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Three Applications of the Cuntz Semigroup

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Building on work of Elliott and coworkers, we present three applications of the Cuntz semigroup:

(i) for many simple C*-algebras, the Thomsen semigroup is recovered functorially from the Elliott invariant, and this yields a new proof of Elliott's classification theorem for simple, unital AI algebras;

(ii) for the algebras in (i), classification of their Hilbert modules is similar to the von Neumann algebra context;

(iii) for the algebras in (i), approximate unitary equivalence of self-adjoint operators is characterised in terms of the Elliott invariant.

1 Introduction

The Cuntz semigroup (see [6], [9], [12], [13] for definitions and basic properties) has recently become quite popular. In this note we extend the main theorem of [3] to stable C*-algebras. By combining this result with those of Coward-Elliott-Ivanescu [5] and Elliott-Ciuperca [4], we obtain the applications of the abstract directly (see

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Theorems 4.1 and 4.2 for the Thomsen semigroup result, Theorem 3.4 for Hilbert modules and Theorem 5.1 for unitary orbits of self-adjoints). For the reader interested primarily in Elliott's classification program, we emphasize that most of our results are formulated in terms of the Elliott invariant—the Cuntz semigroup is a powerful technical tool used only in proofs. This paper is a natural sequel to [3] and [12], and the latter contain the requisite definitions, notation, and basic facts employed herein. Finally, we thank the referee for a number of helpful comments and suggestions.

2 Computation of $\mathrm{W}(A \otimes \mathcal{K})$

Throughout this paper A will denote a unital simple separable C^*-algebra with tracial states. Let $\mathrm{W}(A)$ denote the Cuntz semigroup of A and let $\mathrm{T}(A)$ denote the simplex of tracial states. Since $\mathrm{T}(A) \neq \emptyset$, A is stably finite. It follows that $\mathrm{W}(A)$ can be decomposed into the disjoint union of $\mathrm{V}(A)$ (the Murray-von Neumann semigroup of equivalence classes of projections) and the set $\mathrm{W}(A)_+$ of Cuntz classes of positive elements which are *not* equal to the class of a projection. If $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$ denotes the bounded, lower semicontinuous, affine, strictly positive functions on $\mathrm{T}(A)$, then there is a canonical map

$$\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$$

given by

$$\iota(\langle x \rangle)(\tau) = d_\tau(x),$$

where $d_\tau(x) := \lim_{n\to\infty} \tau \otimes \mathrm{Tr}_k(x^{1/n})$ for an element $x \in A \otimes M_k(\mathbb{C})$. (Here Tr_k is the *non-normalized* trace on $M_k(\mathbb{C})$.) The main theorem of [3] was that ι is an order isomorphism, whence

$$\mathrm{W}(A) \cong \mathrm{V}(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++} \tag{2.1}$$

as partially ordered semigroups for two important classes of C^*-algebras: simple unital exact finite C^*-algebras which absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially, and simple unital AH algebras with slow dimension growth. (We refer the reader to [12] for the definition of the order structure on $\mathrm{V}(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++}$. As usual, $\mathcal{Z}$ denotes the Jiang-Su algebra—see [8].) In this section we prove a structure theorem similar to (2.1) for $\mathrm{W}(A \otimes \mathcal{K})$, with A as above.

Recall that A has *strict comparison* if $x \precsim y$ whenever $d_\tau(x) < d_\tau(y)$ for all $\tau \in \mathrm{T}(A)$. ($\precsim$ denotes Cuntz's relation and $x \sim y$ means $x \precsim y$ and $y \precsim x$.) When A is unital simple exact and has strict comparison, the map ι is an isomorphism whenever it is surjective (cf. [12, Proposition 3.3]).

Lemma 2.1. Let A be a simple unital exact C*-algebra, and let $\langle a \rangle \in \mathrm{W}(A)_+$ be given. It follows that for any $\epsilon > 0$, there exists $\delta > 0$ and a continuous affine function $f : \mathrm{T}(A) \to \mathbb{R}^+$ such that

$$d_\tau((a-\epsilon)_+) < f(\tau) < d_\tau((a-\delta)_+),\ \forall \tau \in \mathrm{T}(A). \qquad \square$$

Proof. First note that zero must be an accumulation point of the spectrum $\sigma(a)$ (otherwise, functional calculus would provide a projection with the same Cuntz class as $\langle a \rangle \in \mathrm{W}(A)_+$, which is impossible). Choose points $\delta < \eta \in (0,\epsilon) \cap \sigma(a)$ so that each of (δ,η) and (η,ϵ) are nonempty. Since A is simple, each trace and hence each lower semicontinuous dimension function is faithful. It follows from a functional calculus argument that

$$d_\tau((a-\epsilon)_+) < d_\tau((a-\eta)_+) < d_\tau((a-\delta)_+),\ \forall \tau \in \mathrm{T}(A).$$

Let μ_τ be the (regular Borel) measure induced on $\sigma(a)$ by $\tau \in \mathrm{T}(A)$. The affine map $h : \mathrm{T}(A) \to \mathbb{R}^+$ given by

$$h(\tau) := \mu_\tau([\epsilon,\infty) \cap \sigma(a))$$

is upper semicontinuous by the Portmanteau Theorem ([1]). From the inclusions

$$(\epsilon,\infty) \cap \sigma(a) \subseteq [\epsilon,\infty) \cap \sigma(a) \subseteq (\eta,\infty) \cap \sigma(a)$$

we have the following inequalities:

$$d_\tau((a-\epsilon)_+) \le h(\tau) \le d_\tau((a-\eta)_+) < d_\tau((a-\delta)_+),\ \forall \tau \in \mathrm{T}(A).$$

The affine map $\tau \mapsto d_\tau((a-\delta)_+)$ is strictly positive and lower semicontinuous. Since $\mathrm{T}(A)$ is a metrizable compact convex set, this map is the pointwise supremum of a strictly increasing sequence of continuous affine maps, say $(f_n)_{n=1}^\infty$. A straightforward argument using compactness then shows that there is some $n_0 \in \mathbb{N}$ such that

$$f_n(\tau) > h(\tau),\ \forall \tau \in \mathrm{T}(A),\ \forall n \ge n_0.$$

Setting $f(\tau) = f_{n_0}(\tau)$ completes the proof. ∎

Let A be a unital C*-algebra and $a \in A \otimes \mathcal{K}$ be positive. Let $\{e_n\} \subset \mathcal{K}$ be an increasing sequence of projections with $\mathrm{rank}(e_n) = n$, and put $P_n = 1 \otimes e_n \in A \otimes \mathcal{K}$. Then,

$$P_1 a P_1 \precsim P_2 a P_2 \precsim P_3 a P_3 \precsim \cdots$$

in $\mathrm{W}(A \otimes \mathcal{K})$ and $P_n a P_n \to a$ in norm. Let $b = \sup_n \langle P_n a P_n \rangle \in \mathrm{W}(A \otimes \mathcal{K})$ (suprema of increasing sequences in $\mathrm{W}(A \otimes \mathcal{K})$ always exist by [5, Theorem 1]). Then, given $\epsilon > 0$, there is some $n \in \mathbb{N}$ such that

$$(a - \epsilon)_+ \precsim P_n a P_n \precsim b.$$

It follows that $a \precsim b$. Since $P_n a P_n \precsim a$ for each n, we also have that $b \precsim a$, which shows $a \sim b$, i.e. $\langle a \rangle = \sup_n \langle P_n a P_n \rangle$.

Lemma 2.2. Let A be a simple unital exact C*-algebra, and let $a \in A \otimes \mathcal{K}$ be a positive element such that $\langle a \rangle \in \mathrm{W}(A \otimes \mathcal{K})_+$. It follows that there is a sequence $(a_n)_{n=1}^{\infty}$ of positive elements in $A \otimes \mathcal{K}$ satisfying the following conditions:

(i) $\langle a \rangle = \sup_n \langle a_n \rangle$;

(ii) $a_n \in A \otimes \mathrm{M}_{k(n)}$ for some $k(n) \in \mathbb{N}$;

(iii) for each n there is a continuous affine function $f_n : \mathrm{T}(A) \to \mathbb{R}$ such that

$$d_\tau(a_n) < f(\tau) < d_\tau(a_{n+1}), \ \forall \tau \in \mathrm{T}(A). \qquad \square$$

Proof. Let P_n be the unit of $A \otimes \mathrm{M}_n$ (as above) and define $b_n := P_n a P_n$. The sequence b_n satisfies parts (i) and (ii) of the conclusion of the lemma by construction. Note that $b_n \precsim b_{n+1}$.

Case I. Let us first address the case where infinitely many of the b_ns are Cuntz equivalent to a projection. By passing to a subsequence, we may assume that every b_n is Cuntz equivalent to a projection (this does not affect the validity of (i) and (ii)). If infinitely many of the b_ns are Cuntz equivalent to a fixed projection $p \in A \otimes \mathcal{K}$, then we have

$$\langle a \rangle = \sup_n \langle b_n \rangle = \langle p \rangle;$$

this contradicts our assumption that a is not Cuntz equivalent to a projection. Thus, each Cuntz class $\langle b_m \rangle$, $m \in \mathbb{N}$ occurs at most finitely many times in the sequence $(\langle b_n \rangle)_{n=1}^{\infty}$. Passing to a subsequence again, we may assume that $\langle b_m \rangle \neq \langle b_n \rangle$ whenever $m \neq n$.

Put $a_n = b_n$. As noted, $(a_n)_{n=1}^{\infty}$ satisfies parts (i) and (ii) of the conclusion of the lemma already. The map $\tau \mapsto d_\tau(a_n)$ is continuous since a_n is Cuntz equivalent to a projection. The fact that a_n is Cuntz equivalent to a projection also means that it is complemented inside a_{n+1}, i.e., there is a projection p_n in $A \otimes \mathrm{M}_{k(n+1)}$ such that $\langle a_n \rangle + \langle p_n \rangle = \langle a_{n+1} \rangle$ ([12, Proposition 2.2]). Since A is simple, the map $\tau \mapsto d_\tau(p_n)$ is continuous and strictly positive on $\mathrm{T}(A)$. Setting $f_n(\tau) = d_\tau(a_n) + (1/2)d_\tau(p_n)$ then gives condition (iii).

Case II. Now we may assume that none of the b_ns is equivalent to a projection. Given any $\epsilon_1 > 0$, we may use Lemma 2.1 to find $\delta_1 > 0$ and a continuous affine map $f_1 : \mathrm{T}(A) \to \mathbb{R}^+$ such that

$$d_\tau((b_1 - \epsilon_1)_+) < f_1(\tau) < d_\tau((b_1 - \delta_1)_+),\ \forall \tau \in \mathrm{T}(A).$$

Assume that we have found sequences $\epsilon_1, \ldots, \epsilon_n$ and $\delta_1, \ldots, \delta_n$ of strictly positive tolerances satisfying the following conditions:

(a) $(b_k - \epsilon_k)_+ \precsim (b_k - \delta_k)_+$, $k \in \{1, \ldots, n\}$;

(b) there is a continuous affine map $f_k : \mathrm{T}(A) \to \mathbb{R}$ such that

$$d_\tau((b_k - \epsilon_k)_+) < f_k(\tau) < d_\tau((b - \delta_k)_+),\ \forall \tau \in \mathrm{T}(A);$$

(c) $(b_k - \epsilon_k/l)_+ \precsim (b_l - \epsilon_l)$, $1 \le k < l$ and $l \in \{1, \ldots, n\}$;

(d) $(b_k - \delta_k)_+ \precsim (b_{k+1} - \epsilon_{k+1})_+$, $k \in \{1, \ldots, n\}$.

Using the basic properties of Cuntz's comparison relation, we can find ϵ_{n+1} satisfying (c) and (d) above (with $n+1$ in place of l and n in place of k, respectively). Applying Lemma 2.1, we can find δ_{n+1} satisfying (a) and (b) with $n+1$ in place of k. Thus, our sequences $\epsilon_1, \ldots, \epsilon_n$ and $\delta_1, \ldots, \delta_n$ can be extended, inductively, to sequences $(\epsilon_i)_{i=1}^{\infty}$ and $(\delta_i)_{i=1}^{\infty}$ satisfying (a)–(d), as appropriate.

Set $a_n = (b_n - \epsilon_n)_+$. Let us verify condition (i) for this choice of a_n. Since $a_n \precsim a$ for each n, we have

$$\sup_n a_n \precsim a.$$

On the other hand, (c) gives $(b_k - \epsilon_k/n)_+ \precsim a_n$ for every $1 \le k < n$ and $n \in \mathbb{N}$. It follows that

$$\sup_n \langle a_n \rangle \ge \langle (b_k - \epsilon_k/n)_+ \rangle,\ \forall n, k \in \mathbb{N}.$$

In particular,

$$\sup_n \langle a_n \rangle \geq \langle b_k \rangle, \ \forall k \in \mathbb{N},$$

and so

$$\sup_n \langle a_n \rangle \geq \sup_k \langle b_k \rangle = \langle a \rangle.$$

Condition (ii) is satisfied by construction, while condition (iii) is satisfied by the functions f_n from (b) above. This completes the proof. ■

Definition 2.3. For every positive element $a \in A \otimes \mathcal{K}$, define an affine function $\iota\langle a \rangle : \mathrm{T}(A) \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\iota\langle a \rangle(\tau) = \sup_n d_\tau(P_n a P_n),$$

for each trace $\tau \in \mathrm{T}(A)$, where P_n are the projections defined before Lemma 2.2.

In analogy with previous notation, we now observe that $a \mapsto \iota\langle a \rangle$ drops to a well-defined map (denoted by ι) on $\mathrm{W}(A \otimes \mathcal{K})$. (Indeed, if $a \in M_n(A \otimes \mathcal{K})_+$ then we can identify it with a positive element in $(A \otimes \mathcal{K})_+$ and hence define $\iota\langle a \rangle$; since $\iota\langle \cdot \rangle$ is independent of the projections used in its definition, it is not hard to check that our recipe for extending ι to $M_n(A \otimes \mathcal{K})_+$ is independent of the identification $M_n(\mathcal{K}) \cong \mathcal{K}$.)

Lemma 2.4. If $a, b \in (A \otimes \mathcal{K})_+$ and $\langle a \rangle = \langle b \rangle \in \mathrm{W}(A \otimes \mathcal{K})$ then $\iota\langle a \rangle = \iota\langle b \rangle$. Moreover, $\iota\langle a \rangle$ is independent of the choice of projections P_n. □

Proof. Assume $a \sim b$. For each $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a $\delta > 0$ and $m \in \mathbb{N}$ such that

$$(P_n a P_n - 2\epsilon)_+ \precsim (a - \epsilon)_+ \precsim (b - \delta)_+ \precsim P_m b P_m.$$

It follows that for any $\tau \in \mathrm{T}(A)$,

$$\iota\langle b \rangle(\tau) \geq d_\tau(P_m b P_m) \geq d_\tau(P_n a P_n - 2\epsilon)_+.$$

Since n and ϵ were arbitrary, we conclude that $\iota\langle b \rangle(\tau) \geq \iota\langle a \rangle(\tau)$. Similarly, $\iota\langle a \rangle(\tau) \geq \iota\langle b \rangle(\tau)$.

For the second assertion, let $\{e_n\}, \{f_n\} \subset \mathcal{K}$ be increasing sequences of projections with $\mathrm{rank}(e_n) = \mathrm{rank}(f_n) = n$, and put $P_n = 1 \otimes e_n, Q_n = 1 \otimes f_n \in A \otimes \mathcal{K}$. Fix $n \in \mathbb{N}$

and $\epsilon > 0$. Since $\lim_{k\to\infty} \|P_n Q_k - P_n\| = 0$, we can find k such that $\|P_n Q_k a Q_k P_n - P_n a P_n\| < \epsilon$. It follows that $(P_n a P_n - \epsilon)_+ \precsim Q_k a Q_k$ for all sufficiently large k. In particular, $d_\tau((P_n a P_n - \epsilon)_+) \leq \sup_k d_\tau(Q_k a Q_k)$ for every $\epsilon > 0$. Since $d_\tau(P_n a P_n) = \sup_\epsilon d_\tau((P_n a P_n - \epsilon)_+)$, the lemma follows. ■

Proposition 2.5. Let A be a unital simple exact C*-algebra with strict comparison of positive elements. If $\langle a\rangle, \langle b\rangle \in \mathrm{W}(A \otimes \mathcal{K})_+$, then $a \sim b$ if and only if $\iota\langle a\rangle = \iota\langle b\rangle$. □

Proof. The forward implication is contained in Lemma 2.4, so suppose that $\iota\langle a\rangle = \iota\langle b\rangle$. Find, using Lemma 2.2, sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ corresponding to a and b, respectively; let f_n and g_n denote, respectively, the functions provided by part (iii) of the conclusion of Lemma 2.2. By a compactness argument, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $\tau \in \mathrm{T}(A)$ we have the following inequalities:

$$d_\tau(a_n) < f_n(\tau) < d_\tau(b_m); \;\; d_\tau(b_n) < g_n(\tau) < d_\tau(a_m).$$

Since A has strict comparison, $a_n \precsim b_m$ and $b_n \precsim a_m$. It follows that

$$\langle a\rangle = \sup_n \langle a_n\rangle = \sup_m \langle b_m\rangle = \langle b\rangle,$$

as desired. ■

Let $\mathrm{SAff}(\mathrm{T}(A))$ denote the set of functions on $\mathrm{T}(A)$ which are pointwise suprema of increasing sequences of continuous, affine, and strictly positive functions on $\mathrm{T}(A)$. Define an addition operation on the disjoint union $\mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A))$ as follows:

(i) if $x, y \in V(A)$, then their sum is the usual sum in $V(A)$;
(ii) if $x, y \in \mathrm{SAff}(\mathrm{T}(A))$, then $x + y$ is the pointwise sum in $\mathrm{SAff}(\mathrm{T}(A))$;
(iii) if $x \in V(A)$ and $y \in \mathrm{SAff}(\mathrm{T}(A))$, then their sum is the usual (pointwise) sum of $\hat{x}$ and y in $\mathrm{SAff}(\mathrm{T}(A))$, where $\hat{x}(\tau) = \tau(p)$ for some projection p with $\langle p\rangle = x$, $\forall \tau \in \mathrm{T}(A)$.

Equip $\mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A))$ with the partial order $\leq$ which restricts to the usual partial orders on $V(A)$ (i.e. Murray-von Neumann) and $\mathrm{SAff}(\mathrm{T}(A))$ (i.e. $f \leq g \Leftrightarrow f(\tau) \leq g(\tau)$ for all $\tau \in \mathrm{T}(A)$), and which satisfies the following conditions for $x \in V(A)$ and $y \in \mathrm{SAff}(\mathrm{T}(A))$:

(i) $x \leq y$ if and only if $\hat{x}(\tau) < y(\tau)$, $\forall \tau \in \mathrm{T}(A)$;
(ii) $y \leq x$ if and only if $y(\tau) \leq \hat{x}(\tau)$, $\forall \tau \in \mathrm{T}(A)$.

Theorem 2.6. Let A be a unital simple exact and tracial C^*-algebra with strict comparison. Assume that $\iota\colon W(A)_+ \to \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ is surjective. It follows that

$$\mathrm{W}(A \otimes \mathcal{K}) \cong \mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A)),$$

as ordered semigroups. □

Proof. Define

$$\phi : \mathrm{W}(A \otimes \mathcal{K}) \to \mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A))$$

by $\mathbf{id}_{\mathrm{V}(A\otimes\mathcal{K})}$ on $\mathrm{V}(A \otimes \mathcal{K})$ and by ι on $\mathrm{W}(A \otimes \mathcal{K})_+$ (that is, $x \mapsto \iota\langle a\rangle$, where $x = \langle a\rangle$). Let us first prove that ϕ is a bijection. Since A is stably finite, $\mathrm{W}(A \otimes \mathcal{K}) = \mathrm{V}(A) \sqcup \mathrm{W}(A \otimes \mathcal{K})_+$, hence it suffices to show $\iota\colon \mathrm{W}(A \otimes \mathcal{K})_+ \to \mathrm{SAff}(\mathrm{T}(A))$ is a bijection.

Injectivity of ι follows from Proposition 2.5. Surjectivity follows from two facts: (i) the range of ι contains $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$ and (ii) $\mathrm{W}(A \otimes \mathcal{K})$ has suprema (cf. [5]). Indeed, given $f \in \mathrm{SAff}(\mathrm{T}(A))$ we find continuous affine functions $f_n \leq f_{n+1} \leq \cdots$ converging up to f pointwise. Letting $a_n \in A \otimes \mathcal{K}$ be positive elements such that $\hat{a}_n = f_n$, we let $x = \sup_n \langle a_n\rangle \in \mathrm{W}(A \otimes \mathcal{K})$ (we have used strict comparison here to ensure $\{\langle a_n\rangle\}$ is an increasing sequence in $\mathrm{W}(A \otimes \mathcal{K})$). Then it is clear that $\iota(x) = f$.

To complete the proof, we must show that ϕ is order preserving. Suppose that $x \leq y$, $x, y \in \mathrm{W}(A \otimes \mathcal{K})$. There are four cases to consider.

(a) If $x, y \in \mathrm{V}(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since $\phi|_{\mathrm{V}(A\otimes\mathcal{K})} = \mathbf{id}_{\mathrm{V}(A\otimes\mathcal{K})}$.

(b) If $x, y \in \mathrm{W}(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since $\phi|_{\mathrm{W}(A\otimes\mathcal{K})} = \iota$ and ι is order-preserving. (The proof of this last fact follows from the proof of the first implication in Proposition 2.5.)

(c) If $x \in \mathrm{V}(A \otimes \mathcal{K})$ and $y \in \mathrm{W}(A \otimes \mathcal{K})_+$, then we apply [12, Proposition 2.2] to find $z \in \mathrm{W}(A \otimes \mathcal{K})$ such that $x + z = y$. It follows that $\iota(x)(\tau) < \iota(y)(\tau)$, $\forall \tau \in \mathrm{T}(A)$ (note that $\iota(x)(\tau) < \infty$ in this case), whence $\phi(x) \leq \phi(y)$.

(d) If $x \in \mathrm{W}(A \otimes \mathcal{K})_+$ and $y \in \mathrm{V}(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since ι is order-preserving. ■

The theorem above holds for all simple unital AH algebras with slow dimension growth, and for the class of simple unital exact stably finite C^*-algebras which absorb $\mathcal{Z}$ ([3, Theorems 5.3 and 5.5], [14, Corollary 4.6], [16, Corollary 4.6]).

3 Classifying Hilbert modules

Let E, F be countably generated Hilbert modules over a separable, unital C*-algebra A. By Kasparov's stabilization theorem, there are projections $P_E, P_F \in L(H_A)$ such that E is isomorphic to $P_E H_A$ and F is isomorphic to $P_F H_A$. (Here $H_A = \ell^2 \otimes A$ is the standard Hilbert module over A and $L(H_A)$ is the set of bounded adjointable operators on H_A; see [10] for more.) Since $L(H_A) = M(A \otimes \mathcal{K})$ (the multiplier algebra of $A \otimes \mathcal{K}$), we can find strictly positive elements $a \in P_E(A \otimes \mathcal{K})P_E$ and $b \in P_F(A \otimes \mathcal{K})P_F$. According to [5, Theorem 3], if we further assume A has stable rank one,

$$E \cong F \text{ if and only if } \langle a \rangle = \langle b \rangle \in \mathrm{W}(A \otimes \mathcal{K}).$$

In this section we'll reformulate this result in terms of the projections P_E and P_F.

First, an alternate formula for $\iota\langle a \rangle \in \mathrm{SAff}(\mathrm{T}(A))$ will be handy. Let $\mathcal{F} \subset \mathcal{K}$ denote the *finite-rank operators* and $A \otimes \mathcal{F}$ be the algebraic tensor product of A and $\mathcal{F}$ (which we identify with the "finite-rank" operators on H_A).

Lemma 3.1. For every $0 \leq a \in A \otimes \mathcal{K}$ and $\tau \in \mathrm{T}(A)$ we have

$$\iota\langle a \rangle(\tau) = \sup\{d_\tau(b) : 0 \leq b \in A \otimes \mathcal{F}, b \precsim a\}. \qquad \square$$

Proof. If $P = 1 \otimes e$ for some finite rank projection $e \in \mathcal{K}$, then $PaP \in A \otimes \mathcal{F}$ and $PaP \precsim a$; hence, the inequality $\leq$ is immediate.

For the other direction, fix $b \in A \otimes \mathcal{F}$ such that $b \precsim a$, and fix $\epsilon > 0$. Choose $\delta > 0$ such that $d_\tau(b) - \epsilon \leq d_\tau((b-\delta)_+)$ and find $x \in A \otimes \mathcal{K}$ such that $\|x^* a x - b\| < \delta$. By density, we may assume $x \in A \otimes M_n(\mathbb{C})$ for some large $n \in \mathbb{N}$. It follows that $(b-\delta)_+ \precsim x^* a x$. Now, let $P_n = 1 \otimes e_n$, for some increasing finite-rank projections e_n, such $P_n x = x = x P_n$ for all n. We have that $(b-\delta)_+ \precsim x^* a x = x^*(P_n a P_n)x$. Hence,

$$d_\tau(b) - \epsilon \leq d_\tau((b-\delta)_+) \leq d_\tau(P_n a P_n),$$

and, by Lemma 2.4, this completes the proof of the lemma. ■

Definition 3.2. For any projection $Q \in M(A \otimes \mathcal{K})$ and tracial state $\tau \in \mathrm{T}(A)$ we define

$$\hat{Q}(\tau) = \sup\{\tau \otimes \mathrm{Tr}(b) : 0 \leq b \in A \otimes \mathcal{F}, b \leq Q\},$$

where Tr is the (unbounded) trace on $\mathcal{F}$.

Lemma 3.3. Assume A is unital with stable rank one. For any projection $Q \in M(A \otimes \mathcal{K})$, strictly positive element $a \in Q(A \otimes \mathcal{K})Q$ and $\tau \in \mathrm{T}(A)$, we have

$$\hat{Q}(\tau) = \iota\langle a\rangle(\tau). \qquad \square$$

Proof. Since $\{b : b \in A \otimes \mathcal{F}, b \leq P\} \subset \{b : b \in A \otimes \mathcal{F}, b \precsim a\}$ (cf. [9, Propostion 2.7(ii)]), and $\tau \otimes \mathrm{Tr}(b) \leq \lim_n \tau \otimes \mathrm{Tr}(b^{1/n}) = d_\tau(b)$, the previous lemma implies that $\hat{Q}(\tau) \leq \iota\langle a\rangle(\tau)$.

For the opposite inequality, fix $b \in A \otimes \mathcal{F}$ such that $b \precsim a$, and $\epsilon > 0$. Choose $\delta > 0$ such that $d_\tau(b) - \epsilon \leq d_\tau((b-\delta)_+)$. Since A has stable rank one, so does $(A \otimes \mathcal{K})^{\sim}$ (the unitization of $A \otimes \mathcal{K}$). Hence, by [13, Proposition 2.4], we can find a unitary $u \in (A \otimes \mathcal{K})^{\sim}$ such that $u^*(b-\delta)_+u \leq Q$. Since $u^*(b-\delta)_+u \in A \otimes \mathcal{F}$, the following inequalities complete the proof:

$$d_\tau(b) - \epsilon \leq d_\tau((b-\delta)_+) = d_\tau(u^*(b-\delta)_+u) = \lim_n \tau \otimes \mathrm{Tr}([u^*(b-\delta)_+u]^{1/n}) \leq \hat{Q}(\tau). \qquad \blacksquare$$

Recall that if $M \subset B(L^2(M))$ is a II_1-factor in standard form, then isomorphism classes of modules over M (i.e. normal representations $M \subset B(H)$) are completely determined by the traces of the corresponding projections in $M'\overline{\otimes}B(H)$. Our next theorem is analogous to this classical result.

Theorem 3.4. Let A be a unital simple exact C*-algebra with strict comparison and stable rank one. Given two countably generated Hilbert modules E, F over A, the following are equivalent:

(i) E is isomorphic to F;

(ii) P_E is Murray-von Neumann equivalent to P_F;

(iii) Either $\langle P_E\rangle = \langle P_F\rangle \in \mathrm{V}(A)$ (in the case $P_E, P_F \in A \otimes \mathcal{K}$), or $\hat{P}_E = \hat{P}_F$.

In particular, if neither E nor F is a finitely generated projective module, then $E \cong F$ if and only if $\hat{P}_E = \hat{P}_F$. $\square$

Proof. In the case that both $P_E, P_F \in A \otimes \mathcal{K}$, the equivalence of the three conditions is a well-known exercise; when neither P_E nor P_F belong to $A \otimes \mathcal{K}$, the first two conditions are easily seen to be equivalent. Hence, we assume neither P_E nor P_F belong to $A \otimes \mathcal{K}$ and will show the first and third statements to be equivalent.

Let a (resp. b) be a strictly positive element in $P_E(A \otimes \mathcal{K})P_E$ (resp. $P_F(A \otimes \mathcal{K})P_F$). If $E \cong F$ then $\langle a\rangle = \langle b\rangle \in \mathrm{W}(A \otimes \mathcal{K})$ (by [5, Theorem 3]), and hence the previous lemma implies that $\hat{P}_E = \hat{P}_F$. Conversely, if we know $\hat{P}_E = \hat{P}_F$ then (by the previous lemma)

$\iota\langle a\rangle = \iota\langle b\rangle$, so Proposition 2.5 ensures that $\langle a\rangle = \langle b\rangle \in \mathrm{W}(A\otimes\mathcal{K})$. Then [5, Theorem 3] implies $E \cong F$. ■

Remark 3.5. The theorem above is, in a certain sense, best possible: we really need strict comparison. More precisely, the hypotheses are satisfied by simple AH algebras with slow dimension growth (and $\mathcal{Z}$-stable algebras—cf. [2, Theorem 1], [14, Corollary 4.6], [16, Corollary 4.6]), but the result cannot be extended to all AH algebras. Indeed, the reader will find in [17] a pair of positive elements in a simple unital AH algebra of stable rank one such that the corresponding Hilbert modules, say E and F, are not isomorphic but do satisfy $\hat{P}_E = \hat{P}_F$.

It is also worth remarking that the result above gives a complete parametrization of isomorphism classes of countably generated Hilbert modules over A in terms of K_0 and traces.

4 From Elliott to Thomsen and the classification of simple AI algebras

Theorem 4.1. Let A be a unital simple C*-algebra of stable rank one for which $\mathrm{W}(A\otimes\mathcal{K}) \cong \mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A))$. Then, the Thomsen semigroup of A (cf. [15]) can be functorially recovered from the Elliott invariant of A. □

This theorem follows immediately from [4, Theorems 4 and 10]. The result applies to any algebra satisfying the hypotheses of Theorem 2.6—in particular, by [3, Theorems 5.3 and 5.5], A could be a simple unital AH algebra with slow dimension growth, or a simple unital exact and $\mathcal{Z}$-stable C*-algebra. The assumption of simplicity in the theorem is actually redundant. The assumption on the structure of $\mathrm{W}(A\otimes\mathcal{K})$ guarantees that every trace on A is faithful, whence A is simple.

Theorem 4.2 (Elliott, [7])**.** Let A and B be simple unital inductive limits of algebras of the form $F\otimes C[0,1]$, where F is finite dimensional. Then $A\cong B$ if and only if $\mathbf{Ell}(A)\cong\mathbf{Ell}(B)$. □

Proof. If $\mathbf{Ell}(A)\cong\mathbf{Ell}(B)$ then $\mathrm{W}(A\otimes\mathcal{K})\cong\mathrm{W}(B\otimes\mathcal{K})$, by Theorem 2.6 and [3, Theorem 5.3] (since AI algebras have no dimension growth). From [4, Theorem 4] it follows that the Thomsen semigroups of A and B are isomorphic too. Hence, by [15, Theorem 1.5], $A\cong B$. ■

This theorem is the best possible in the sense that the Elliott invariant is not complete for *non-simple* AI algebras (cf. [15, pg. 48]). The Cuntz semigroup, however, is a complete invariant in the non-simple case, as shown in [4, Theorem 11].

5 Unitary orbits of self-adjoints in simple, unital, exact C*-algebras

Let $a \in A$ be self-adjoint with spectrum $\sigma(a)$. Let $\phi_a : \mathrm{C}(\sigma(a)) \to A$ be the canonical homomorphism induced by sending the generator z of $\mathrm{C}(\sigma(a))$ to $a \in A$, and denote by $\mathbf{Ell}(a)$ the following pair of induced maps:

$$\mathrm{K}_*(\phi_a) : \mathrm{K}_*(C(\sigma(a))) \to \mathrm{K}_*(A);\ \phi_a^\sharp : \mathrm{T}(A) \to \mathrm{T}(C(\sigma(a))).$$

As in Theorem 4.1, the hypotheses of the next result guarantee the simplicity of A.

Theorem 5.1. Let A be a simple unital exact C*-algebra with strict comparison and stable rank one. Let $a, b \in A$ be self-adjoint. It follows that a and b are approximately unitarily equivalent if and only if $\sigma(a) = \sigma(b)$ and $\mathbf{Ell}(a) = \mathbf{Ell}(b)$. □

Proof. The "only if" statement is routine, so assume $\sigma(a) = \sigma(b)$ and $\mathbf{Ell}(a) = \mathbf{Ell}(b)$.

First, we handle the case that $\sigma(a) = \sigma(b) \subset (0, \infty)$, i.e., that both a and b are positive and invertible. Let $X = \sigma(a) = \sigma(b)$ and $\mathrm{W}_a \colon \mathrm{W}(C(X)) \to \mathrm{W}(A \otimes \mathcal{K})$ (resp. $\mathrm{W}_b \colon \mathrm{W}(C(X)) \to \mathrm{W}(A \otimes \mathcal{K})$) denote the Cuntz-semigroup map induced by the canonical homomorphism $C(X) \to A \otimes \mathcal{K}$ sending $z \mapsto a \otimes e_{1,1}$ (resp. $z \mapsto b \otimes e_{1,1}$). We claim that $\mathrm{W}_a = \mathrm{W}_b$.

So, let $h \in M_n(C(X))$ be positive and $h_a \in M_n(A)$ (resp. $h_b \in M_n(A)$) denote the image of h under the canonical inclusion $M_n(C(X)) \subset M_n(A)$ sending $C(X) \to C^*(a)$ (resp. $C(X) \to C^*(b)$). If $h \sim p$ for some projection in matrices over $C(X)$, then $h_a \sim p_a$ and $h_b \sim p_b$ (where p_a and p_b are the respective images of p under the maps induced by a and b). Since $\mathbf{Ell}(a) = \mathbf{Ell}(b)$, $[p_a] = [p_b] \in \mathrm{V}(A)$ and thus $\langle p_a \rangle = \langle p_b \rangle \in \mathrm{W}(A \otimes \mathcal{K})$—i.e. $\mathrm{W}_a(h) = \mathrm{W}_b(h)$.

If h is not equivalent to a projection in matrices over $C(X)$, then neither h_a nor h_b are equivalent to projections (in matrices over A); indeed, since A has stable rank one, [11, Proposition 3.12] implies that if h_a was equivalent to a projection then zero would not be an accumulation point of $\sigma(h_a) = \sigma(h)$, hence h would have to be equivalent to a projection as well, contrary to our assumption. In other words, $\langle h_a \rangle, \langle h_b \rangle \in \mathrm{W}(A \otimes \mathcal{K})_+$ and hence Proposition 2.5 implies that it suffices to show $d_\tau(h_a) = d_\tau(h_b)$ for every $\tau \in \mathrm{T}(A)$. However, if μ is a measure on $\sigma(h)$ then $d_\mu(h) = \mu(\sigma(h) \setminus \{0\})$. Since $\mathbf{Ell}(a) = \mathbf{Ell}(b)$, the maps on tracial spaces agree—i.e. for each $\tau \in \mathrm{T}(A)$ the measures induced by restriction agree on $\sigma(h_a) = \sigma(h_b)$—and hence $d_\tau(h_a) = d_\tau(h_b)$ for every $\tau \in \mathrm{T}(A)$, as desired.

Knowing that $\mathrm{W}_a = \mathrm{W}_b$, it now follows from [4] that $a \otimes e_{1,1}$ is approximately unitarily equivalent to $b \otimes e_{1,1}$ in the unitization of $A \otimes \mathcal{K}$. So, let $v_n \in (A \otimes \mathcal{K})^+$ be unitaries such that $v_n(a \otimes e_{1,1})v_n^* \to b \otimes e_{1,1}$. Since a is invertible, for every $\varepsilon > 0$ there exists a polynomial p such that $\|p(a) - 1\| < \varepsilon$; since $\sigma(a) = \sigma(b)$, $\|p(b) - 1\| < \varepsilon$ as well. Hence, for large n, $\|v_n(1 \otimes e_{1,1})v_n^* - 1 \otimes e_{1,1}\| < C\varepsilon$ for some constant C depending only on $\sigma(a)$. If ε is sufficiently small, this implies that $(1 \otimes e_{1,1})v_n(1 \otimes e_{1,1})$ is almost a unitary in A—hence can be perturbed to an honest unitary u_n. A routine exercise now confirms that a is approximately unitarily equivalent to b (in A).

For the case of general self-adjoints $a, b \in A$, we deduce the theorem from a simple trick. Namely, fix some constant c such that $a + c1$ is positive and invertible. Then $b + c1$ is also positive and invertible. By the case handled above, $a + c1$ and $b + c1$ are approximately unitarily equivalent, hence the same is true of a and b. ■

The theorem above holds for all simple unital AH algebras with slow dimension growth, and for the class of simple unital exact stably finite $\mathcal{Z}$-stable C*-algebras (see [2, Theorem 1], [14, Corollary 4.6], [16, Corollary 4.6]).

Another version of Theorem 5.1 holds for simple unital exact and stably finite C*-algebras (without the strict comparison or stable rank assumptions):

Theorem 5.2. Let a and b be self-adjoint elements of a simple unital exact and stably finite C*-algebra A. Then a and b are approximately unitarily equivalent in $A \otimes \mathcal{Z}$—i.e. there exist unitaries $u_n \in A \otimes \mathcal{Z}$ such that $\|u_n(a \otimes 1)u_n^* - b \otimes 1\| \to 0$ —if and only if $\sigma(a) = \sigma(b)$ and $\mathbf{Ell}(a) = \mathbf{Ell}(b)$. □

The proof of this result is a tiny perturbation of the proof of Theorem 5.1. The result is also, in some sense the best possible: in [17] a pair of positive elements in a simple unital AH algebra were constructed which have identical Elliott data but which are not Cuntz equivalent (hence not unitarily equivalent). For the interested reader, the elements in question are $f(\tau^*(\xi) \times \tau^*(\xi))$ and $f\theta_1 \oplus f\theta_1$, constructed in Section 3 of [17].

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