(y) = \mathrm{diag}\,(f(\gamma_1(y)), \ldots, f(\gamma_n(y)))$$

for continuous maps $\gamma_i\colon Y \to X, 1 \leq i \leq n$. The γ_i are called *eigenvalue maps*.

Let X be a nonempty, separable, and compact Hausdorff space. Let $A = \lim_{i\to\infty}(A_i, \phi_i)$ be a unital inductive limit C*-algebra where, for each $i \in \mathbb{N}$, $A_i \cong \mathrm{M}_{n_i}(\mathrm{C}(X))$ for some $n_i \in \mathbb{N}$ with $n_i | n_{i+1}$, ϕ_i is diagonal, and the eigenvalue maps of ϕ_i are either the identity map on X, or have range equal to a point. Such an algebra will be called a *Goodearl algebra*. This definition generalises slightly the one provided by Goodearl in [15].

If each ϕ_i in the inductive sequence for A has every eigenvalue map equal to the identity map on X, then we will say that A is *degenerate*. In this case one obtains a (in general non-simple) algebra isomorphic to the tensor product $\mathrm{C}(X) \otimes \mathfrak{U}$, where $\mathfrak{U}$ is the UHF algebra whose K_0-group is the subgroup of the rationals whose denominators, when in lowest terms, divide some n_i. This subgroup is dense in $\mathbb{R}$ whenever $n_i \to \infty$ as $i \to \infty$. In this case, $\mathrm{T}(A)$ may be identified with the Bauer simplex $M_1^+(X)$ of positive probability measures on X, hence its extreme boundary $\partial_e \mathrm{T}(A)$ is homeomorphic to X. (Recall that a Bauer simplex is a Choquet simplex with closed extreme boundary—see [1] for details.)

If X is a compact Hausdorff space, denote by $\mathrm{L}(X)$ the semigroup of lower semicontinuous real-valued functions defined on X, by $\mathrm{L}(X)^{++}$ the subsemigroup of $\mathrm{L}(X)$ consisting of strictly positive elements, and by $\mathrm{L}_b(X)$ the subsemigroup of bounded functions. Let A be a unital C*-algebra such that $\mathrm{T}(A)$ is a non-empty Bauer simplex. Then, there is a semigroup isomorphism between $\mathrm{LAff}_b(\mathrm{T}(A))$ and $\mathrm{L}_b(\partial_e \mathrm{T}(A))$—the behaviour of $f \in \mathrm{LAff}(\mathrm{T}(A))$ is determined by the behaviour of its restriction to $\partial_e \mathrm{T}(A)$ (cf. [16, Lemma 7.2]). It follows that proving the surjectivity of ι for such an algebra only requires proving that every $f \in \mathrm{L}_b(\partial_e \mathrm{T}(A))^{++}$ can be realised as the image of some $a \in \mathrm{M}_\infty(A)_{++}$ under the map

$$\iota_e\colon W(A)_+ \to \mathrm{L}_b(\partial_e \mathrm{T}(A))^{++}$$

given by

$$\iota_e(\langle a\rangle) = d_\tau(a), \quad \text{for all } \tau \in \partial_e \mathrm{T}(A).$$

Clearly, it will suffice to prove the above for functions f such that $\|f\| \leq 1$.

Theorem 8.1 *Let A be a degenerate Goodearl algebra. Then ι is surjective.*

Proof We identify $\mathrm{T}(A)$ with the Bauer simplex $M_1^+(X)$, whence $\partial_e(\mathrm{T}(A))$ is homeomorphic to X. Let us write τ_x for the trace that corresponds to a point x in X. This, in turn, corresponds to the point mass measure δ_x at x.

Let $f \in \mathrm{L}_b(X)^{++}$ be given, and assume that $\|f\| \leq 1$. We prove that f is the image of an element $a \in A_+$ under the map ι_e defined above.

Define, for each $i \in \mathbb{N}$, a function f_i as follows: put

$$F_{i,k} := \left\{ x \in X \mid f(x) \leq \frac{k}{n_i} \right\}, \quad 1 \leq k \leq n_i,$$
$$f_i(x) = 0, \quad \text{for all } x \in F_{1,k},$$

and

$$f_i(x) := \frac{k-1}{n_i} \quad \text{whenever } x \in F_{i,k} \setminus F_{i,k-1}.$$

Let us check that f_i converges pointwise to f, and that $f_j \geq f_i$ whenever $j \geq i$. Let $x \in F_{i,k} \setminus F_{i,k-1}$ for $1 \leq k \leq n_i$, and take $j \geq i$. Then $f_i(x) = \frac{k-1}{n_i}$. Write $n_j = n_i n_i'$, and note that $f(x) \leq \frac{k}{n_i} = \frac{kn_i'}{n_j}$. Thus $x \in F_{j,kn_i'}$. Let $l \geq 0$ be such that $x \in F_{j,kn_i'-l} \setminus F_{j,kn_i'-l-1}$. Since

$$\frac{k-1}{n_i} < f(x) \leq \frac{kn_i' - l}{n_j},$$

it is easy to check now that $f_j(x) = \frac{kn_i'-l-1}{n_j} \geq \frac{k-1}{n_i} = f_i(x)$.

Note that for $x \in F_{i,k} \setminus F_{i,k-1}$ we have $f(x) - f_i(x) \leq \frac{1}{n_i}$, whence clearly $f_i \to f$.

We will construct an increasing sequence $a_1 \leq a_2 \leq \cdots$ of positive elements in A converging to a positive element a, such that $d_{\tau_x}(a_i) = f_i(x)$, for all $x \in X$. It will follow that $d_{\tau_x}(a) = f(x)$, for all $x \in X$.

For each $i \in \mathbb{N}$, choose n_i continuous functions $f_{i,k}\colon X \to [0, 1/2^i]$ as follows: $f_{i,1} \equiv 0$, and $f_{i,k}$ is supported on the open set $F^c_{i,k-1}$, for $2 \leq k \leq n_i$. Put

$$\tilde{a}_i := \mathrm{diag}(f_{i,1}, \ldots, f_{i,n_i}) \in A_i.$$

Define $a_1 := \tilde{a}_1 \in A_1$. Suppose that we have constructed $a_1, \ldots, a_i$ such that $a_j \in A_j$ and also $a_1 \leq a_2 \leq \cdots \leq a_i$ when viewed in A_i (through the natural maps). We now construct a_{i+1}.

Consider the image of a_i in A_{i+1} under ϕ_i. It is a diagonal element, and its diagonal entries consist of n_{i+1}/n_i copies of $f_{i,k}$ for each $1 \leq k \leq n_i$. Now, for any such k, notice that the open set $F_{i,k}^c$ is contained in $F_{i+1,l}^c$ for every $(k-1)(n_{i+1}/n_i) + 1 \leq l \leq k(n_{i+1}/n_i)$. Assume, by permuting the diagonal entries of $\tilde{a}_{i+1}$ if necessary, that the entries of $\phi(a_i)$ equal to $f_{i,k}$ correspond to the entries of $f_{i+1,l}$ of $\tilde{a}_{i+1}$ for which $(k-1)(n_{i+1}/n_i) + 1 \leq l \leq k(n_{i+1}/n_i)$. Now define a_{i+1} to be the diagonal element whose entries are the pointwise maximum of the entries of $\phi_i(a_i)$ and $\tilde{a}_{i+1}$.

Since $F_{i,k}^c \subseteq F_{i+1,l}^c$, we have that $\mathrm{Coz}(\max\{f_{i,k}, f_{i+1,l}\}) = \mathrm{Coz}(f_{i+1,l}) = F_{i+1,l}^c$ ($\mathrm{Coz}(f)$ denotes the cozero set of a function f). For any $x \in X$, we have

$$d_{\tau_x}(a_{i+1}) = d_{\tau_x}(\tilde{a}_{i+1}) = \frac{1}{n_{i+1}} \sum_{j=1}^{n_{i+1}} \delta_x(F_{i+1,j}^c) = \frac{k}{n_{i+1}},$$

where k is such that $x \in F_{i+1,k}^c \setminus F_{i+1,k+1}^c$. Hence $d_{\tau_x}(a_{i+1}) = f_{i+1}(x)$. Observe that $\phi_i(a_i) \leq a_{i+1}$ and $\|a_i - a_{i-1}\| < 1/2^i$ by construction.

Continue in this way and identify the a_i's with their images in A. Then the sequence $(a_i)_{i=1}^{\infty} \subseteq A$ has the following properties:

(i) $a_i \leq a_{i+1}$ for all i;
(ii) $\|a_i - a_{i-1}\| < 1/2^i$;
(iii) $d_{\tau_x}(a_i) = f_i(x)$, for all $x \in X$.

It follows that $a := \lim_{j\to\infty} a_j$ has the desired property:

$$d_{\tau_x}(a) = f(x), \quad \text{for all } x \in X.$$

□

Simple Goodearl algebras [15] are either of real rank zero or real rank one, and are known to be approximately divisible (see [11]). It follows from Theorem 2.3 of [36] that they are $\mathcal{Z}$-stable, and so, in the real rank zero case, the surjectivity of ι is given by Corollary 7.4. In the real rank one case, it is known that the connecting $*$-homomorphisms ϕ_i in the inductive sequence for the given algebra must contain a vanishingly small proportion of eigenvalue maps with range equal to a point—the connecting maps are very nearly those of a degenerate Goodearl algebra [15]. Combining this fact with the construction of Theorem 8.1 yields the surjectivity of ι for simple Goodearl algebras of real rank one. The details are left to the reader.

9 Concluding remarks

Although $\mathcal{Z}$-stability is a useful tool in the proofs of Theorem 6.5 and Corollary 7.4, it is by no means a necessary condition for the surjectivity of ι.

A calculation akin to the proof of Theorem 8.1 shows that ι is surjective for the non-$\mathcal{Z}$-stable AH algebra constructed in Theorem 1.1 of [34]. Also:

Proposition 9.1 *Let A be a unital C*-algebra with unique tracial state τ. Suppose that there exists $a \in A^+$ such that* $\mathrm{Sp}(a) = [0, 1]$*, and that τ induces an atom-free measure on* $\mathrm{Sp}(a)$*. Then, ι is surjective.*

Proof We need only produce, for every $\lambda \in (0, 1]$, positive elements $g_\lambda \in A$ such that $d_\tau(g_\lambda) = \lambda$. This is straightforward: let O_λ be an open set of measure λ with respect to τ (such a set exists since said measure is an atom-free probability measure on $[0, 1]$), and let g_λ be a positive function supported on O_λ. □

The results of Sects. 6, 7, and 8 suggest a closing question:

Question 9.2 *Is ι surjective for any unital and stably finite C*-algebra A having no nonzero finite-dimensional representations?*

An affirmative answer will extend the equivalence of (EC) and (WEC) to all simple, separable, unital, nuclear, finite, and $\mathcal{Z}$-stable C*-algebras.

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