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Stability in the Cuntz semigroup of a commutative C*-algebra

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Abstract

Let A be a C*-algebra. The Cuntz semigroup $W(A)$ is an analogue for positive elements of the semigroup $V(A)$ of Murray–von Neumann equivalence classes of projections in matrices over A. We prove stability theorems for the Cuntz semigroup of a commutative C*-algebra which are analogues of classical stability theorems for topological vector bundles over compact Hausdorff spaces.

Let $\mathcal{SDG}$ denote the class of simple, unital, and infinite-dimensional AH algebras with slow dimension growth, and let A be an element of $\mathcal{SDG}$. We apply our stability theorems to obtain the following:

(i) A has strict comparison of positive elements;
(ii) $W(A)$ is recovered functorially from the Elliott invariant of A;
(iii) the lower semicontinuous dimension functions on A are weak-$*$ dense in the dimension functions on A;
(iv) the dimension functions on A form a Choquet simplex.

Statement (ii) confirms a conjecture of Perera and the author, while statements (iii) and (iv) confirm, for $\mathcal{SDG}$, conjectures of Blackadar and Handelman from the early 1980s.

1. *Introduction*

In 1978, Cuntz introduced a generalisation of Murray–von Neumann comparison: given positive elements a and b in a C*-algebra A, write $a \precsim b$ if there is a sequence $(v_i)_{i=1}^{\infty}$ in A such that

$$\|v_i b v_i^* - a\| \xrightarrow{i\to\infty} 0$$

[7]. We say that a is *Cuntz subequivalent* to b. The relation $\sim$ given by

$$a \sim b \quad \Longleftrightarrow \quad a \precsim b \ \wedge \ b \precsim a$$

is an equivalence relation known as *Cuntz equivalence*. If A is unital, then one can mimic the construction of the ordered semigroup $V(A)$ of Murray–von Neumann equivalence classes of projections in matrices over A by substituting positive elements for projections and Cuntz equivalence for Murray–von Neumann equivalence. This yields a positively ordered Abelian monoid $W(A)$ called the *Cuntz semigroup* of A, and its partially ordered Grothendieck envelope $\mathrm{K}_0^*(A)$. If A is stably finite, then each tracial state on A gives rise to an order-preserving state on $W(A)$. If the partial order on $W(A)$ is determined by these states, then A is said to have *strict comparison of positive elements*, or simply *strict comparison*.

The study of the Cuntz semigroup has been resurgent of late. Rørdam has proved that simple, unital, exact, and finite C*-algebras which absorb the Jiang–Su algebra tensorially have strict comparison [21], whence, by results of W. Winter and the author, this last property is enjoyed by all classes of simple, separable, and nuclear C*-algebras currently known to satisfy Elliott's classification conjecture [25, 26]. Coward, Elliott, and Ivanescu have recently identified a category of partially ordered semigroups into which the Cuntz semigroup is a continuous

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functor with respect to inductive limits. Significantly, the Cuntz semigroup is deeply connected to the classification programme for separable and nuclear C*-algebras: such algebras cannot be classified by their K-theory in the absence of strict comparison [22], and there is some evidence that the converse will hold [27]. Brown, Perera, and the author recently proved a structure theorem for the Cuntz semigroup which applies to most of our stock-in-trade simple, separable, and nuclear C*-algebras [6, 20].

In this paper we study the structure of the Cuntz semigroup for commutative and approximately homogeneous (AH) C*-algebras. Our main result is a positive answer to the following question, raised in [23].

Question 1.1. Does there exist a constant $K > 0$ such that for any compact metrisable Hausdorff space X, $n \in \mathbb{N}$, and positive elements $a, b \in \mathrm{M}_n(\mathrm{C}(X))$ for which

$$\mathrm{rank}(a)(x) + K\dim(X) \leqslant \mathrm{rank}(b)(x), \qquad \text{for all } x \in X,$$

one has $a \precsim b$?

This question asks for an analogue in positive elements of the following well-known stability theorem for vector bundles.

Theorem 1.2 [17, Chapter 8, Theorems 1.2 and 1.5]. *Let X be a compact metrisable Hausdorff space, and let ω and ξ be complex vector bundles over X. If the fibre dimension of ω exceeds the fibre dimension of ξ by at least $\lceil \dim(X)/2 \rceil$ over every point in X, then ξ is isomorphic to a sub-bundle of ω.*

Our positive answer to Question 1.1 has several applications to AH algebras. Recall that an AH algebra is an inductive limit C*-algebra $A = \lim_{i\to\infty}(A_i, \phi_i)$, where

$$A_i = \bigoplus_{l=1}^{n_i} p_{i,l}(\mathrm{C}(X_{i,l}) \otimes \mathcal{K})p_{i,l} \tag{1.1}$$

for compact connected Hausdorff spaces $X_{i,l}$, projections $p_{i,l} \in \mathrm{C}(X_{i,l}) \otimes \mathcal{K}$, and natural numbers n_i. If A is separable, then one may assume that the $X_{i,l}$ are finite CW-complexes [1, 16]. The algebras A_i are called *semi-homogeneous*, and the inductive system (A_i, ϕ_i) is referred to as a *decomposition* for A. All AH algebras in this paper are assumed to be separable.

If an AH algebra A admits a decomposition as in (1.1) for which

$$\max_{1\leqslant l\leqslant n_i} \left\{ \frac{\dim(X_{i,1})}{\mathrm{rank}(p_{i,1})}, \ldots, \frac{\dim(X_{i,n_i})}{\mathrm{rank}(p_{i,n_i})} \right\} \xrightarrow{i\to\infty} 0,$$

then we say that A has *slow dimension growth*. Let $\mathcal{SDG}$ denote the class of simple, unital, and infinite-dimensional AH algebras with slow dimension growth. The class $\mathcal{SDG}$ was studied intensively during the early to mid 1990s by Blackadar, Bratteli, Dădărlat, Elliott, Gong, Kumjian, Li, Rørdam, Thomsen, and others (see [2, 3, 5, 8–11, 13, 15]). The crowning achievement of this study was the confirmation of Elliott's classification conjecture for $\mathcal{SDG}$ under the additional assumption of real rank zero [8, 10, 15], and the same confirmation in the real-rank-one case under the stronger hypothesis of *very slow dimension growth* for A [13]:

$$\max_{1\leqslant l\leqslant n_i} \left\{ \frac{\dim(X_{i,1})^3}{\mathrm{rank}(p_{i,1})}, \ldots, \frac{\dim(X_{i,n_i})^3}{\mathrm{rank}(p_{i,n_i})} \right\} \xrightarrow{i\to\infty} 0.$$

This strengthened hypothesis is thought by experts to be unnecessary, but there has been no progress on this problem. In fact, there has been no progress on the basic structure of elements of $\mathcal{SDG}$ since the late 1990s.

In this paper we obtain significant new results on the structure of slow dimension growth AH algebras. We use our positive answer to Question 1.1 to prove that if $A \in \mathcal{SDG}$, then A has strict comparison of positive elements. There is evidence that this powerful K-theoretic condition will characterise those simple, separable, and nuclear C*-algebras which are amenable to K-theoretic classification — it already does so for a class of AH algebras intersecting substantially with $\mathcal{SDG}$ (see [27]) — and so our result supports the belief that $\mathcal{SDG}$ will satisfy the Elliott conjecture without the assumption of very slow dimension growth. By appealing to our recent work with Brown and Perera [6], we confirm several conjectures pertaining to the structure of the Cuntz semigroup for $A \in \mathcal{SDG}$. First, $W(A)$ is recovered functorially from the Elliott invariant of A, allowing one to attack the Elliott conjecture for $\mathcal{SDG}$ with considerably more information than would appear to be contained in the Elliott invariant alone. (This functorial recovery was conjectured for a class of algebras containing $\mathcal{SDG}$ by Perera and the author in [20].) Second, the states on $W(A)$ coming from traces on A are weak-$*$ dense in the state space of $W(A)$. Third, the state space of $W(A)$ is a Choquet simplex. The results concerning the state space of $W(A)$ were conjectured by Blackadar and Handelman in 1982 (see [4]), but were only known to hold for commutative C*-algebras at the time.

In a separate paper we will apply our positive answer to Question 1.1 to answer, affirmatively, a question of N. C. Phillips: 'Do there exist simple, separable, nuclear, and non-$\mathcal{Z}$-stable C*-algebras which agree on the Elliott invariant, yet are not isomorphic?' (See [24].)

This paper is organised as follows: Section 2 contains the definition of the Cuntz semigroup and recalls some essential facts about Cuntz subequivalence; Section 3 contains the stability theorems which answer Question 1.1; in Section 4 we establish results (i)–(iv) of the abstract.

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2. *Preliminaries*

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. If $A = \mathbb{C}$, then we simply write M_n. Let $\mathrm{M}_\infty(A)$ denote the algebraic limit of the direct system $(\mathrm{M}_n(A), \phi_n)$, where

$$\phi_n : \mathrm{M}_n(A) \longrightarrow \mathrm{M}_{n+1}(A)$$

is given by

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathrm{M}_\infty(A)_+$ and $\mathrm{M}_n(A)_+$ denote the positive elements in $\mathrm{M}_\infty(A)$ and $\mathrm{M}_n(A)$ respectively. Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\|v_n b v_n^* - a\| \xrightarrow{n \to \infty} 0.$$

We say that a and b are *Cuntz equivalent* (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. This relation is an equivalence relation, and we write $\langle a \rangle$ for the equivalence class of a. The set

$$W(A) := \mathrm{M}_\infty(A)_+/\sim$$

becomes a positively ordered Abelian monoid when equipped with the operation

$$\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$$

and the partial order

$$\langle a\rangle \leqslant \langle b\rangle \quad \Longleftrightarrow \quad a \precsim b.$$

In the sequel, we refer to this object as the *Cuntz semigroup* of A. The Grothendieck enveloping group of $W(A)$ is denoted $\mathrm{K}_0^*(A)$.

Given $a \in \mathrm{M}_\infty(A)_+$ and $\epsilon > 0$, we denote by $(a-\epsilon)_+$ the element of $C^*(a)$ corresponding (via the functional calculus) to the function

$$f(t) = \max\{0, t-\epsilon\}, \qquad \text{for } t \in \sigma(a).$$

(Here $\sigma(a)$ denotes the spectrum of a.) The proposition below collects some facts about Cuntz subequivalence due to Kirchberg and Rørdam.

Proposition 2.1 (Kirchberg and Rørdam [**18**], Rørdam [**21**]). *Let A be a C^*-algebra, and $a, b \in A_+$.*

(i) *We have $(a-\epsilon)_+ \precsim a$ for every $\epsilon > 0$.*
(ii) *The following are equivalent:*
  (a) *$a \precsim b$;*
  (b) *for all $\epsilon > 0$, $(a-\epsilon)_+ \precsim b$;*
  (c) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $(a-\epsilon)_+ \precsim (b-\delta)_+$.*
(iii) *If $\epsilon > 0$ and $\|a-b\| < \epsilon$, then $(a-\epsilon)_+ \precsim b$.*

Now suppose that A is unital and stably finite, and denote by $\mathrm{QT}(A)$ the space of normalised 2-quasitraces on A (see [**4**, Definition II.1.1]). Let $S(W(A))$ denote the set of additive and order-preserving maps s from $W(A)$ to $\mathbb{R}^+$ having the property that $s(\langle 1_A\rangle) = 1$. Such maps are called *states*. Given $\tau \in \mathrm{QT}(A)$, one may define a map

$$s_\tau : \mathrm{M}_\infty(A)_+ \longrightarrow \mathbb{R}^+$$

by

$$s_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}). \tag{2.1}$$

This map is lower semicontinuous, and depends only on the Cuntz equivalence class of a. It moreover has the following properties:

(i) if $a \precsim b$, then $s_\tau(a) \leqslant s_\tau(b)$;
(ii) if a and b are mutually orthogonal, then $s_\tau(a+b) = s_\tau(a) + s_\tau(b)$;
(iii) $s_\tau((a-\epsilon)_+) \nearrow s_\tau(a)$ as $\epsilon \to 0$.

Thus, s_τ defines a state on $W(A)$. Such states are called *lower semicontinuous dimension functions*, and the set of them is denoted $\mathrm{LDF}(A)$. The space $\mathrm{QT}(A)$ is a simplex (see [**4**, Theorem II.4.4]), and the map from $\mathrm{QT}(A)$ to $\mathrm{LDF}(A)$ defined by (2.1) is bijective and affine [**4**, Theorem II.2.2]. A *dimension function* on A is a state on $\mathrm{K}_0^*(A)$, where it is assumed that the latter has been equipped with the usual order coming from the Grothendieck map. The set of dimension functions is denoted $\mathrm{DF}(A)$. The set $\mathrm{LDF}(A)$ is a (generally proper) face of $\mathrm{DF}(A)$. If A has the property that $a \precsim b$ whenever $s(a) < s(b)$ for every $s \in \mathrm{LDF}(A)$, then we say that A has *strict comparison of positive elements*.

3. *A stability theorem for* $W(\mathrm{C}(X))$

3.1. *Strategy*

In this section we provide a positive answer to Question 1.1. The proof is long and proceeds in several steps, so an overview of our strategy is in order. We first generalise the concept of a trivial vector bundle by introducing trivial positive elements in matrices over a commutative C*-algebra, and prove that our question need only be answered in the case that X is a finite simplicial complex (Proposition 3.7). For a finite simplicial complex X, we prove that any positive element in $\mathrm{M}_n(\mathrm{C}(X))$ can be approximated in norm from below by a particularly tractable kind of positive element (Theorem 3.9). Such an approximant is then shown to be dominated (in the sense of Cuntz comparison) by a trivial element whose rank over each point of X exceeds the rank of the approximant by an amount no greater than some fixed multiple of the dimension of X (Theorem 3.13). Finally, we prove that any positive element dominates a trivial element provided that the rank of the given element exceeds the rank of the trivial element by $\dim(X)+1$ over every point in X (Theorem 3.14). Combining the last two theorems yields a positive answer to Question 1.1.

REMARK 3.1. We are seeking the constant K of Question 1.1. The reader will notice that we make little effort to find the smallest possible value of K. This is deliberate. The applications of Section 4 require only that some K exists, and any effort to find the optimal K introduces more complexity into an already difficult proof. Of course, it is interesting to ask what the optimal value of K might be. We suspect that it is the same as it is for projections, namely, about half the covering dimension of X.

3.2. *Notation and background*

Let X be a topological space. By an upper (respectively lower) semicontinuous function $f : X \to \mathrm{M}_n(\mathbb{C})_+$ we will mean a function such that for every vector $\xi \in \mathbb{C}^n$, the real-valued function

$$x \longmapsto \langle f(x)\xi \mid \xi\rangle$$

is upper (respectively lower) semicontinuous (cf. [5]). The following result of Bratteli and Elliott, based on earlier work of Dădărlat, Nagy, Némethi and Pasnicu [9, Proposition 3.2], will be used extensively in the sequel.

THEOREM 3.2 (Bratteli and Elliott [5, Theorem 3.1]). *Let X be a compact metrisable Hausdorff space of dimension d, and let P and Q be maps from X into the projections of M_n such that P is lower semicontinuous and Q is upper semicontinuous. Suppose that*

$$P(x) \geqslant Q(x), \qquad \text{for all } x \in X,$$

and that, furthermore, there exists a natural number k such that

$$\operatorname{rank}(P(x)) > k + \tfrac{1}{2}(d+1), \qquad \text{for all } x \in X,$$

and

$$\operatorname{rank}(Q(x)) < k - \tfrac{1}{2}(d+1), \qquad \text{for all } x \in X.$$

Then, there is a continuous map R from X into the rank-k projections of M_n such that

$$P(x) \geqslant R(x) \geqslant Q(x), \qquad \text{for all } x \in X.$$

For a natural number $l \in \mathbb{N}$, let θ_l denote the trivial vector bundle of complex fibre dimension l. In a metric space X we will use the notation $B_r(x)$ for the open ball of radius $r > 0$ about $x \in X$. The following well-known theorem is a direct consequence of [**17**, Chapter 8, Theorems 1.2 and 1.5].

THEOREM 3.3. *Let X be a compact metrisable Hausdorff space of dimension d, and let ω be a complex vector bundle over X. Then, there is a complex vector bundle $\overline{\omega}$ on X of rank less than or equal to d with the property that $\omega \oplus \overline{\omega}$ is a trivial vector bundle.*

Let $\Gamma_n(X)$ denote the set of n-multisets (sets of n elements, allowing multiplicity) whose entries are elements of a metric space X. Equip $\Gamma_n(X)$ with the following metric: for $A, B \in \Gamma_n$, let $\mathcal{P}$ denote the set of all possible pairings of the elements of A with the elements of B; for $P \in \mathcal{P}$, let $\Delta(P)$ denote the maximum distance between two paired elements in P; set $\mathrm{dist}(A, B) = \min_{P \in \mathcal{P}} \Delta(P)$.

3.3. *Trivial positive elements*

DEFINITION 3.4. Let X be a compact metrisable Hausdorff space, and let $a \in \mathrm{M}_n(\mathrm{C}(X))$ be positive with (lower semicontinuous) rank function $f : X \to \mathbb{Z}^+$ taking values in $\{n_1, \ldots, n_k\}$, where $n_1 < n_2 < \ldots < n_k$.

(i) For each $1 \leqslant i \leqslant k$ define sets

$$G_{i,a} := \{x \in X \mid f(x) > n_i\},$$

$$F_{i,a} := \{x \in X \mid f(x) = n_i\},$$

and

$$H_{i,a} := \{x \in X \mid f(x) \leqslant n_i\}.$$

Note that $G_{i,a}$ is open.

(ii) Say that a is *trivial* if there exist a natural number n, n_k mutually orthogonal projections $p_1, \ldots, p_{n_k} \in \mathrm{M}_n(\mathrm{C}(X))$, each corresponding to a trivial complex line bundle, and positive continuous functions $g_{i,a} : X \to \mathbb{R}\mathbf{1}_{\mathrm{M}_n(\mathrm{C}(X))}$ with $\mathrm{supp}(g_{i,a}) = G_{i,a}$ such that a is Cuntz equivalent to

$$\left(\bigoplus_{j=1}^{n_1} g_{1,a} p_j \right) \oplus \ldots \oplus \left(\bigoplus_{j=n_{k-1}+1}^{n_k} g_{k,a} p_j \right).$$

(iii) Say that a is *well supported* if, for each $1 \leqslant i \leqslant k$, there is a projection $p_i \in \mathrm{M}_n(\mathrm{C}(\overline{F_{i,a}}))$ such that

$$\lim_{r \to \infty} a(x)^{1/r} = p_i(x), \qquad \text{for all } x \in F_{i,a},$$

and $p_i(x) \leqslant p_j(x)$ whenever $x \in \overline{F_{i,a}} \cap \overline{F_{j,a}}$ and $i \leqslant j$.

Clearly, if a above is a projection, then it is trivial in the sense we have defined if and only if it corresponds to a trivial vector bundle. When there is no danger of confusion, we will omit the second subscript for the sets $G_{i,a}$, $F_{i,a}$ and $H_{i,a}$.

The following lemma was observed in [**19**]. The proof is an easy exercise.

LEMMA 3.5. *Let $a, b \in \mathrm{M}_n$ be positive. Then, $a \precsim b$ if and only if* $\mathrm{rank}(a) \leqslant \mathrm{rank}(b)$.

An important analogy between trivial positive elements and trivial vector bundles is the fact that Cuntz comparison for trivial positive elements is encoded by rank functions.

Proposition 3.6. *Let X be a compact Hausdorff space, and let $a, b \in \mathrm{M}_n(\mathrm{C}(X))_+$ be trivial. Then, $a \precsim b$ if and only if* $\mathrm{rank}(a)(x) \leqslant \mathrm{rank}(b)(x)$, *for all $x \in X$.*

Proof. If $a \precsim b$, then $a(x) \precsim b(x)$ for every x in X. It follows from Lemma 3.5 that

$$\mathrm{rank}(a)(x) \leqslant \mathrm{rank}(b)(x), \qquad \text{for all } x \in X.$$

Now suppose that the rank inequality above holds. Let

$$n_1 < n_2 < \ldots < n_s \qquad \text{and} \qquad m_1 < m_2 < \ldots < m_t$$

be the rank values taken by a and b respectively. By assumption there exist projections $p_1, \ldots, p_{n_s}$ and $q_1, \ldots, q_{m_t}$ in some $\mathrm{M}_n(\mathrm{C}(X))$, each corresponding to a trivial line bundle, and positive continuous functions

$$g_{i,a} : X \longrightarrow \mathbb{R}\mathbf{1}_{\mathrm{M}_n(\mathrm{C}(X))}, \qquad \text{for } 1 \leqslant i \leqslant s,$$

and

$$g_{j,b} : X \longrightarrow \mathbb{R}\mathbf{1}_{\mathrm{M}_n(\mathrm{C}(X))}, \qquad \text{for } 1 \leqslant j \leqslant t,$$

with $\mathrm{supp}(g_{i,a}) = G_{i,a}$ and $\mathrm{supp}(g_{j,b}) = G_{j,b}$ such that

$$a \sim \left(\bigoplus_{j=1}^{n_1} g_{1,a} p_j \right) \oplus \ldots \oplus \left(\bigoplus_{j=n_{s-1}+1}^{n_s} g_{s,a} p_j \right)$$

and

$$b \sim \left(\bigoplus_{j=1}^{m_1} g_{1,b} q_j \right) \oplus \ldots \oplus \left(\bigoplus_{j=m_{t-1}+1}^{m_t} g_{t,b} q_j \right).$$

The projections p_i and q_i are Murray–von Neumann equivalent for every $1 \leqslant i \leqslant n_s$, as are their complements inside $\mathrm{M}_n(\mathrm{C}(X))$. It follows that there is a unitary U_1 in $\mathrm{M}_n(\mathrm{C}(X))$ such that $u_1 q_1 u_1* = p_1$. We can repeat this argument to find a unitary u_2 in $(1-p_1)\mathrm{M}_n(\mathrm{C}(X))(1-p_1)$ such that $u_2 q_2 u_2* = p_2$. Then $U_2 = (u_2 \oplus p_1)u_1$ is a unitary in $\mathrm{M}_n(\mathrm{C}(X))$ such that $U_2 q_i U_2^* = p_i$ for $i = 1, 2$. Iterating this process we arrive at a unitary U in $\mathrm{M}_n(\mathrm{C}(X))$ such that

$$U q_i U^* = p_i, \qquad \text{for } 1 \leqslant i \leqslant n_s.$$

Our rank inequality implies that

$$\mathrm{supp}(g_{i,a}) \subseteq \mathrm{supp}(g_{i,b}), \qquad \text{for } 1 \leqslant i \leqslant n_s,$$

whence $g_{i,a} p_i$ is in the hereditary subalgebra of $\mathrm{M}_n(\mathrm{C}(X))$ generated by $g_{i,b} p_i$. It follows that

$$a \sim \left(\bigoplus_{j=1}^{n_1} g_{1,a} p_j \right) \oplus \ldots \oplus \left(\bigoplus_{j=n_{s-1}+1}^{n_s} g_{s,a} p_j \right)$$

is in the hereditary subalgebra of $\mathrm{M}_n(\mathrm{C}(X))$ generated by

$$U \left[\left(\bigoplus_{j=1}^{m_1} g_{1,b} q_j \right) \oplus \ldots \oplus \left(\bigoplus_{j=m_{t-1}+1}^{m_t} g_{t,b} q_j \right) \right] U^* \sim b,$$

and that $a \precsim b$, as desired. $\square$

3.4. *Reduction to finite simplicial complexes*

Proposition 3.7. *Suppose that there exists a constant $K > 0$ such that for any finite simplicial complex X and positive elements $a, b \in \mathrm{M}_\infty(\mathrm{C}(X))$ for which*

$$\mathrm{rank}(a)(x) + K\dim(X) \leqslant \mathrm{rank}(b)(x), \qquad \textit{for all } x \in X,$$

one has $a \precsim b$. Then, this same statement holds (with the same value for K) upon replacing finite simplicial complexes with arbitrary compact metric spaces.

Proof. Let a, b, and $K > 0$ be as in the hypotheses of the theorem. Let there be given a compact metric space Z, and positive elements $a, b \in \mathrm{M}_\infty(\mathrm{C}(Z))$ such that

$$\mathrm{rank}(a)(z) + K\dim(Z) \leqslant \mathrm{rank}(b)(z), \qquad \text{for all } z \in Z.$$

(This implies, in particular, that Z has finite covering dimension, but Theorem 3.15 holds vacuously if $\dim(Z) = \infty$; there is no loss of generality here.)

A central theorem in the dimension theory of topological spaces asserts that if Z is a compact metric space of finite covering dimension, then Z is the limit of an inverse system $(Y_i, \pi_{i,j})$, where each Y_i is a finite simplicial complex of dimension less than or equal to the dimension of Z (cf. [**14**, Theorem 1.13.2]). Thus, we have an inductive limit decomposition for $\mathrm{M}_\infty(\mathrm{C}(Z))$:

$$\mathrm{M}_\infty(\mathrm{C}(Z)) = \lim_{i\to\infty} \left(\mathrm{M}_\infty(\mathrm{C}(Y_i)), \phi_i\right),$$

where $\phi_i : \mathrm{M}_\infty(\mathrm{C}(Y_i)) \to \mathrm{M}_\infty(\mathrm{C}(Y_{i+1}))$ is induced by $\pi_{i,i+1} : Y_{i+1} \to Y_i$.

We claim that for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mathrm{rank}(a-\epsilon)_+(z) + K\dim(Z) \leqslant \mathrm{rank}(b-\delta)_+(z), \qquad \text{for all } z \in Z.$$

Notice that if $\epsilon_1 \leqslant \epsilon_2$, then $\mathrm{rank}(a-\epsilon_1)_+ \geqslant \mathrm{rank}(a-\epsilon_2)_+$. Let $\epsilon > 0$ be given, and fix $z \in Z$. Put $A = \mathrm{rank}(a)(z)$ and $B = \mathrm{rank}(b)(z)$ for convenience. Let $\eta(z)$ denote the smallest non-zero eigenvalue of $a(z)$. For the purpose of proving our claim, we view a and b as being contained in some $\mathrm{M}_n(\mathrm{C}(Z))$. The map

$$\sigma : \mathrm{M}_n(\mathrm{C}(Z)) \times Z \longrightarrow \Gamma_n,$$

which assigns to an ordered pair (d, x) the multiset whose elements are the eigenvalues of $d(x)$, is continuous in both variables, and so there is a neighbourhood $V(z)$ of z upon which a has precisely A eigenvalues greater than or equal to $\min\{\eta(z)/2, \epsilon\}$. It follows from the functional calculus that $\mathrm{rank}(a-\epsilon)_+$ is less than or equal to A on $V(z)$. Now $\sigma(b, \cdot\,)$ is continuous, and so there is a neighbourhood $U(z)$ of z upon which b has at least B non-zero eigenvalues. In fact, more is true: $U(z)$ may be chosen so that there is a continuous choice of B non-zero eigenvalues of b on $U(z)$ which coincides with the spectrum of $b(z)$ at z. Let δ_z be half the smallest eigenvalue occurring in this continuous choice of eigenvalues. Then, $\mathrm{rank}(b-\delta_z)_+ \geqslant B$ on $U(z)$, and this remains true if δ_z is replaced by some smaller δ'. Put $W(z) = V(z) \cap U(z)$. Then,

$$\mathrm{rank}(a-\epsilon)_+(x) \leqslant A + K\dim(Z) \leqslant B \leqslant \mathrm{rank}(b-\delta_z)_+(x), \qquad \text{for all } x \in W(z).$$

The space Z is compact, so we may find $z_1, \ldots, z_k \in Z$ such that

$$Z = W(z_1) \cup \ldots \cup W(z_k).$$

Put $\delta = \min\{\delta_{z_1}, \ldots, \delta_{z_k}\}$, and let $x \in Z$ be given. Then $x \in W(z_l)$ for some $1 \leqslant l \leqslant k$, and applying the preceding inequality we have

$$\mathrm{rank}(a-\epsilon)_+(x) + K\dim(Z) \leqslant \mathrm{rank}(b-\delta_{z_l})_+(x) \leqslant \mathrm{rank}(b-\delta)_+(x).$$

This proves the claim.

Given $\eta > 0$ and $a \in \mathrm{M}_\infty(\mathrm{C}(Z))_+$ one can always find an element $\tilde{a}$ satisfying:
(i) $\tilde{a} \precsim a$;
(ii) $(a-\eta)_+ \precsim \tilde{a}$;
(iii) $\tilde{a}$ is in the image of $\phi_{i,\infty}$ for some $i \in \mathbb{N}$.
(This follows from Proposition 2.1 and the inductive limit decomposition of $\mathrm{M}_\infty(\mathrm{C}(Z))$.) Let $\epsilon > 0$ be given, and find $\delta > 0$ so that

$$\mathrm{rank}(a-\epsilon/2)_+(z) + K\dim(Z) \leqslant \mathrm{rank}(b-\delta)_+(z), \qquad \text{for all } z \in Z.$$

Find an approximant $\tilde{b}$ for b satisfying (i)–(iii) above with $\eta = \delta$. Similarly, find an approximant $\tilde{a}$ for $(a-\epsilon/2)_+$ with $\eta = \epsilon/2$. Then, for all $z \in Z$,

$$\begin{aligned} \mathrm{rank}(\tilde{a})(z) + K\dim(Z) &\leqslant \mathrm{rank}(a-\epsilon/2)_+(z) + K\dim(Z) \\ &\leqslant \mathrm{rank}(b-\delta)_+(z) \\ &\leqslant \mathrm{rank}(\tilde{b})(z). \end{aligned}$$

We may assume that both $\tilde{a}$ and $\tilde{b}$ are the images under $\phi_{i,\infty}$ of elements $\hat{a}$ and $\hat{b}$ in $\mathrm{M}_\infty(\mathrm{C}(Y_i))$, respectively. These pre-images satisfy

$$\mathrm{rank}(\hat{a})(y) + K\dim(Z) \leqslant \mathrm{rank}(\hat{b})(y), \qquad \text{for all } y \in \mathrm{Im}(\pi_{i,\infty}),$$

where $\pi_{i,\infty} : Z \to Y_i$ is the continuous map which induces $\phi_{i,\infty}$. We cannot apply the hypothesis of the proposition unless the inequality above holds for all $y \in Y_i$, and so we modify the pre-image $\hat{b}$. We may, as before, view $\hat{a}$ and $\hat{b}$ as lying in some $\mathrm{M}_n(\mathrm{C}(Y_i))$. Choose a continuous function $f : Y_i \to [0,1]$ supported on the complement of $\mathrm{Im}(\pi_{i,\infty})$. Put

$$\hat{\hat{b}} = \hat{b} \oplus f \cdot \mathbf{1}_{\lceil K\dim(Z)\rceil}.$$

Since $\hat{\hat{b}}$ and $\hat{b}$ agree on $\mathrm{Im}(\pi_{i,\infty})$, we have $\phi_{i,\infty}(\hat{\hat{b}}) = \tilde{b}$. But clearly

$$\mathrm{rank}(\hat{a})(y) + K\dim(Z) \leqslant \mathrm{rank}(\hat{\hat{b}})(y), \qquad \text{for all } y \in Y_i,$$

whence $\hat{a} \precsim \hat{\hat{b}}$ by our hypothesis. Cuntz subequivalence is preserved under $*$-homomorphisms, whence $\tilde{a} \precsim \tilde{b}$. Now

$$(a-\epsilon)_+ \precsim \tilde{a} \precsim \tilde{b} \precsim b;$$

ϵ was arbitrary, and the proposition follows. □

3.5. *Well supported approximants*

LEMMA 3.8. *Let X be a finite simplicial complex, V be an open subset of X, and U be a closed subset of V. Then, there are a refinement of the simplicial structure on X and a subcomplex Y of this refined structure satisfying:*
(i) $Y \supseteq V^c$;
(ii) $U \cap Y = \emptyset$;
(iii) Y is the closure of its interior.
Moreover, $\overline{Y^c}$ and $\partial Y = \partial Y^c$ are subcomplexes of this refined structure.

Proof. We first define a precursor $\widetilde{Y}$ to Y, whose definition we later refine to obtain Y proper. By assumption, $U \cap V^c = \emptyset$. Since U and V^c are compact, there is a $\delta > 0$ such that

$$\mathrm{dist}(U, x) > \delta, \qquad \text{for all } x \in V^c.$$

Refine the simplicial structure on X through repeated barycentric subdivision until the largest diameter of any simplex is less than $\delta/2$. Let $\widetilde{Y}$ be the subcomplex consisting of all simplices whose intersection with V^c is non-empty. The distance from any point in $\widetilde{Y}$ to V^c is at most $\delta/2$, and so $\widetilde{Y} \cap U = \emptyset$. Every $x \in V^c$ is contained in some simplex of X, so $\widetilde{Y} \supseteq V^c$.

Now choose $x \in \widetilde{Y}^c$, and let Θ_x be the smallest simplex of the refined simplicial structure on X containing x. Suppose that Θ_x is not contained in $\overline{\widetilde{Y}^c}$. Then, Θ_x contains a point $y \in \widetilde{Y}^\circ$, and there is an open set $U \subseteq \widetilde{Y}^\circ$ such that $y \in U$. Thus, there is a point $y' \in U$ which is in the (relative) interior of Θ_x, and the smallest simplex of X containing y' is Θ_x. This implies that Θ_x is contained in $\widetilde{Y}$ by construction, contradicting $x \in \Theta_x$. We conclude that $\Theta_x \subseteq \overline{\widetilde{Y}^c}$. Now

$$\widetilde{Y}^c \subseteq \bigcup_{x \in \widetilde{Y}^c} \Theta_x \subseteq \overline{\widetilde{Y}^c},$$

and so the second containment above is in fact equality. We conclude that $\overline{\widetilde{Y}^c}$ is a subcomplex of X. Now $\widetilde{Y}^c$ is open, whence $\overline{\widetilde{Y}^c}$ is the closure of its interior.

Notice that $\overline{\widetilde{Y}^c}$ is defined in the same manner as Y: for each point in a given open set, one finds the smallest simplex containing the said point, and then takes the union of these simplices over all points in the open set. We may therefore define

$$Y := \overline{\left(\overline{\widetilde{Y}^c}\right)^c},$$

and apply the arguments above to conclude that Y is a subcomplex. Now Y both contains V and is contained in $\widetilde{Y}$, and so satisfies conclusions (i) and (ii) of the lemma; Y satisfies conclusion (iii) by construction. Repeating the arguments above one last time, we find that $\overline{Y^c}$ and hence $\partial Y^c = \partial Y$ are subcomplexes. □

Theorem 3.9. *Let X be a finite simplicial complex, $a \in \mathrm{M}_n(\mathrm{C}(X))_+$, and $\epsilon > 0$ be given. Then, there exists a well supported element $f \in \mathrm{M}_n(\mathrm{C}(X))_+$ such that $f \leqslant a$ and $\|f - a\| < \epsilon$. Moreover, f takes the same rank values as a, and the sets $\overline{F_i}$ corresponding to f (see Definition 3.4) may be assumed, upon refining the simplicial structure of X, to be subcomplexes of X.*

Proof. Let $a \in \mathrm{M}_n(\mathrm{C}(X))_+$, and let H_i be the set $H_{i,a}$ of Definition 3.4. Let $\epsilon > 0$ be given.

Step 1. We will show that there exist open sets $V_i \supseteq H_i$, for $1 \leqslant i \leqslant k$, and an upper semicontinuous function $g : X \to \mathrm{M}_n(\mathbb{C})_+$ satisfying:

(i) $g(x) = a(x)$, for all $x \in H_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$;
(ii) $g(x)$ is continuous on $V_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$, for $1 \leqslant i \leqslant k$;
(iii) $\|g(x) - a(x)\| < \epsilon$, for all $x \in X$;
(iv) $\mathrm{rank}(g)(x) = n_i$, for all $x \in V_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$;
(v) with

$$D_i := \overline{V_i} \setminus \left(\bigcup_{j=1}^{i-1} V_j\right),$$

there is a continuous projection-valued function $p_i \in \mathrm{M}_n(\mathrm{C}(D_i))$ such that

$$\lim_{r\to\infty} g(x)^{1/r} = p_i(x), \qquad \text{for all } x \in V_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right),$$

and $p_i(x) \leqslant p_j(x)$ whenever $x \in D_i \cap D_j$ and $i \leqslant j$.

Moreover, upon refining the simplicial structure of X, we may assume that both V_i and $\overline{V_i^c}$ are subcomplexes of X.

Let $n_1 < \ldots < n_k$ be the rank values taken by a. The function which assigns to each point $x \in \partial H_1$ the minimum non-zero eigenvalue of $a(x)$ is continuous on a compact set, and so achieves a minimum, say $\eta_1 > 0$. For each $x \in \partial H_1$, find $\delta_x > 0$ such that, for every $y \in B_{\delta_x}(x)$, the eigenvalues of $a(y)$ are all either greater than

$$L_1^u := \max\left\{\tfrac{2}{3}\eta_1, \eta_1 - \epsilon\right\},$$

or less than

$$L_1^l := \min\left\{\tfrac{1}{3}\eta_1, \epsilon\right\}.$$

Let U_x be the connected component of $B_{\delta_x}(x)$ containing x. Put

$$W_1 = H_1 \cup \left(\bigcup_{x \in \partial H_1} U_x \right).$$

Then W_1 is open. Use Lemma 3.8 to find a subcomplex Y_1 of X such that $Y_1 \supseteq \widetilde{W_1}^{\,c}$ and $Y_1 \cap H_1 = \emptyset$. Put $V_1 = Y_1^c$, and note that V_1 so defined is a subcomplex of some (possibly refined) simplicial structure on X. Define $p_1(x)$ to be the support projection of $a(x)$ for $x \in H_1$, and the support projection of the eigenvectors of $a(x)$ corresponding to eigenvalues which are greater than or equal to L_1^u for $x \in \overline{V_1} \setminus H_1$. For each $x \in V_1$, put

$$g(x) = p_1(x)a(x).$$

Claim. *The following hold:*

(i) $g(x) = a(x)$, *for all* $x \in H_1$;
(ii) $g(x)$ *is continuous on* V_1;
(iii) $\|g(x) - a(x)\| < \epsilon$, *for all* $x \in V_1$;
(iv) $\operatorname{rank}(g)(x) = n_1$, *for all* $x \in V_1$;
(v) p_1 *is a continuous projection-valued function on* $\overline{V_1}$ *such that*

$$\lim_{r \to \infty} g(x)^{1/r} = p_i(x), \qquad \text{for all } x \in V_1.$$

Proof. Part (i) is clear from the definition of g on H_1.

For (ii), let $x_n \to x$ in V_1. If x is an interior point of H_1, then $g(x_n) \to g(x)$ since a is continuous and $g = a$ on H_1. Otherwise, $x \in V_1$ is an interior point of some U_y, and we may find $N \in \mathbb{N}$ such that $x_n \in U_y$, for all $n \geqslant N$. It will thus suffice to prove that $g(x)$ is continuous on U_y, with $y \in \partial H_1$. Let $\sigma(a(x))$ denote the spectrum of a at the point $x \in X$. The map $s : X \to \Gamma_n$ given by $s(x) = \sigma(a(x))$ is continuous. Thus, $s(x_n) \to s(x)$. In particular, the submultiset of $s(x_n)$ corresponding to elements larger than L_1^u converges to the similar submultiset of $s(x)$; the eigenvectors of $a(x_n)$ corresponding to $p_1(x_n)$ converge to the eigenvectors of $a(x)$ corresponding to $p(x)$, and so $g(x_n) \to g(x)$, as required.

Part (iii) follows from the functional calculus and the definition of g because $a(x) - g(x)$ is zero if $x \in H_1$, and is equal, in the functional calculus, to $h(t) = t$ on the part of the spectrum of a which is less than $L_1^l \leqslant \epsilon$ and zero otherwise.

Part (iv) is trivial for $x \in H_1$, so suppose that $x \in U_y$ for some $y \in \partial H_1$. The rank of $g(x)$ is equal to the number of eigenvalues of $a(x)$ which are greater than L_1^u, and this number is n_1 for $a(y)$. Let $M \subseteq \Gamma_n$ consist of those multisets with the property that each element of the multiset is either greater than L_1^u or less than L_1^l. Let $A \subseteq M$ consist of those multisets for which the number of elements greater than L_1^u is exactly n_1, and let B be the complement of A relative to M. Then, A and B are separated. Since $s(z) \subseteq M$, for all $z \in U_y$, $s(y) \in A$, and U_y is connected, we conclude that $s(z) \in A$, for all $z \in U_y$. In particular, the rank of $g(x)$ is n_1, as required.

Observe that (iv) implies that the function which assigns to a point $x \in V_1$ the submultiset of $s(x)$ consisting of those eigenvalues which are greater than or equal to L_1^u is continuous on V_1. Note that $p_1(x)$ is the spectral projection on this submultiset for each $x \in V_1$, and is thus continuous merely by the existence of the continuous functional calculus. This proves (v), and hence the claim. $\square$

The claim above (now proved) is the base case of an inductive argument. We now describe the construction of V_2, p_2, and the definition of g on $V_2 \setminus V_1$. The construction of the subsequent V_i and p_i, and the definition of g on $V_i \setminus (\bigcup_{j=1}^{i-1} V_j)$ will be similar: all of the essential difficulties have already been encountered for $i = 2$.

Let $\partial(H_2 \setminus V_1)$ be the boundary of $H_2 \setminus V_1$ inside V_1^c (we may assume that this boundary is non-empty by shrinking V_1, if necessary). Find, as before, the minimum value $\eta_2 > 0$ occurring as a non-zero eigenvalue of $a|_{\partial(H_2\setminus V_1)}$. Define

$$L_2^u = \max\left\{\tfrac{2}{3}\eta_2, \eta_2 - \epsilon\right\} \qquad \text{and} \qquad L_2^l = \min\left\{\tfrac{1}{3}\eta_2, \tfrac{1}{3}\eta_1, \epsilon\right\}.$$

(Note the dependence of L_2^l on η_1.) Find, for each $x \in \partial(H_2 \setminus V_1)$, a connected open (rel V_1^c) set $U_x \subseteq V_1^c$ containing x and with the property that for every $y \in U_x$, the eigenvalues of $a(y)$ are all either greater than L_2^u or less than L_2^l. Let $\widetilde{U_x}$ be an open set in X such that $U_x = \widetilde{U_x} \cap V_1^c$. Put

$$W_2 = H_2 \cup \left(\bigcup_{x \in \partial(H_2 \setminus V_1)} \widetilde{U_x} \right) \subseteq X.$$

Refine the simplicial structure on X so that we may find (using Lemma 3.8) a subcomplex Y_2 of V_1^c which is disjoint from W_2^c and whose interior (rel V_1^c) contains $H_2 \setminus V_1$. Put $V_2 = Y_2^\circ \cup V_1$ (interior taken (rel V_1^c)). Thus, V_2^c and $\overline{V_2}$ are subcomplexes of X. Define $p_2(x)$ to be the support projection of $a(x)$ for $x \in H_2 \setminus V_1$. For any $y \in (U_x \setminus H_2) \cap V_2$ and some $x \in \partial(H_2 \setminus V_1)$, let $p_2(x)$ be the support projection of those eigenvectors of $a(y)$ corresponding to eigenvalues greater than L_2^u. Put $g(x) = p_2(x)a(x)$, for all $x \in V_2 \setminus V_1$.

Claim. *The p_2 and $g(x)$ so defined have the desired properties, namely:*
 (i) *$g(x) = a(x)$, for all $x \in H_2 \setminus V_1$;*
 (ii) *$g(x)$ is continuous on $V_2 \setminus V_1$;*
 (iii) *$\|g(x) - a(x)\| < \epsilon$, for all $x \in V_2$;*
 (iv) *$\mathrm{rank}(g)(x) = n_2$, for all $x \in V_2 \setminus V_1$;*
 (v) *p_2 extends to a continuous projection-valued function on $D_2 := \overline{V_2} \setminus V_1$ such that*

$$\lim_{r\to\infty} g(x)^{1/r} = p_i(x), \qquad \text{for all } x \in V_1,$$

and $p_1(x) \leqslant p_2(x)$ whenever $x \in \overline{V_1} \cap (\overline{V_2} \setminus V_1)$.

Proof. The proofs of all but the last part of (v) are identical to the proofs of the corresponding statements for p_1 above. We must show that $p_1(x) \leqslant p_2(x)$ whenever $x \in \overline{V_1} \cap (\overline{V_2} \setminus V_1)$. This follows from the fact that the eigenvalues of $(1 - p_2(x))a(x)(1 - p_2(x))$ are all less than or equal to $L_2^l \leqslant L_1^l$, and so correspond to eigenvectors in the complement of the range of $p_1(x)$. $\square$

The remaining p_i, for $i < k$, may be constructed inductively in a manner similar to the construction of p_2: let η_i be the minimum eigenvalue taken by a on $\partial\big(H_i \setminus \big(\bigcup_{j=1}^{i-1} V_j\big)\big)$ (boundary relative to $X \setminus \big(\bigcup_{j=1}^{i-1} V_j\big)$ both here and below, and assumed to be non-empty by shrinking $V_1, \ldots, V_{i-1}$ if necessary) and put

$$L_i^u = \max\left\{\tfrac{2}{3}\eta_i, \eta_i - \epsilon\right\} \qquad \text{and} \qquad L_i^l = \min\left\{\tfrac{1}{3}\eta_i, L_{i-1}^l\right\};$$

find connected open (rel $X \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$) sets U_x for each $x \in \partial\left(H_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)\right)$ which contain x, and have the property that every eigenvalue of $a(y)$, for $y \in U_x$, is either greater than L_i^u or less than L_i^l; find open sets $\widetilde{U_x} \subseteq X$ such that $U_x = \widetilde{U_x} \cap X \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$, and define

$$W_i = H_i \cup \left(\bigcup_{x \in \partial(H_i \setminus (\bigcup_{j=1}^{i-1} V_j))} \widetilde{U_x} \right) \subseteq X;$$

refine the simplicial structure on X and find (using Lemma 3.8) a subcomplex Y_i of V_{i-1}^c which is disjoint from W_i^c and whose interior (rel V_{i-1}^c) contains $H_i \setminus V_{i-1}$; put $V_i = Y_i^\circ \cup V_{i-1}$ (interior taken (rel V_{i-1}^c)) so that V_i^c and $\overline{V_i}$ are subcomplexes of X; define $p_i(x)$ to be the support projection of the eigenvectors of $a(x)$ with non-zero eigenvalues for $x \in H_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$, and the support projection of those eigenvectors of $a(x)$ having eigenvalues greater than or equal to L_i^u for $x \in \overline{V_i} \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$; put $g = p_i(x)a(x)$, for all $x \in V_i \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$.

The p_i and g so defined have the desired properties. As before, all but the last part of statement (v) follow from the proofs of the corresponding facts for p_1 and p_2. So suppose that $x \in D_i \cap D_j$, where $j \leqslant i$. The range of $1 - p_i(x)$ corresponds to the span of eigenvectors of $a(x)$ with eigenvalues less than or equal to $L_i^l \leqslant L_j^l$. This range is contained in the range of $1 - p_j(x)$, since the latter corresponds to the span of eigenvectors of $a(x)$ with eigenvalues less than L_j^l. It follows that $p_j(x) \leqslant p_i(x)$, as required.

For $i = k$, the situation is straightforward. Simply put $g(x) = a(x)$, for all $x \in V_{k-1}^c$.

Step 2. We have $V_1 \subseteq \ldots \subseteq V_k = X$. Since the H_i are closed, we may find open sets $U_1 \subseteq \ldots \subseteq U_k = X$ such that $H_i \subseteq U_i \subseteq \overline{U_i} \subseteq V_i$. Moreover, we may assume, using Lemma 3.8, that U_i^c and $\overline{U_i}$ are subcomplexes (possibly empty) of X for $1 \leqslant i \leqslant k$. Let us define a positive upper semicontinuous function $\tilde{f} : X \to \mathrm{M}_n(\mathrm{C}(X))$ as follows: $\tilde{f}(x) = p_i(x)a(x)$, for all $x \in U_i \setminus \left(\bigcup_{j=1}^{i-1} U_j\right)$. Then, $\tilde{f}(x) \geqslant g$, for all $x \in X$. Moreover, $\tilde{f}$ is well supported by the p_i. To see this, one need only check the coherence condition. Put $E_i = \overline{U_i} \setminus \left(\bigcup_{j=1}^{i-1} U_j\right)$, and let $x \in E_i \cap E_j$, with $j \leqslant i$. The range of $1 - p_i(x)$ corresponds to the span of eigenvectors of $a(x)$ with eigenvalues less than or equal to $L_i^l \leqslant L_j^l$ (since $U_i \subseteq V_i$). This range is contained in the range of $1 - p_j(x)$, since the latter corresponds to the span of eigenvectors of $a(x)$ with eigenvalues less than L_j^l. It follows that $p_j(x) \leqslant p_i(x)$, as required.

We now describe a smoothing process which will transform $\tilde{f}$ into the function f required by the theorem. For the first step in our process we work inside $X \setminus U_{k-2}$. Find a continuous function $s : X \setminus U_{k-2} \to [0,1]$ which is zero on $\overline{U_{k-1}} \setminus U_{k-2}$, equal to 1 on $X \setminus V_{k-1}$, and non-zero off $\overline{U_{k-1}} \setminus U_{k-2}$. Define f^{k-1} on $X \setminus U_{k-2}$ as follows:

$$f^{k-2}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \overline{U_{k-1}} \setminus U_{k-2} \cup (X \setminus V_{k-1}), \\ g(x) \oplus s(x)(\tilde{f} \ominus g)(x) & \text{if } x \in V_{k-1} \setminus \overline{U_{k-1}}. \end{cases}$$

Thus f^{k-2} is continuous on $X \setminus U_{k-2}$ and dominates g on $X \setminus U_{k-2}$. It is also subordinate to a.

For the generic step in our process, we work inside $X \setminus U_{k-i}$, with $i \geqslant 3$. Assume that we have found a continuous function $f^{k-i+1} : X \setminus U_{k-i+1} \to \mathrm{M}_n(\mathbb{C})_+$ such that

$$g(x) \leqslant f^{k-i+1}(x), \qquad \text{for all } x \in (X \setminus U_{k-i+1}) \cap V_{k-i+1}.$$

(This is the key property required to go from one stage in the smoothing process to the next.) Find a continuous function $s : X \setminus U_{k-i} \to [0,1]$ which is zero on $\overline{U_{k-i+1}} \setminus U_{k-i}$, equal to 1 on

$X \setminus V_{k-i+1}$, and non-zero off $\overline{U_{k-i+1}} \setminus U_{k-i}$. Define f^{k-i} on $X \setminus U_{k-i}$ as follows:

$$f^{k-i}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \overline{U_{k-i+1}} \setminus U_{k-i}, \\ f^{k-i+1}(x) & \text{if } x \in (X \setminus V_{k-i+1}), \\ g(x) \oplus s(x)(\tilde{f}^{k-i+1} \ominus g)(x) & \text{if } x \in V_{k-i+1} \setminus \overline{U_{k-i+1}}. \end{cases}$$

Then, as before, f^{k-i} is continuous on $X \setminus U_{k-i}$, dominates g, and is subordinate to a. This smoothing process terminates when $i = k-1$, and the resulting continuous function $f : X \to \mathrm{M}_n(\mathbb{C})_+$ has the desired properties. (Note that the sets H_i of Definition 3.4 corresponding to f are precisely the $\overline{U_i}$. The sets F_i of Definition 3.4 corresponding to f are thus the $\overline{U_i} \cap U_{i-1}^c$, and so are subcomplexes of the refined simplicial structure on X.) $\square$

3.6. *Trivial majorants*

Lemma 3.10. *Let X be a finite simplicial complex of dimension d, and let Y be a subcomplex. Let $p \in \mathrm{M}_n(\mathrm{C}(Y))$ be a projection of (complex) fibre dimension $l \in \mathbb{N}$ corresponding to a trivial vector bundle, and suppose that $l \geqslant \lceil d/2 \rceil + 1$. Then, there is a projection $q \in A := \mathrm{M}_n(\mathrm{C}(X))$ such that*

(i) *q corresponds to a trivial vector bundle of fibre dimension l;*
(ii) *$q(y) = p(y)$, for all $y \in Y$.*

Proof. The projection p may be viewed as a vector bundle (E, r, Y). Here E is trivial, and so admits l mutually orthogonal and everywhere non-zero cross-sections $s_i : Y \to E$, for $1 \leqslant i \leqslant l$. We will prove that each s_i can be extended to a continuous map v_i defined on all of X which takes values in $\mathbb{C}^n \setminus \{0\}$, and that these extensions can be chosen to be mutually orthogonal. The projection whose range at a point $x \in X$ is $\mathrm{span}\{v_1(x), \ldots, v_l(x)\}$ is then the projection q that we seek.

We proceed by induction. First consider s_1, which may be viewed as a continuous map from Y to $\mathbb{C}^n \setminus \{0\} \cong \mathbb{R}^{2n} \setminus \{0\}$. Theorem 2.2 in [**17**, Chapter 1] states that if (A, B) is a relative CW-complex and R is a space which is connected in each dimension for which A has cells, then every continuous map $f : B \to R$ extends to a continuous map $g : A \to R$. Now (X, Y) is a relative CW-complex, $\mathbb{C}^n$ is $(2n-1)$-connected, and $2n - 1 \geqslant d$, so s_1 extends to a continuous map $v_1 : X \to \mathbb{C}^n \setminus \{0\}$, as desired.

Suppose now that we have found mutually orthogonal and continuous extensions v_i of s_i for each $i < k \leqslant l$. We wish to extend s_k to a continuous map v_k on X taking values in E and pointwise orthogonal to $v_1, \ldots, v_{k-1}$. Let $Q_{k-1} \in \mathrm{M}_n(\mathrm{C}(X))$ be the projection whose range at a point $x \in X$ is $\mathrm{span}\{v_1(x), \ldots, v_{k-1}(x)\}$. To find our extension v_k, it will suffice to extend s_k to an everywhere non-zero cross-section inside

$$A_{k-1} := (\mathbf{1}_A - Q_{k-1})(\mathrm{M}_n(\mathrm{C}(X)))(\mathbf{1}_A - Q_{k-1}) \cong \mathrm{M}_{n-k}(\mathrm{C}(X))$$

(the last isomorphism follows from the fact that $\mathbf{1}_A - Q_{k-1}$ is trivial). This problem is identical to the problem of extending s_1, except that we are now extending a map from Y into $\mathbb{C}^{n-k} \setminus \{0\}$ rather than into $\mathbb{C}^n \setminus \{0\}$. Since $n - l \geqslant \lceil d/2 \rceil + 1$, we have $2(n-k) - 1 \geqslant d$. Since $\mathbb{C}^{n-k} \setminus \{0\}$ is $2(n-k) - 1$-connected, we may use [**17**, Chapter 1, Theorem 2.2], to find the desired extension v_k. $\square$

Lemma 3.11. *Let X be a finite simplicial complex of dimension d, and let $U_1, \ldots, U_k$ be subcomplexes. Suppose that for each $1 \leqslant i \leqslant k$ there is a constant-rank projection-valued map $p_i \in \mathrm{M}_\infty(\mathrm{C}(U_i))$ satisfying:*

(i) *$p_i(x) \leqslant p_j(x)$ whenever $x \in U_i \cap U_j$ and $i \leqslant j$;*
(ii) *$\mathrm{rank}(p_i) < \mathrm{rank}(p_j)$ whenever $i < j$.*

Then, there exist a partition of $\{1,2,\ldots,k\}$ *into non-empty subsets* $J_1,\ldots,J_s$*, subcomplexes* $V_l := \bigcup_{i\in J_l} U_i$*, and projections* $R_l \in \mathrm{M}_\infty(\mathrm{C}(V_l))$*, with* $1 \leqslant l \leqslant s$*, such that:*

(i) *each* R_l *corresponds to a trivial constant-rank vector bundle on* V_l*;*
(ii) $3d+3 \leqslant \mathrm{rank}(R_l) - \mathrm{rank}(p_i) < 4d+3$*, for all* $i \in J_l$*;*
(iii) $\mathrm{rank}(R_l) + d \leqslant \mathrm{rank}(R_{l+1})$*, for* $1 \leqslant l < s$*;*
(iv) $p_i(x) \leqslant R_l(x)$ *for each* $x \in U_i \cap V_l$*;*
(v) *if* $x \in V_l \cap V_t$ *and* $l \leqslant t$*, then* $R_l(x) \leqslant R_t(x)$*.*

Proof. Put $n_i = \mathrm{rank}(p_i)$. For each $n \in \mathbb{N}$, put

$$D_n = \{m \in \mathbb{N} \mid (n-1)d \leqslant m < nd\}.$$

Let $\widetilde{J_1},\ldots,\widetilde{J_s}$ be the list of the D_n which have non-empty intersection with $\{n_1,\ldots,n_k\}$, ordered so that some (and hence every) element of $\widetilde{J_l}$ is less than every element of $\widetilde{J_{l+1}}$. For $1 \leqslant j \leqslant s$, let J_j be the set of indices of the n_i appearing in $\widetilde{J_j}$.

For $A \subseteq X$ put

$$P_A(x) := \bigvee_{1\leqslant i\leqslant k} p_i(x), \qquad \text{for all } x \in A.$$

The condition that $p_i(x) \leqslant p_j(x)$ whenever $x \in U_i \cap U_j$ and $i \leqslant j$ implies that $P_A(x)$ is an upper semicontinuous projection-valued function for every $A \subseteq X$.

An application of [**5**, Theorem 3.1] allows us to find a constant-rank projection-valued map $Q_s \in \mathrm{M}_\infty(\mathrm{C}(V_s))$ such that

$$\mathrm{rank}(Q_s) - \mathrm{rank}(p_k) \leqslant d$$

and

$$p_i(x) \leqslant Q_s(x), \qquad \text{for all } i \in J_s \text{ and all } x \in U_i.$$

It follows that $\mathrm{rank}(Q_s) - \mathrm{rank}(r_i) < 2d$ for every $i \in J_s$. For every $N \geqslant d$, there is a projection $\overline{Q_{s,N}} \in \mathrm{M}_\infty(\mathrm{C}(V_s))$ of rank n such that $Q_s \oplus \overline{Q_{s,N}}$ corresponds to a trivial vector bundle (cf. Theorem 3.3). Put

$$N_s = 3d + 3 + \mathrm{rank}(r_k) - \mathrm{rank}(Q_s)$$

and

$$R_s = Q_s \oplus \overline{Q_{s,N_s}}.$$

Now R_s so chosen satisfies conditions (i), (ii), and (iv) in the conclusion of the lemma; conditions (iii) and (v) are not yet relevant.

Now suppose that we have found, for each $m < l \leqslant s$, constant-rank projections $R_l \in \mathrm{M}_\infty(\mathrm{C}(V_l))$ satisfying conditions (i)–(iv) of the conclusion of the lemma, and satisfying condition (v) whenever $t, l > m$. We will construct R_m on V_m so that $R_1,\ldots,R_m$ satisfy (i)–(iv), and satisfy (v) when $t, l \geqslant m$. Proceeding inductively then yields the lemma.

Define a projection-valued map

$$\widetilde{R}_m : \bigcup_{m\leqslant l\leqslant s} V_l \longrightarrow \mathrm{M}_\infty(\mathbb{C})$$

by setting $\widetilde{R}_m(x) = R_l(x)$ if $l > m$ is the smallest index such that $x \in V_l$, and setting $\widetilde{R}_m(x)$ equal to the unit of the (arbitrarily large) matrix algebra which constitutes the target space of all of our projection-valued maps otherwise. One easily checks that $\widetilde{R}_m$ is lower semicontinuous, and that

$$\mathrm{rank}(\widetilde{R}_m - P_{V_m})(x) \geqslant 3d+3, \qquad \text{for all } x \in V_m.$$

An application of [5, Theorem 3.1] then yields a constant-rank projection-valued map $Q_m \in \mathrm{M}_\infty(\mathrm{C}(V_m))$ such that

$$P_{V_m}(x) \leqslant Q_m(x) \leqslant \widetilde{R}_m(x), \qquad \text{for all } x \in V_m,$$

and

$$\mathrm{rank}(\widetilde{R}_m - Q_m)(x) \geqslant 2d+2, \qquad \text{for all } x \in V_m.$$

Applying [5, Theorem 3.1] to $\widetilde{R}_m$ and Q_m yields a constant-rank projection-valued map $Q'_m \in \mathrm{M}_\infty(\mathrm{C}(V_m))$ such that

$$Q_m(x) \leqslant Q'_m(x) \leqslant \widetilde{R}_m(x), \qquad \text{for all } x \in V_m,$$

and

$$\mathrm{rank}(Q'_m - Q_m)(x) \geqslant d+1, \qquad \text{for all } x \in V_m.$$

By [17, Chapter 8, Theorem 1.2], there is a subprojection $\overline{Q_m}$ of $Q'_m - Q_m$ with constant rank at most d such that $T_m := Q_m \oplus \overline{Q_m}$ corresponds to a trivial vector bundle over V_m. By definition we have

$$P_{V_m}(x) \leqslant T_m(x) \leqslant \widetilde{R}_m(x), \qquad \text{for all } x \in V_m.$$

From the definition of $\widetilde{R}_m(x)$ we have

$$P_{V_m}(x) \leqslant T_m(x) \leqslant \bigvee_{m<l\leqslant s} R_l(x), \qquad \text{for all } x \in V_m \cap (V_{m+l} \cup \ldots \cup V_s).$$

Now $R_{m+1} - T_m$ is a trivial projection on $V_m \cap V_{m+1}$, and so can be extended to a trivial projection of the same rank on $V_m \cap (V_{m+1} \cup V_{m+2})$ by Lemma 3.10 (this requires the fact that the R_j, for $m < j \leqslant s$, satisfy condition (iii) in the conclusion of the lemma). We can repeat this extension process until $R_{m+1} - T_m$ has been extended to a trivial constant-rank projection T'_m defined on $V_m \cap (V_{m+1} \cup \ldots \cup V_s)$ and satisfying

$$T'_m(x) \leqslant \left(\bigvee_{m<l\leqslant s} R_l(x) \right) - T_m(x), \qquad \text{for all } x \in V_m \cap (V_{m+l} \cup \ldots \cup V_s).$$

Note that the rank of T'_m is at least d. Choose a trivial subprojection T''_m of T'_m with the property that $\mathrm{rank}(T_m \oplus T''_m) = \mathrm{rank}(R_{m+1}) - d$. Apply Lemma 3.10 to extend T''_m to V_m inside the complement of T_m, and put

$$R_m(x) = T_m(x) \oplus T''_m(x), \qquad \text{for all } x \in V_m.$$

The space $R_m(x)$ so defined has the desired properties. □

Lemma 3.12. *Let X be a finite simplicial complex of dimension d, and let $U_1, \ldots, U_k$ be subcomplexes. Suppose that for each $1 \leqslant i \leqslant k$, there exists a constant-rank projection-valued function $R_i \in \mathrm{M}_\infty(\mathrm{C}(U_i))$ satisfying:*

(i) *R_i corresponds to a trivial vector bundle over U_i;*
(ii) *$\mathrm{rank}(R_1) \geqslant \lceil d/2 \rceil + 1$;*
(iii) *$\mathrm{rank}(R_i) + \lceil d/2 \rceil + 1 \leqslant \mathrm{rank}(R_{i+1})$, for $1 \leqslant i < k$;*
(iv) *$R_i(x) \leqslant R_j(x)$ whenever $x \in U_i \cap U_j$ and $i \leqslant j$.*

Finally, suppose that $\mathrm{rank}(R_i) = n_i$, with $n_1 < \ldots < n_k$, and put $n_0 = 0$. Then, there exist mutually orthogonal rank-one projections $p_1, \ldots, p_{n_k} \in \mathrm{M}_\infty(\mathrm{C}(X))$, each corresponding to a trivial vector bundle on X, such that

$$R = \bigvee_{1\leqslant i\leqslant k} R_i = \bigoplus_{i=1}^{k} \left(\bigoplus_{j=n_{i-1}+1}^{n_i} \chi(U_i \cup \ldots \cup U_k) p_j \right).$$

Proof. We proceed by induction on k. Consider the case $k=1$. Since we may view R_1 as an upper semicontinuous projection-valued map from X into $\mathrm{M}_n(\mathbb{C})$ for n large, and since (X, U_1) is a relative cell complex with cells of dimension at most $d \ll n$, we may apply Lemma 3.10 to extend R_1 to a projection $\widetilde{R}_1$ on X corresponding to a trivial vector bundle. Write $\widetilde{R}_1 = p_1 + \ldots + p_{n_1}$ for line bundles $p_1, \ldots, p_{n_1}$, each of which corresponds to a trivial vector bundle. Then,

$$R_1 = \chi(U_1)\widetilde{R}_1 = \bigoplus_{j=1}^{n_1} \chi(U_1) p_j,$$

as desired.

Now suppose that the lemma holds for $i < k$. Conditions (i) and (iii) in the hypotheses of the lemma, together with Lemma 3.10, allow us to extend R_1 to a trivial projection on $U_1 \cup U_2$ subordinate to $R_1 \vee R_2$. Iterating this process, we extend R_1 to a trivial projection $\widetilde{R}_1$ on $U_1 \cup \ldots \cup U_k$ which is subordinate to $R_1 \vee \ldots \vee R_k$. Finally, apply Lemma 3.10 once more to extend $\widetilde{R}_1$ to a trivial projection on all of X. Write $\widetilde{R}_1 = p_1 + \ldots + p_{n_1}$, where each p_j is a projection corresponding to a one-dimensional trivial vector bundle on X. Then, as before,

$$R_1 = \chi(U_1)\widetilde{R}_1 = \bigoplus_{j=1}^{n_1} \chi(U_1) p_j.$$

By [**17**, Chapter 8, Theorem 1.5], we find that $R_i - R_1$ is trivial for $1 \leqslant i \leqslant k$. If $i > 1$, then $R_i - R_1$ has rank greater than or equal to $\lceil d/2 \rceil + 1$. It follows that the projections $(R_2 - \widetilde{R}_1), (R_3 - \widetilde{R}_1), \ldots, (R_k - \widetilde{R}_1)$ over the subcomplexes $U_2, U_3, \ldots, U_k$, respectively, satisfy the hypotheses of the lemma. Moreover, these projections may be viewed as maps from X into the orthogonal complement of $\widetilde{R}_1$. It follows that there exist rank-one projections $p_{n_1+1}, \ldots, p_{n_k} \in \mathrm{M}_\infty(\mathrm{C}(X))$, each corresponding to a trivial vector bundle and each orthogonal to $\widetilde{R}_1$, which satisfy

$$\bigvee_{2 \leqslant i \leqslant k} (R_i - \widetilde{R}_1) = \bigoplus_{i=2}^{k} \left(\bigoplus_{j=n_{i-1}+1}^{n_i} \chi(U_i \cup \ldots \cup U_k) p_j \right).$$

We have

$$R = \bigvee_{1 \leqslant i \leqslant k} R_i = R_1 \vee \left(\bigvee_{2 \leqslant i \leqslant k} (R_i - \widetilde{R}_1) \right),$$

and the lemma follows. □

Theorem 3.13. *Let X be a finite simplicial complex, and let $a \in \mathrm{M}_\infty(\mathrm{C}(X))_+$. Then, $a \precsim d$ for any $d \in \mathrm{M}_\infty(\mathrm{C}(X))_+$ which is trivial and satisfies*

$$\operatorname{rank}(a)(x) + 4\dim(X) + 3 \leqslant \operatorname{rank}(d)(x), \qquad \textit{for all } x \in X.$$

Proof. We may assume that $\|a\| \leqslant 1$. Let $n_1 < \ldots < n_k$ be the rank values taken by a, and let $\epsilon > 0$ be given. Form an approximant f to a satisfying the conclusions of Theorem 3.9, where the sets $\overline{F_1}, \ldots, \overline{F_k}$ corresponding to f (cf. Definition 3.4) are subcomplexes of X satisfying the conclusion of Lemma 3.8. Let $p_1, \ldots, p_k$ be the supporting projections for f, and notice that these satisfy the hypotheses of Lemma 3.11. The conclusion of Lemma 3.11 then provides a family of constant-rank projections $R_l \in \mathrm{M}_\infty(\mathrm{C}(V_l))$, where $1 \leqslant l \leqslant s$ (each V_l is a union of consecutive sets $\overline{F_i}$). These, in addition to satisfying the hypotheses of Lemma 3.12, have the properties that

$$R(x) := \bigvee_{1 \leqslant l \leqslant s} R_l(x) \geqslant \bigvee_{1 \leqslant i \leqslant k} p_i(x) \geqslant f(x), \qquad \text{for all } x \in X,$$

and

$$\mathrm{rank}(R)(x) - \mathrm{rank}\left(\bigvee_{1 \leqslant i \leqslant k} p_i(x) \right) \leqslant 4\dim(X) + 3.$$

Set $m_0 = 0$ and $m_l = \mathrm{rank}(R_l)$. Let $q_1, \ldots, q_{n_k}$ be the family of mutually orthogonal rank-one projections, each corresponding to a trivial bundle, which are provided by the conclusion of Lemma 3.12, that is,

$$R = \bigoplus_{l=1}^{s} \left(\bigoplus_{j=m_{l-1}+1}^{m_l} \chi(V_l \cup \ldots \cup V_s) q_j \right).$$

By construction (see the proof of Theorem 3.9) we have

$$H_{i,a} = \{x \in X \mid \mathrm{rank}(a)(x) \leqslant n_i\} \subseteq (\overline{F_1} \cup \ldots \cup \overline{F_i})^\circ.$$

Put

$$F_{i,a} = \{x \in X \mid \mathrm{rank}(a)(x) = n_i\}.$$

Use Lemma 3.8 to find, inductively (and upon refining the simplicial structure of X if necessary), subcomplexes $U_1, \ldots, U_k$ of X satisfying

$$\overline{F_i} \cup \ldots \cup \overline{F_k} \subseteq (U_i \cup \ldots \cup U_k)^\circ \subseteq U_i \cup \ldots \cup U_k \subseteq H^c_{i-1,a}, \qquad \text{for } 1 \leqslant i \leqslant k.$$

The set V_l is a union of consecutive $\overline{F_i}$; let J_l be the set of indices occurring among these $\overline{F_i}$, and let M_l be the largest element of J_l. Put

$$\widetilde{V_l} = \bigcup_{i \in J_l} U_i.$$

For each $1 \leqslant l \leqslant s$, choose a continuous function $g_l : X \to [0,1]$ which is identically 1 on $V_l \cup \ldots \cup V_s \subseteq H^c_{M_{l-1},a}$, identically zero on $H_{M_{l-1},a}$, and non-zero off $H_{M_{l-1},a}$. Then,

$$\widetilde{R}(x) = \bigoplus_{l=1}^{s} \left(\bigoplus_{j=m_{l-1}+1}^{m_l} g_l(x) q_j \right)$$

is a trivial element of $\mathrm{M}_\infty(\mathrm{C}(X))_+$ such that $\widetilde{R}(x) \geqslant R(x)$, for all $x \in X$.

We claim that for all $x \in X$, we have

$$\mathrm{rank}(\widetilde{R})(x) - \mathrm{rank}(a)(x) \leqslant 3\dim(X) + 3.$$

Indeed,

$$A_l := \{x \in X \mid \mathrm{rank}(\widetilde{R})(x) = m_l\} = V_l \cap (V_1 \cup \ldots \cup V_{l-1})^c,$$

and g_j is non-zero on A_l if and only if $j \leqslant l$. We have chosen g_j to satisfy

$$g_j(x) \neq 0 \quad \Longleftrightarrow \quad x \in H^c_{M_{j-1},a},$$

so $x \in A_l$ if and only if

$$x \in H^c_{M_{j-l},a} \cap H_{M_l,a}.$$

Thus, for $x \in A_l$, $\operatorname{rank}(a)(x) = n_i$ for some $i \in J_l$. We have

$$\operatorname{rank}(R)(x) - \operatorname{rank}\left(\bigvee_{1\leqslant i\leqslant k} p_i(x)\right) \leqslant 4\dim(X) + 3,$$

whence $m_l - n_i \leqslant 4\dim(X) + 3$, for all $i \in J_l$. This proves the claim.

If d is trivial and satisfies the hypotheses of the theorem, then

$$\operatorname{rank}(d)(x) - \operatorname{rank}(\widetilde{R})(x) \geqslant 0, \qquad \text{for all } x \in X.$$

It follows from Proposition 3.6 that

$$f \leqslant \widetilde{R} \precsim d.$$

Thus, for every $\epsilon > 0$, there exists $v \in \mathrm{M}_\infty(\mathrm{C}(X))$ such that

$$\|vdv^* - f\| < \epsilon.$$

It follows that

$$\|vdv^* - a\| \leqslant \epsilon + \|vdv^* - f\| < 2\epsilon;$$

ϵ was arbitrary, and the theorem follows. □

3.7. *Trivial minorants*

Theorem 3.14. *Let X be a compact metric space, and $a \in \mathrm{M}_\infty(\mathrm{C}(X))_+$. Then, $d \precsim a$ whenever d is trivial and satisfies*

$$\operatorname{rank}(d)(x) \leqslant \max\{\operatorname{rank}(a)(x) - \dim(X) - 1, 0\}, \qquad \textit{for all } x \in X.$$

Proof. The hypotheses of the theorem imply that $d = 0$ if $\dim(X) = \infty$, in which case the theorem holds.

Suppose that $\dim(X) < \infty$. We proceed by induction on k. Suppose that a takes rank values $n_1 < \ldots < n_k$, and, with $n_0 = 0$, put

$$G_i = \{x \in X \mid \operatorname{rank}(a)(x) > n_{i-1}\}.$$

Suppose that $k = 1$. If $n_1 \leqslant \dim(X) + 1$, then there is nothing to prove, so assume that $n_1 > \dim(X) + 1$. By [**17**, Chapter 8, Theorem 1.5], a (which, since it only takes one rank value, is Cuntz equivalent to a projection) is Cuntz equivalent to $\theta_{n_1-\dim(X)+1} \oplus p$ for some projection $p \in \mathrm{M}_\infty(\mathrm{C}(X))$. The projection $\theta_{n_1-\dim(X)+1}$ is trivial, and so dominates any trivial d satisfying the hypotheses of the theorem by Proposition 3.6.

Now suppose that we have proved the theorem when a takes $i < k$ rank values. We treat two cases: $n_1 \leqslant \dim(X) + 1$ and $n_1 > \dim(X) + 1$.

Suppose first that $n_1 \leqslant \dim(X) + 1$. Then, any trivial d satisfying the hypotheses of the theorem necessarily satisfies

$$\operatorname{rank}(d)(x) = 0, \qquad \text{for all } x \in G_2^c.$$

Put $\epsilon_n = 1/2^n$. Since $\|d(x)\|$ is uniformly continuous on $\overline{G_2}$, there exists a sequence (δ_n) of non-negative reals such that $\|d(x) - d(y)\| < \epsilon_n/3$ whenever $\operatorname{dist}(x, y) < \delta_n$. For each $n \in \mathbb{N}$, let $V_n \subseteq \overline{G_2}$ be the closed set which is the complement of the set of points in $\overline{G_2}$ whose distance from $\overline{G_2} \setminus G_2$ is strictly less than δ_n, and let U_n be the closed set consisting of those points in $\overline{G_2}$ whose distance from $\overline{G_2} \setminus G_2$ is less than or equal to $\delta_n/2$. Choose a function $f_n : \overline{G_2} \to [0, 1]$ which is identically zero on U_n and identically 1 on V_n. Notice that $\|d(x)\| < \epsilon$, for all $x \in \overline{G_2} \setminus V_n$. Upon restriction to $\overline{U_n^c}$, a takes at most $k - 1$ rank values. Since $a|_{\overline{U_n^c}}$ and $d|_{\overline{U_n^c}}$ satisfy the hypotheses of the theorem *a fortiori*, there is an element $w_n \in \mathrm{M}_\infty(\mathrm{C}(\overline{U_n^c}))$

such that

$$\left\| w_n (a|_{\overline{U_n^c}}) w_n^* - d|_{\overline{U_n^c}} \right\| < \epsilon_n.$$

Put $g_n = f_n \cdot w_n$, and note that since f_n is zero off $\overline{U_n^c}$, we may view g_n as an element of $\mathrm{M}_\infty(\mathrm{C}(\overline{G_2}))$. Now,

$$\|g_n(x)a(x)g_n^*(x) - d(x)\| < \epsilon_n, \qquad \text{for all } x \in \overline{V_n},$$

and, for every $x \in V_n^c$,

$$\begin{aligned} \|g_n(x)a(x)g_n^*(x) - d(x)\| &\leqslant \|f_n(x)w_n(x)a(x)w_n^*(x)f_n(x)\| + \|d(x)\| \\ &\leqslant \|f_n(x)^2(w_n(x)a(x)w_n^*(x) - d(x))\| + 2\|d(x)\| \\ &\leqslant 3\|d(x)\| < \epsilon_n \end{aligned}$$

(the second-to-last inequality uses the fact that $f_n(x) = 0$, for all $x \in U_n$). Thus,

$$g_n a g_n^* \xrightarrow{n\to\infty} d,$$

and $d \precsim a$, as desired.

Now suppose that $n_1 > \dim(X) + 1$. We will reduce to the case $n_1 \leqslant \dim(X) + 1$, proving the theorem. By [9, Proposition 3.2], there is a projection p on X such that

$$\mathrm{rank}(p) \geqslant \lceil \dim(X)/2 \rceil + (n_1 - \dim(X) - 1)$$

and $p \precsim a$. An application of [17, Chapter 8, Theorem 1.2] yields a trivial subprojection θ_m of p, where $m = n_1 - \dim(X) - 1$. By [20, Proposition 2.2], there is a positive element $b \in \mathrm{M}_\infty(\mathrm{C}(X))$ such that $a \sim b \oplus \theta_m$. Let $p_1, \ldots, p_m$ be the first m trivial rank-one projections supporting d (cf. Definition 3.4). Then, $p_1 \oplus \ldots \oplus p_m \sim \theta_m$ (where $\sim$ is in fact true Murray–von Neumann equivalence), and

$$d' := (1 - (p_1 \oplus \ldots \oplus p_m))d(1 - (p_1 \oplus \ldots \oplus p_m))$$

and

$$d'' := (p_1 \oplus \ldots \oplus p_m)d(p_1 \oplus \ldots \oplus p_m)$$

are trivial. Moreover, d' and b (substituted for d and a, respectively) satisfy the hypotheses of the theorem, b takes k rank values, and the lowest rank value taken by b is less than or equal to $\dim(X) + 1$. We may thus apply our proof above to conclude that $d' \precsim b$. Since $d'' \precsim \theta_m$, we have

$$d = d' \oplus d'' \precsim b \oplus \theta_m \sim a,$$

as desired. □

3.8. *The main theorem*

THEOREM 3.15. *Let X be a compact metric space of covering dimension $d \in \mathbb{N}$. Let $a, b \in \mathrm{M}_n(\mathrm{C}(X))$ be positive, and suppose that*

$$\mathrm{rank}(a)(x) + 9d \leqslant \mathrm{rank}(b)(x), \qquad \textit{for all } x \in X.$$

Then, $a \precsim b$.

Proof. Combining Theorems 3.13 and 3.14 yields the theorem with $9d$ replaced by $5d + 4$. But if $d = 0$, then

$$\mathrm{rank}(a)(x) \leqslant \mathrm{rank}(b)(x), \qquad \text{for all } x \in X,$$

implies that $a \precsim b$ by [19, Theorem 3.3]. If $d \geqslant 1$, then $5d + 4 \leqslant 9d$. □

Theorem 3.15 applies equally well to locally compact second countable Hausdorff spaces whose one-point compactifications have finite covering dimension. We fully expect that it will generalise to recursive subhomogeneous C*-algebras.

4. *Applications to AH algebras*

4.1. *The dimension-rank ratio vs. the radius of comparison*

Recall the terminology concerning AH algebras from Section 1. A unital AH algebra A has *flat dimension growth* [**23**, Definition 1.2] if it admits a decomposition for which

$$\limsup_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} < \infty. \tag{4.1}$$

The study of unital AH algebras with flat dimension growth was suggested by Blackadar [**1**] in 1991, but there were no non-trivial examples of such algebras — algebras with flat dimension growth but *not* slow dimension growth — until the pioneering work of Villadsen in 1997 [**28**]. We initiated the study of such algebras in earnest in [**23**]. Our key tool was the *dimension-rank ratio* of a unital AH algebra A (write $\mathrm{drr}(A)$), an isomorphism invariant which is defined to be the infimum of the set of non-negative reals c such that A has a decomposition satisfying

$$\limsup_{i\to\infty} \max_{1\leqslant l\leqslant m_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} \leqslant c. \tag{4.2}$$

This invariant may be thought of as a measure of the ratio of the topological dimension of A to its matricial size, despite the fact that both quantities may be infinite. Its effectiveness in studying AH algebras of flat dimension growth leads one naturally to consider whether it has an analogue defined for all unital and stably finite C*-algebras. In [**23**] we introduced the following candidate.

Definition 4.1 [**23**, Definition 6.1]. Say that A *has r-comparison* if whenever one has positive elements $a, b \in \mathrm{M}_\infty(A)$ such that

$$s(\langle a\rangle) + r < s(\langle b\rangle), \qquad \text{for all } s \in \mathrm{LDF}(A),$$

then $\langle a\rangle \leqslant \langle b\rangle$ in $W(A)$. Define the radius of comparison of A, denoted $\mathrm{rc}(A)$, to be

$$\inf\{r \in \mathbb{R}^+ \mid (W(A), \langle 1_A\rangle) \text{ has } r\text{-comparison}\}$$

if it exists, and ∞ otherwise.

Theorem 3.15 confirms the radius of comparison as the proper abstraction of the dimension-rank ratio for semi-homogeneous algebras.

Theorem 4.2. *There exist constants $K_1, K_2 > 0$ such that for any semi-homogeneous C^*-algebra*

$$A = \bigoplus_{i=1}^{n} p_i(\mathrm{C}(X_i) \otimes \mathcal{K})p_i$$

with each X_i a connected finite-dimensional CW-complex, one has the following inequalities:

$$\mathrm{drr}(A) \leqslant K_1\,\mathrm{rc}(A)$$

and

$$\mathrm{rc}(A) \leqslant K_2\,\mathrm{drr}(A).$$

Proof. Suppose first that $n = 1$. The first inequality is [23, Theorem 6.6]. For the second inequality, use [23, Theorem 2.2] to conclude that $\mathrm{drr}(A) = \dim(X)/\operatorname{rank}(p)$; the inequality now follows from the definition of the radius of comparison and Theorem 3.15. To prove the theorem for general n use the following facts:

$$\mathrm{drr}(A \oplus B) = \max\{\mathrm{drr}(A), \mathrm{drr}(B)\}$$

and

$$\mathrm{rc}(A \oplus B) = \max\{\mathrm{rc}(A), \mathrm{rc}(B)\}$$

for any unital AH algebras A and B (see [23, Proposition 2.2(ii)] and [23, Proposition 6.2(ii)], respectively). □

We will address the relationship between the dimension-rank ratio and the radius of comparison for general AH algebras in a separate paper.

4.2. *Strict comparison of positive elements*

The next lemma is due to M. Rørdam. We are grateful for his permission to use it here.

LEMMA 4.3 (Rørdam, private communication). *Let A be a C^*-algebra and $\{A_i\}_{i\in I}$ a collection of C^*-subalgebras whose union is dense. Then, for every $a \in \mathrm{M}_\infty(A)_+$ and $\epsilon > 0$ there exist $i \in I$ and $\tilde{a} \in \mathrm{M}_\infty(A_i)$ such that*

$$(a - \epsilon)_+ \precsim \tilde{a} \precsim (a - \epsilon/2)_+ \precsim a$$

in $\mathrm{M}_\infty(A)$.

Proof. First find a positive element b in some $\mathrm{M}_\infty(A_i)$, with $i \in I$, such that

$$\|b - (a - \epsilon/2)_+\| < \epsilon/4.$$

Put $\tilde{a} := (b - \epsilon/4)_+$. The conclusion follows from Proposition 2.1 and the estimate $\|a - \tilde{a}\| < \epsilon$. □

LEMMA 4.4. *Let A be the limit of an inductive system $(A_i, \phi_i)_{i\in\mathbb{N}}$ of C^*-algebras, where ϕ_i is injective for each $i \in \mathbb{N}$. Let $a, b \in \mathrm{M}_\infty(A_i)$ be positive elements such that $\phi_{i\infty}(a) \precsim \phi_{i\infty}(b)$ in $\mathrm{M}_\infty(A)$. Then, for every $\epsilon > 0$ there is a $j > i$ such that*

$$(\phi_{ij}(a) - \epsilon)_+ \precsim \phi_{ij}(b)$$

inside $\mathrm{M}_\infty(A_j)$.

Proof. By working in a matrix algebra over A, we may assume that $a, b \in A$. Since the ϕ_i are injective, we simply identify a and b with their forward images in A_j, for $j \geqslant i$, and in A itself. We have $a \precsim b$ in $\mathrm{M}_\infty(A)$, so there is a sequence (v_n) in $\mathrm{M}_\infty(A)$ such that

$$v_n b v_n^* \xrightarrow{n\to\infty} a.$$

This sequence may be chosen to lie in the dense local C*-algebra $\bigcup_{i=1}^\infty A_i$. Indeed, for any $w_n \in A$ we have

$$\begin{aligned}\|w_n b w_n * - v_n b v_n^*\| &= \|(w_n - v_n + v_n)b(w_n - v_n + v_n)^* - v_n b v_n^*\| \\ &= \|(w_n - v_n)b(w_n - v_n)^* + (w_n - v_n)b v_n * + v_n b(w_n - v_n)^*\| \\ &\leqslant \|(w_n - v_n)\|(\|b(w_n - v_n)^*\| + \|b v_n^*\| + \|b v_n\|),\end{aligned}$$

so choosing $w_n \in \bigcup_{i=1}^{\infty} A_i$ sufficiently close to v_n yields

$$w_n b w_n^* \xrightarrow{n\to\infty} a.$$

Let $\epsilon > 0$ be given. Find $i, n \in \mathbb{N}$ such that

$$\|w_n b w_n^* - a\| < \epsilon$$

and $a, b, w_n \in \mathrm{M}_\infty A_i$. It then follows from Proposition 2.1 that $(a-\epsilon)_+ \precsim b$ inside $\mathrm{M}_\infty(A_i)$, as desired. □

THEOREM 4.5. *Let (A_i, ϕ_i) be an inductive sequence of unital, exact, and stably finite C^*-algebras with simple limit A. Suppose further that each ϕ_i is injective and that*

$$\liminf_{i\to\infty} \mathrm{rc}(A_i) = 0.$$

Then, $\mathrm{rc}(A) = 0$. In particular, A has strict comparison of positive elements.

Proof. We will prove that $W(A)$ is almost unperforated, that is, that whenever one has $\langle a\rangle, \langle b\rangle \in W(A)$ such that $(n+1)\langle a\rangle \leqslant n\langle b\rangle$, then $\langle a\rangle \leqslant \langle b\rangle$. It then follows from [**21**, Corollary 4.6] that A has strict comparison of positive elements.

Let $a, b \in \mathrm{M}_\infty(A)$ be positive, and suppose that $(n+1)a \precsim nb$ for some $n \in \mathbb{N}$. Let $\epsilon > 0$ be given, and find $\delta > 0$ such that

$$((n+1)a - \epsilon/2)_+ = (n+1)(a-\epsilon/2)_+ \precsim n(b-\delta)_+ = (nb-\delta)_+.$$

Use Lemma 4.3 to find some $i \in \mathbb{N}$ and $\tilde{a}, \tilde{b} \in \mathrm{M}_\infty(A_i)_+$ such that

$$(a - 3\epsilon/4)_+ \precsim \tilde{a} \precsim (a-\epsilon/2)_+ \qquad \text{and} \qquad (b-\delta)_+ \precsim \tilde{b} \precsim b. \tag{4.3}$$

It follows that

$$(n+1)(a-\epsilon)_+ \precsim (n+1)(\tilde{a} - \epsilon/4)_+ \precsim n\tilde{b} \precsim nb.$$

By Lemma 4.4 we may, by increasing i if necessary, assume that

$$(n+1)(\tilde{a}-\epsilon/4)_+ \precsim n\tilde{b} \tag{4.4}$$

inside $\mathrm{M}_\infty(A_i)$. Since A is simple, we may assume that the images of both $(\tilde{a}-\epsilon/4)_+$ and $\tilde{b}$ under any $\tau \in \mathrm{T}(A_i)$ are non-zero. It follows that

$$s((\tilde{a}-\epsilon/4)_+) \neq 0 \qquad \text{and} \qquad s(\tilde{b}) \neq 0, \qquad \text{for all } s \in \mathrm{LDF}(A_i).$$

Equation (4.4) shows that for any $\tau \in \mathrm{T}(A_i)$,

$$s_\tau(\tilde{b}) - s_\tau((\tilde{a}-\epsilon/4)_+) \geqslant (1/n)s_\tau(\tilde{b}).$$

The map $\tau \mapsto s_\tau(\tilde{b})$ is strictly positive and lower semicontinuous on the compact space $\mathrm{T}(A_i)$. It therefore achieves a minimum value $c > 0$, and

$$s_\tau((\tilde{a}-\epsilon/4)_+) + c/2 < s_\tau(\tilde{b}), \qquad \text{for all } \tau \in \mathrm{T}(A_i).$$

Increasing i if necessary, we may assume that $\mathrm{rc}(A_i) < c/2$, whence

$$(\tilde{a}-\epsilon/4)_+ \precsim \tilde{b} \precsim b \tag{4.5}$$

by the definition of $\mathrm{rc}(\,\cdot\,)$. From (4.3) we have the inequality

$$(a-3\epsilon/4)_+ \precsim \tilde{a},$$

whence

$$(a-\epsilon)_+ \precsim (\tilde{a}-\epsilon/4)_+ \tag{4.6}$$

by the functional calculus. Combining (4.5) and (4.6) we have

$$(a-\epsilon)_+ \precsim b.$$

Since ϵ was arbitrary, the theorem follows from Proposition 2.1. □

COROLLARY 4.6. *Let A be a simple unital AH algebra with slow dimension growth. Then, $W(A)$ is almost unperforated. In particular, A has strict comparison of positive elements.*

Proof. As in equation (1.1) we have $A = \lim_{i\to\infty}(A_i, \phi_i)$, where

$$A_i = \bigoplus_{l=1}^{n_i} p_{i,l}(\mathrm{C}(X_{i,l}) \otimes \mathcal{K})p_{i,l}.$$

Now A has slow dimension growth, so we may assume that the ϕ_i are injective (this is the main result of [12]) and that

$$\max_{1\leqslant l\leqslant n_i} \left\{ \frac{\dim(X_{i,1})}{\mathrm{rank}(p_{i,1})}, \ldots, \frac{\dim(X_{i,n_i})}{\mathrm{rank}(p_{i,n_i})} \right\} \xrightarrow{i\to\infty} 0.$$

This last condition, by [23, Theorem 2.3], implies that $\mathrm{drr}(A_i) \to 0$ as $i \to \infty$. Then Proposition 4.2 shows that $\mathrm{rc}(A_i) \to 0$ as $i \to \infty$, and we have collected the hypotheses of Theorem 4.5. □

Combining Corollary 4.6 with earlier work of Rørdam we recover the characterisation of real rank zero for simple unital AH algebras of slow dimension growth obtained in [3].

COROLLARY 4.7 (Blackadar, Dădărlat and Rørdam, [3]). *Let A be a simple unital AH algebra with slow dimension growth. Then, the following are equivalent:*
(i) *A has real rank zero;*
(ii) *the projections in A separate tracial states;*
(iii) *the image of $\mathrm{K}_0(A)$ is uniformly dense in the space of continuous affine functions on the tracial state space $\mathrm{T}(A)$.*

Proof. The algebra A has stable rank 1 by the results of [3], and a weakly unperforated Cuntz semigroup by Corollary 4.6. Apply [21, Theorem 7.2]. □

Finally, we note that several theorems on $\mathcal{Z}$-stable C*-algebras recently proved by Brown, Perera, and the author can be extended to AH algebras with slow dimension growth.

COROLLARY 4.8. *Let A be an element of $\mathcal{SDG}$. The following statements hold:*
(i) *$W(A)$ is recovered functorially from the Elliott invariant of A;*
(ii) *$\mathrm{LDF}(A)$ is weak-$*$ dense in $\mathrm{DF}(A)$;*
(iii) *$\mathrm{DF}(A)$ is a Choquet simplex.*

Proof. Since A has strict comparison of positive elements, we may appeal to [6, Theorem 5.4 and Corollary 6.9]. □

We refer the reader to [20, Section 4] for an explicit description of the functor which reconstructs $W(A)$ from the Elliott invariant of A in (i) above.

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