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The Elliott conjecture for Villadsen algebras of the first type ☆

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Abstract

We study the class of simple C^*-algebras introduced by Villadsen in his pioneering work on perforated ordered K-theory. We establish six equivalent characterisations of the proper subclass which satisfies the strong form of Elliott's classification conjecture: two C^*-algebraic ($\mathcal{Z}$-stability and approximate divisibility), one K-theoretic (strict comparison of positive elements), and three topological (finite decomposition rank, slow dimension growth, and bounded dimension growth). The equivalence of $\mathcal{Z}$-stability and strict comparison constitutes a stably finite version of Kirchberg's characterisation of purely infinite C^*-algebras. The other equivalences confirm, for Villadsen's algebras, heretofore conjectural relationships between various notions of good behaviour for nuclear C^*-algebras.

Keywords: Nuclear C^*-algebras; Classification

1. Introduction

The classification theory of norm-separable C^*-algebras began with Glimm's study of UHF algebras in 1960 [13], and was expanded by Bratteli's 1972 classification of approximately finite-dimensional (AF) algebras via certain directed graphs [4]. It was with the work of El-

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liott, however, that the theory grew exponentially. His classification of both AF and A$\mathbb{T}$ algebras of real rank zero via their scaled ordered K-theory suggested a deep truth about the structure of separable and nuclear C^*-algebras [6,7]. He articulated this idea in the late 1980s, and formalised it in his 1994 ICM address [9]: simple, separable, and nuclear C^*-algebras should be classified up to $*$-isomorphism by their topological K-theory and traces. This prediction came to be known as the Elliott conjecture.

The 1990s and early 2000s saw Elliott's conjecture confirmed in remarkable generality, cf. [23]. Kirchberg and Phillips established it for purely infinite C^*-algebras satisfying the Universal Coefficient Theorem [17,21], and Lin did the same for his C^*-algebras of tracial topological rank zero [19]. Elliott, Gong, and Li, confirmed the conjecture for unital approximately homogeneous (AH) algebras of bounded dimension growth [12]. These results cover many natural examples of C^*-algebras, including those arising from certain graphs, dynamical systems, and shift spaces.

In the midst of these successes, Villadsen produced a strange thing: a simple, separable, and nuclear C^*-algebra whose ordered K_0-group was perforated, i.e., contained a non-positive element x such that nx was positive and non-zero for some $n \in \mathbb{N}$ [33]. (This answered a long-standing question of Blackadar concerning the comparison theory of projections in C^*-algebras.) The techniques used by Villadsen to study the K-theory of his algebra were drawn from differential topology, and it took time for the functional analysts of the classification community to digest them. Then, in 2002, Rørdam found a way to adapt Villadsen's techniques to construct the first counterexample to the Elliott conjecture [24]. Other counterexamples followed [26,27].

The success of Elliott's conjecture, however, is no accident. It is a deep and fascinating phenomenon, and one must ask whether there is a regularity property lurking in those algebras for which the Elliott conjecture is confirmed. Various candidates exist: stability under tensoring with the Jiang–Su algebra $\mathcal{Z}$, finite decomposition rank, and, for approximately subhomogeneous (ASH) algebras, the notion of strict slow dimension growth. The first property—known as $\mathcal{Z}$-stability—is perhaps the most natural candidate, since tensoring with $\mathcal{Z}$ does not affect K-theory or traces of a simple unital C^*-algebra with weakly unperforated K_0-group. Elliott's conjecture thus predicts that all such algebras will be $\mathcal{Z}$-stable. It is this very prediction which forms the basis for the counterexamples of Rørdam and the first named author: one produces pairs of simple unital C^*-algebras with weakly unperforated K_0-groups, one of which is not $\mathcal{Z}$-stable. These examples have legitimised the assumption of $\mathcal{Z}$-stability in Elliott's classification program, leading to the wide-ranging classification theorem of the second named author for ASH algebras of real rank zero [37,38].

The problem with $\mathcal{Z}$-stability in relation to Elliott's classification program is that its ability to *characterise* those algebras which are amenable to classification is an article of faith. In all cases where $\mathcal{Z}$-stability is sufficient for classification (e.g., simple unital ASH algebras of real rank zero), it may also be automatic; when it is known to be necessary for classification (e.g., AH algebras), it is not known to suffice. In this paper we prove that $\mathcal{Z}$-stability does characterise those algebras which satisfy Elliott's conjecture in an ambient class where the assumption of $\mathcal{Z}$-stability is truly necessary. The class considered is at once substantial and the natural starting point for establishing such a characterisation: Villadsen's algebras. In fact we will prove much more. $\mathcal{Z}$-stability is not only the hoped for necessary and sufficient condition for classification, but is furthermore equivalent to a topological condition (finite decomposition rank) and to a K-theoretic condition (strict comparison of positive elements). These three conditions, all of which make sense for an arbitrary nuclear C^*-algebra, are equivalent to three further conditions which are to varying extents native to the class of algebras we consider: approximate divisibility, slow dimension growth, and bounded dimension growth.

Some comments on our characterisations are in order. Nuclear C^*-algebras can be viewed from several angles. They are evidently analytic objects, but can be seen as ordered algebraic objects through their K-theory, or as topological objects via the decomposition rank of Kirchberg and the second named author. Our main result says that from each of these viewpoints, there is a natural way to characterise those C^*-algebras which satisfy the Elliott conjecture. The equivalence of $\mathcal{Z}$-stability, approximate divisibility, finite decomposition rank, slow dimension growth, and bounded dimension growth is a satisfying confirmation of the expectations of experts. The equivalence of these conditions with strict comparison of positive elements, however, is unexpected and exciting for several reasons. First, the very idea of there being a K-theoretic characterisation of those algebras which will satisfy Elliott's conjecture is new. Second, it is a condition that can be verified for large classes of examples generally suspected to be amenable to classification [25,29]. Third, and most remarkably, this equivalence is a stably finite version of Kirchberg's celebrated characterisation of purely infinite C^*-algebras.

Our paper is organised as follows: in Section 2 we recall the definitions of the regularity properties which appear in our main result; Section 3 introduces Villadsen algebras of the first type, and states our main result; Sections 4–7 contain the proof of the main result; Section 8 gives some examples of non-$\mathcal{Z}$-stable Villadsen algebras.

2. Preliminaries and notation

2.1. AH algebras and dimension growth

Below we recall the concepts of (separable unital) AH algebras and their dimension growth.

Definition 2.1. A separable unital C^*-algebra A is called approximately homogeneous, or AH, if it can be written as an inductive limit

$$A = \lim_{i\to\infty} (A_i, \phi_i)$$

where each A_i is a C^*-algebra of the form

$$A_i = \bigoplus_{j=1}^{m_i} p_{i,j}\big(C(X_{i,j}) \otimes M_{r_{i,j}}\big)p_{i,j}$$

for natural numbers m_i and $r_{i,j}$, compact metrisable spaces $X_{i,j}$ and projections $p_{i,j} \in C(X_{i,j}) \otimes M_{r_{i,j}}$. We refer to the inductive system $(A_i, \phi_i)_{i\in\mathbb{N}}$ as an AH decomposition for A.

We say the AH decomposition $(A_i, \phi_i)_{\mathbf{N}}$ has slow dimension growth, if

$$\lim_{i\to\infty} \max_{j=1,\dots,m_i} \frac{\dim X_{i,j}}{\operatorname{rank} p_{i,j}} = 0;$$

it has very slow dimension growth, if

$$\lim_{i\to\infty} \max_{j=1,\dots,m_i} \frac{(\dim X_{i,j})^3}{\operatorname{rank} p_{i,j}} = 0$$

and it has bounded dimension growth, if

$$\sup_{i \in \mathbf{N}} \max_{j=1,\dots,m_i} \dim X_{i,j} = d < \infty.$$

The AH algebra A has slow (very slow or bounded, respectively) dimension growth, if it has an AH decomposition which has slow (very slow or bounded, respectively) dimension growth.

Remark 2.2. Slow dimension growth is obviously entailed by very slow dimension growth. Moreover, it is easy to see that if A is simple, then bounded dimension growth implies very slow dimension growth. One of the remarkable results of [15] says that, for simple AH algebras, very slow dimension growth also implies bounded dimension growth.

2.2. Approximate divisibility and the Jiang–Su algebra

Let p, q and n be natural numbers with p and q dividing n. C^*-algebras of the form

$$I[p,n,q] = \bigl\{ f \in M_n\bigl(C\bigl([0,1]\bigr)\bigr) \bigm| f(0) = \mathbf{1}_{n/p} \otimes a,\ f(1) = b \otimes \mathbf{1}_{n/q},\ a \in M_p,\ b \in M_q \bigr\}$$

are commonly referred to as dimension drop intervals. If $n = pq$ and $\gcd(p,q) = 1$, then the dimension drop interval is said to be prime.

In [16], Jiang and Su construct a C^*-algebra $\mathcal{Z}$, which is the unique simple unital inductive limit of dimension drop intervals having $\mathrm{K}_0 = \mathcal{Z}$, $\mathrm{K}_1 = 0$ and a unique normalised trace. It is a limit of prime dimension drop intervals where the matrix dimensions tend to infinity, and there is a unital embedding of any prime dimension drop interval into $\mathcal{Z}$. Jiang and Su show that $\mathcal{Z}$ is strongly self-absorbing in the sense of [31].

A C^*-algebra A is said to be $\mathcal{Z}$-stable, if it is isomorphic to $A \otimes \mathcal{Z}$ (since $\mathcal{Z}$ is nuclear, there is no need to specify which tensor product we use). It was shown in [16] and [32] that all classes of simple C^*-algebras for which the Elliott conjecture has been verified so far consist of $\mathcal{Z}$-stable C^*-algebras.

Using semiprojectivity of prime dimension drop intervals, it is not too hard to see that a separable unital C^*-algebra A is $\mathcal{Z}$-stable if and only if the following holds (cf. [32]): for any $n \in \mathbb{N}$ there is a sequence of unital completely positive contractions $\phi_i : M_n \oplus M_{n+1} \to A$ such that the restrictions of ϕ_i to M_n and M_{n+1} both preserve orthogonality (i.e., have order zero in the sense of [35]) and such that $\|[\phi_i((x,0)), \phi_i((0,y))]\| \to 0$ and $\|[\phi_i((x,y)), a]\| \to 0$ as i goes to infinity for every $x \in M_n$, $y \in M_{n+1}$ and $a \in A$. This characterisation shows that $\mathcal{Z}$-stability generalises the concept of approximate divisibility:

Following [3], we say a separable unital C^*-algebra A is approximately divisible, if for any $n \in \mathbb{N}$ there is a sequence of unital $*$-homomorphisms $\phi_i : M_n \oplus M_{n+1} \to A$ such that $\|[\phi_i(x), a]\| \to 0$ as i goes to infinity for every $x \in M_n \oplus M_{n+1}$ and $a \in A$.

It was shown in [32] that approximate divisibility indeed implies $\mathcal{Z}$-stability. The converse cannot hold in general (approximate divisibility asks for the existence of an abumdance of projections). However, using the classification result of [12], it was shown in [11] that simple AH algebras of bounded dimension growth are approximately divisible.

2.3. The Cuntz semigroup

Let A be a C^*-algebra, and let $\mathrm{M}_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. If $A = \mathbb{C}$, then we simply write M_n. Let $\mathrm{M}_\infty(A)$ denote the algebraic limit of the direct system $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathrm{M}_\infty(A)_+$ (resp. $\mathrm{M}_n(A)_+$) denote the positive elements in $\mathrm{M}_\infty(A)$ (resp. $\mathrm{M}_n(A)$). Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\left\| v_n b v_n^* - a \right\| \xrightarrow{n \to \infty} 0.$$

We say that a and b are *Cuntz equivalent* (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. This relation is an equivalence relation, and we write $\langle a \rangle$ for the equivalence class of a. The set

$$W(A) := \mathrm{M}_\infty(A)_+/\sim$$

becomes a positively ordered Abelian monoid when equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leqslant \langle b \rangle \quad \Leftrightarrow \quad a \precsim b.$$

In the sequel, we refer to this object as the *Cuntz semigroup* of A.

Given $a \in \mathrm{M}_\infty(A)_+$ and $\epsilon > 0$, we denote by $(a - \epsilon)_+$ the element of $C^*(a)$ corresponding (via the functional calculus) to the function

$$f(t) = \max\{0, t - \epsilon\}, \quad t \in \sigma(a).$$

(Here $\sigma(a)$ denotes the spectrum of a.)

2.4. Dimension functions and strict comparison

Now suppose that A is unital and stably finite, and denote by $\mathrm{QT}(A)$ the space of normalised 2-quasitraces on A (v. [2, Definition II.1.1]). Let $S(W(A))$ denote the set of additive and order preserving maps d from $W(A)$ to $\mathbb{R}^+$ having the property that $d(\langle 1_A \rangle) = 1$. Such maps are called *states*. Given $\tau \in \mathrm{QT}(A)$, one may define a map $d_\tau : \mathrm{M}_\infty(A)_+ \to \mathbb{R}^+$ by

$$d_\tau(a) = \lim_{n \to \infty} \tau\big(a^{1/n}\big). \tag{1}$$

This map is lower semicontinuous, and depends only on the Cuntz equivalence class of a. It moreover has the following properties:

(i) if $a \precsim b$, then $d_\tau(a) \leqslant d_\tau(b)$;
(ii) if a and b are orthogonal, then $d_\tau(a+b) = d_\tau(a) + d_\tau(b)$;
(iii) $d_\tau((a-\epsilon)_+) \nearrow d_\tau(a)$ as $\epsilon \to 0$.

Thus, d_τ defines a state on $W(A)$. Such states are called *lower semicontinuous dimension functions*, and the set of them is denoted $\mathrm{LDF}(A)$. If A has the property that $a \precsim b$ whenever $d(a) < d(b)$ for every $d \in \mathrm{LDF}(A)$, then we say that A has *strict comparison of positive elements* or simply *strict comparison*.

2.5. Decomposition rank

Based on the completely positive approximation property for nuclear C^*-algebras, one may define a noncommutative version of covering dimension as follows:

Definition 2.3. (See [18, Definitions 2.2 and 3.1].) Let A be a separable C^*-algebra.

(i) A completely positive map $\varphi : \bigoplus_{i=1}^{s} M_{r_i} \to A$ is n-decomposable, if there is a decomposition $\{1,\ldots,s\} = \coprod_{j=0}^{n} I_j$ such that the restriction of φ to $\bigoplus_{i\in I_j} M_{r_i}$ preserves orthogonality for each $j \in \{0,\ldots,n\}$.
(ii) A has decomposition rank n, $\mathrm{dr}\, A = n$, if n is the least integer such that the following holds: Given $\{b_1,\ldots,b_m\} \subset A$ and $\epsilon > 0$, there is a completely positive approximation (F,ψ,φ) for $b_1,\ldots,b_m$ within ϵ (i.e., $\psi : A \to F$ and $\varphi : F \to A$ are completely positive contractions and $\|\varphi\psi(b_i) - b_i\| < \epsilon$) such that φ is n-decomposable. If no such n exists, we write $\mathrm{dr}\, A = \infty$.

This notion has good permanence properties; for example, it behaves well with respect to quotients, inductive limits, hereditary subalgebras, unitization and stabilization. It generalises topological covering dimension, i.e., if X is a locally compact second countable space, then $\mathrm{dr}\, C_0(X) = \dim X$; see [18] for details. Moreover, if A is an AH algebra of bounded dimension growth, then $\mathrm{dr}\, A$ is finite.

3. $\mathcal{V}$I Algebras and the main result

3.1. Villadsen algebras of the first type

The class of algebras we consider is an interpolated family of AH algebras. At their simplest they are the UHF algebras of Glimm, while at their most complex they are the algebras introduced by Villadsen in his work on perforated ordered K-theory. In between these extremes they span the full spectrum of complexity for simple, separable, nuclear, and stably finite C^*-algebras. We call these algebras *Villadsen algebras of the first type* as they are defined by a generalisation of Villadsen's construction in [33]. (Villadsen used a second and quite distinct construction in his subsequent work on stable rank, cf. [34].)

Let X and Y be compact Hausdorff spaces. Recall that a $*$-homomorphism

$$\phi : C(X) \to \mathrm{M}_n \otimes C(Y)$$

is said to be *diagonal* if it has the form

$$f \mapsto \operatorname{diag}(f \circ \lambda_1, \dots, f \circ \lambda_n),$$

where $\lambda_i : Y \to X$ is a continuous map for each $1 \leqslant i \leqslant n$. The maps $\lambda_1, \dots, \lambda_n$ are called the *eigenvalue maps* of ϕ. Amplifications of diagonal maps are also called diagonal.

Definition 3.1. Let X be a compact Hausdorff space and $n, m, k \in \mathbb{N}$. A unital diagonal $*$-homomorphism

$$\phi : \mathrm{M}_n \otimes C(X) \to \mathrm{M}_k \otimes C\big(X^{\times m}\big)$$

is said to be a Villadsen map of the first type (a $\mathcal{V}$I map) if each eigenvalue map is either a co-ordinate projection or has range equal to a point.

Definition 3.2. Let X be a compact Hausdorff space, and let $(n_i)_{i=1}^{\infty}$ and $(m_i)_{i=1}^{\infty}$ be sequences of natural numbers with $n_1 = 1$. Fix a compact Hausdorff space X, and put $X_i = X^{\times n_i}$. A unital C^*-algebra A is said to be a Villadsen algebra of the first type (a $\mathcal{V}$I algebra), if it can be written as an inductive limit algebra

$$A \cong \lim_{i \to \infty} \big(\mathrm{M}_{m_i} \otimes \mathrm{C}\big(X^{\times n_i}\big), \phi_i\big)$$

where each ϕ_i is a $\mathcal{V}$I map.

We will refer to the inductive system in Definition 3.2 as a *standard decomposition* for A with *seed space* $X_1 (= X)$. Clearly, such decompositions are not unique.

For $j > i$, put

$$\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let $N_{i,j}$ be the number of distinct co-ordinate projections from $X_j = X_i^{\times n_j/n_i}$ to X_i occuring as eigenvalue maps of $\phi_{i,j}$, and let $M_{i,j}$ denote the multiplicity (number of eigenvalue maps) of $\phi_{i,j}$. Notice that

$$M_{i,j} = M_{j-1,j} M_{i,j-1}, \quad \text{that } N_{i,j} = N_{j-1,j} N_{i,j-1}$$

and that

$$0 \leqslant \frac{N_{i,j}}{M_{i,j}} \leqslant 1.$$

From these relations it follows in particular that the sequence

$$\left(\frac{N_{i,j}}{M_{i,j}}\right)_{j>i}$$

is decreasing and converges for any fixed i.

A is said to have *slow* (*very slow, or bounded*) *dimension growth as a* $\mathcal{V}$I *algebra*, if it admits a standard decomposition as above which has slow (very slow, or bounded, respectively) dimension growth in the sense of Definition 2.1.

Remarks 3.3. Despite its simple definition, the class of $\mathcal{V}$I algebras is surprisingly broad:

- By taking $X_1 = \{*\}$, we can obtain any UHF algebra. If instead we take X_1 to be a finite set, then we obtain a good supply of AF algebras.
- With each X_i equal to a disjoint union of finitely many circles, we obtain a large collection of A$\mathbb{T}$ algebras of real rank zero and real rank one.
- If each X_i is equal to the same compact Hausdorff space X, then we obtain the class of Goodearl algebras.
- If we impose the condition that $n_i/m_i \to 0$, then we obtain AH algebras of slow dimension growth exhibiting a full range of complexity in their Elliott invariants: torsion in K_0 or K_1, and arbitrary tracial state spaces.
- By taking "most" of the eigenvalue maps in each ϕ_i to be distinct co-ordinate projections and setting $X_1 = \mathrm{S}^2$ we obtain Villadsen's example of a simple, separable, and nuclear C^*-algebra with perforated ordered K_0-group [33]. A variation on Villadsen's construction yields the counterexample to Elliott's classification conjecture discovered by the first named author in [26].

The first three examples above are special cases of the fourth, and the latter is a class of algebras for which the Elliott conjecture can be shown to hold. Proving this, however, requires both the most powerful available classification results for stably finite C^*-algebras and the detailed analysis of $\mathcal{V}$I algebras provided in the sequel. Thus, from the standpoint of trying to confirm the Elliott conjecture, $\mathcal{V}$I algebras are no less complex than the class of all simple unital AH algebras. The fifth example demonstrates that $\mathcal{V}$I algebras include non-$\mathcal{Z}$-stable algebras which, in general, cannot be detected with classical K-theory. The sequel will show that there are in fact a tremendous number of such $\mathcal{V}$I algebras (see Section 8).

3.2. The main theorem

Theorem 3.4. *Let A be a simple $\mathcal{V}$I algebra admitting a standard decomposition with seed space a finite-dimensional CW complex. The following are equivalent*:

(i) *A is $\mathcal{Z}$-stable*;
(ii) *A has strict comparison of positive elements*;
(iii) *A has finite decomposition rank*;
(iv) *A has slow dimension growth* (*as an AH algebra*);
(v) *A has bounded dimension growth* (*as an AH algebra*);
(vi) *A is approximately divisible.*

If, moreover, A has real rank zero, then A satisfies the equivalent conditions above.

For Theorem 3.4, the following implications are already known:

$$(\mathrm{v}) \Rightarrow (\mathrm{vi}) \Rightarrow (\mathrm{i}) \Rightarrow (\mathrm{ii}); \qquad (\mathrm{v}) \Rightarrow (\mathrm{iii}); \qquad (\mathrm{iv}) \Rightarrow (\mathrm{ii}).$$

More precisely, (v) ⇒ (vi) was shown in [11] (based on the results of [12]), (vi) ⇒ (i) is a result of [32], (i) ⇒ (ii) is [25, Corollary 4.6], (iv) ⇒ (ii) is [29, Corollary 4.6], and (v) ⇒ (iii) is an easy observation of [18]. We will prove (ii) ⇒ (iv), (ii) ⇒ (v), (iii) ⇒ (ii), and the statement about real rank zero. In the real rank zero setting, the implication (iv) ⇒ (v) is due independently to Dadarlat and Gong [5,14].

Remarks 3.5.

- In the absence of Theorem 3.4, no two of conditions (i)–(vi) are known to be equivalent for an arbitrary simple unital AH algebra. However, (i), (ii), (iv), (v) and (vi) are known to be equivalent for simple unital AH algebras of real rank zero (a combination of results from [8,10,11,19,32,38]). One of the central open questions in the classification theory of nuclear C^*-algebras is whether any of these conditions is actually necessary in the real rank zero case. Theorem 3.4 makes some progress on this question—see the third remark below.
- Conditions (i)–(iii) should remain equivalent in much larger classes of simple, separable, nuclear, and stably finite C^*-algebras. Conditions (iv), (v), and (vi) cannot be expected to hold in general (conditions (iv) and (v) exclude non-AH algebras, and (vi) excludes projectionless algebras, such as $\mathcal{Z}$ itself), but it remains possible that(i)–(vi) are equivalent for simple unital ASH algebras, when (iv) and (v) are adapted in the obvious way to this class.
- It is not known whether real rank zero implies conditions (i)–(vi) above for larger classes of simple nuclear C^*-algebras (clearly, (iii), (iv), and (v) can only hold in the stably finite case). Every existing classification theorem for real rank zero C^*-algebras assumes at least one of conditions (i)–(vi). It thus remarkable that in the class of $\mathcal{V}$I algebras, real rank zero entails classifiability without assuming any of these conditions.
- Our proof of (iv) ⇒ (v) is the first instance of such among simple unital AH algebras of unconstrained real rank.
- The proof of Theorem 3.4 yields new examples of simple C^*-algebras with infinite decomposition rank. Previous examples all had a unique trace, while our examples can exhibit a wide variety of structure in the tracial state space. (See Section 8.)
- Simple $\mathcal{V}$I algebras all have stable rank one by an argument similar to that of [33, Proposition 10]. They may, however, have quite fast dimension growth—[28, Theorem 5.1] exhibits a simple $\mathcal{V}$I algebra for which every AH decomposition has the property that

$$\liminf_{i\to\infty} \max_{j=1,\dots,m_i} \frac{\dim X_{i,j}}{\operatorname{rank} p_{i,j}} = \infty.$$

For completeness, we note that the class of $\mathcal{V}$I algebras which satisfy the conditions of Theorem 3.4 indeed satisfies the Elliott conjecture. Let $\mathrm{Ell}(\bullet)$ denote the Elliott invariant of a unital, exact, and stably finite C^*-algebra. We then have:

Corollary 3.6. *(See Gong [15], Elliott, Gong and Li [12].) Let A and B be simple $\mathcal{V}$I algebras as in Theorem* 3.4 *which satisfy conditions* (i)–(vi)*, and suppose that there is an isomorphism*

$$\phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

Then, there is a $*$-isomorphism $\Phi : A \to B$ which induces ϕ.

3.3. An analogue of Kirchberg's first Geneva theorem

The most interesting aspect of Theorem 3.4 is that it provides an analogue among $\mathcal{V}$I algebras of Kirchberg's characterisation of purely infinite algebras. The latter states that for a simple, separable, and nuclear C^*-algebra A we have

$$A \otimes \mathcal{O}_\infty \cong A \quad \Leftrightarrow \quad A \text{ is purely infinite.}$$

If we suppose that A is a priori traceless, then a result of Rørdam (see [25]) says that $\mathcal{Z}$-stability and $\mathcal{O}_\infty$-stability are equivalent, and the definition of strict comparison reduces to the very definition of pure infiniteness. Thus, we see that Kirchberg's characterisation is equivalent to the statement

$$A \otimes \mathcal{Z} \cong A \quad \Leftrightarrow \quad A \text{ has strict comparison of positive elements.}$$

This statement makes sense even if A has a trace, and is moreover true for the simple $\mathcal{V}$I algebras of Theorem 3.4. Were the statement to hold for all simple, separable, and nuclear C^*-algebras—a distinct possibility—it would be a deep and striking generalisation of Kirchberg's characterisation. In light of this possibility, we suggest that simple and $\mathcal{Z}$-stable C^*-algebras be termed "purely finite."

4. Villadsen's obstruction in the Cuntz semigroup

In this section we prove that under a technical assumption, a simple $\mathcal{V}$I algebra fails to have strict comparison of positive elements. We shall see later that this failure is dramatic enough to ensure that the algebra also has infinite decomposition rank.

4.1. Vector bundles and characteristic class obstructions

All vector bundles considered in this paper are topological and complex. For any connected topological space X, we let θ_l denote the trivial vector bundle of fibre dimension $l \in \mathbb{N}$. If ω is a vector bundle over X, then we denote by $\bigoplus_{i=1}^k \omega$ the k-fold Whitney sum of ω with itself, and by $\omega^{\otimes k}$ its k-fold external tensor product (over X^k). We use $\mathrm{rank}(\omega)$ to denote the fibre dimension of ω. If Y is a second topological space and $f: Y \to X$ is continuous, then $f^*(\omega)$ denotes the induced bundle over Y. By Swan's theorem, ω can be represented by a (non-unique) projection in a matrix algebra over $\mathrm{C}(X)$; we will use vector bundles and projections interchangeably in the sequel.

Recall that the Chern class $c(\omega)$ is an element of the integral cohomology ring $H^*(X)$ of the form

$$c(\omega) = \sum_{i=0}^{\infty} c_i(\omega),$$

where $c_i(\omega) \in H^{2i}(X)$ and $c_i(\omega) = 0$ whenever $i > \mathrm{rank}(\omega)$. Let γ be a second vector bundle over X. We will make use of the following properties of the Chern class:

(i) $c(\theta_l) = 1 \in H^0(X)$;

(ii) $c(\gamma \oplus \omega) = c(\gamma)c(\omega)$, where the product is the cup product;
(iii) If Y is another topological space and $f : Y \to X$ is continuous, then $c(f^*(\omega)) = f^*(c(\omega))$.

Let ξ be the Hopf line bundle over S^2. The following Chern class obstruction argument, due essentially to Villadsen, shows that θ_k is not isomorphic to a sub-bundle of $\bigoplus_{i=1}^{l} \xi^{\otimes l}$ whenever $1 \leqslant k < l$. The top Chern class $c_l(\bigoplus_{i=1}^{l} \xi^{\otimes l})$ (equal, in this case, to the Euler class of $\bigoplus_{i=1}^{l} \xi^{\otimes l}$) is not zero by [24, Proposition 3.2]. If θ_k is isomorphic to a sub-bundle of $\bigoplus_{i=1}^{l} \xi^{\otimes l}$, then there exists a vector bundle γ of rank $l - k$ over $(\mathrm{S}^2)^l$ such that

$$\theta_k \oplus \gamma \cong \bigoplus_{i=1}^{l} \xi^{\otimes l}.$$

Applying the Chern class to this equation yields

$$c(\gamma) = c\left(\bigoplus_{i=1}^{l} \xi^{\otimes l}\right).$$

But then $c_l(\gamma) \neq 0$, contradicting the fact that $c_i(\gamma) = 0$ whenever $i > \mathrm{rank}(\gamma) = k$.

We review for future use some structural aspects of the integral cohomology ring $H^*((\mathrm{S}^2)^n)$. It is well known that

$$H^0(\mathrm{S}^2) \cong H^2(\mathrm{S}^2) \cong \mathbb{Z}$$

and

$$H^i(\mathrm{S}^2) = 0, \quad i \neq 0, 2.$$

It follows from the Künneth formula that

$$H^*((\mathrm{S}^2)^n) \cong H^*(\mathrm{S}^2)^{\otimes n}$$

as graded rings. Let e_i denote the generator of $H^2(\mathrm{S}^2)$ in the ith tensor factor of $H^*(\mathrm{S}^2)^{\otimes n}$. Then,

$$H^*((\mathrm{S}^2)^n) \cong \mathbb{Z}[1, e_1, \ldots, e_n]/R,$$

where

$$R = \{e_i^2 = 0 \mid 1 \leqslant i \leqslant n\}.$$

If $n = Nl$ for some $N \in \mathbb{N}$, then

$$H^*((\mathrm{S}^2)^{Nl}) = H^*((\mathrm{S}^2)^l)^{\otimes N}.$$

Let $e_{i,j}$ denote the generator of the ith copy of $H^2(\mathrm{S}^2)$, $i \in \{1, \ldots, l\}$, in the jth tensor factor of the right-hand side above.

4.2. A failure of strict comparison in $\mathrm{C}(X)$

Villadsen's Chern class obstruction argument may be viewed as a statement about projections in a matrix algebra over $\mathrm{C}(X)$. We present below an analogue of his argument for certain non-projections in $\mathrm{M}_n(\mathrm{C}(X))$.

Let X be a CW-complex with $\dim(X) \geqslant 6$, and let there be given a natural number l satisfying $2 \leqslant l \leqslant \lfloor \dim(X)/3 \rfloor$. Choose an open set $O \subseteq X$ homeomorphic to $(-1,1)^{\dim(X)}$. Define

$$\tilde{A} := \{x \in (-1,1)^3 \mid \operatorname{dist}(x,(0,0,0)) = 1/2\} \cong \mathrm{S}^2$$

and

$$\tilde{B} := \{x \in (-1,1)^3 \mid 1/3 < \operatorname{dist}(x,(0,0,0)) < 2/3\},$$

and let $\pi : \tilde{B} \to \tilde{A}$ be the continuous projection along rays emanating from $(0,0,0)$. Now define a closed subset

$$A = \tilde{A}^l \times \{0\}^{\dim(X)-3l}$$

and an open subset

$$B = \tilde{B}^l \times (-1,1)^{\dim(X)-3l}$$

of O. Define a continuous map $\varPi : B \to A$ by

$$\varPi = \underbrace{\pi \times \cdots \times \pi}_{l \text{ times}} \times \underbrace{ev_0 \times \cdots \times ev_0}_{\dim(X)-3l \text{ times}},$$

where $ev_0(x) = 0$ for every $x \in (-1,1)$. Let $f : X \to [0,1]$ be a continuous map which is identically one on A and identically zero off B.

Notice that $A \cong (\mathrm{S}^2)^l$, so $\xi^{\otimes l}$ may be viewed as a vector bundle over A. Define positive elements

$$P = f \cdot \varPi^*\left(\bigoplus_{i=1}^{l} \xi^{\otimes l}\right) \tag{2}$$

and

$$\varTheta_k = f \cdot \theta_k \tag{3}$$

in $\mathrm{M}_\infty(\mathrm{C}(X))$. For every $x \in X$ and $n \in \mathbb{N}$ we have either

$$\operatorname{rank}(P(x)) = \operatorname{rank}(\varTheta_k(x)) = 0$$

or

$$\operatorname{rank}(P(x)) = l, \qquad \operatorname{rank}(\varTheta_k(x)) = k.$$

If $\tau \in \mathrm{T}(\mathrm{C}(X))$ and $a \in \mathrm{M}_\infty(\mathrm{C}(X))_+$, then $d_\tau(a)$ is obtained by integrating the rank function of a against the probability measure on X corresponding to τ. Thus, if $k < l$, we have

$$d_\tau(\Theta_k) \leqslant d_\tau(P), \quad \forall \tau \in \mathrm{T}(A),$$

and this inequality is strict if $\mu_\tau(B) > 0$, where μ_τ is the measure induced on X by $\tau \in \mathrm{T}(A)$. On the other hand, $\langle \Theta_k \rangle \not\leqslant \langle P \rangle$. To see this suppose, on the contrary, that there exists a sequence $(v_i)_{i=1}^\infty$ in $\mathrm{M}_\infty(\mathrm{C}(X))$ such that

$$\left\| v_i P v_i^* - \Theta_k \right\| \xrightarrow{i \to \infty} 0.$$

Then, the same is true upon restriction to $A \subseteq X$, i.e.,

$$\Theta_k|_A \cong \theta_k \precsim \bigoplus_{i=1}^{l} \xi^{\otimes l} \cong P|_A.$$

This amounts to saying that θ_k is isomorphic to a sub-bundle of $\bigoplus_{i=1}^{l} \xi^{\otimes l}$, contradicting our choice of ξ.

We are now ready to prove a key lemma. Its proof is inspired by the proof of [26, Theorem 1.1].

Lemma 4.1. *Let A be a simple $\mathcal{V}$I algebra with standard decomposition (A_i, ϕ_i) and seed space a CW-complex X_1 of dimension strictly greater than zero. Suppose that for any $\epsilon > 0$ there exists $i \in \mathbb{N}$ such that*

$$\frac{N_{i,j}}{M_{i,j}} > 1 - \epsilon, \quad \forall j > i. \tag{4}$$

Then, for any $n \in \mathbb{N}$ there exist pairwise orthogonal elements $a, b_1, \ldots, b_n \in \mathrm{M}_\infty(A)_+$ such that for each $s \in \{1, \ldots, n\}$

$$d_\tau(a) < d_\tau(b_s), \quad \forall \tau \in \mathrm{T}(A),$$

and

$$\langle a \rangle \not\leqslant \langle b_1 \rangle + \cdots + \langle b_n \rangle.$$

In particular, A does not have strict comparison of positive elements.

Proof. First observe that the simplicity of A combined with the non-zero dimension of X_1 imply that $m_i \to \infty$ as $i \to \infty$—the number of point evaluations appearing as eigenvalue maps in $\phi_{i,j}$ is unbounded as $j \to \infty$. It then follows from our assumption on $N_{i,j}/M_{i,j}$ that $\dim(X_i) \to \infty$ as $i \to \infty$. We may thus assume that $\dim(X_i) \neq 0$, $\forall i \in \mathbb{N}$. Since ϕ_i always contains at least one eigenvalue map which is not a point evaluation, it is injective.

Let $n \in \mathbb{N}$ be given. Find, using the hypotheses of the lemma, an $i \in \mathbb{N}$ such that

$$\frac{N_{i,j}}{M_{i,j}} > \frac{6n-1}{6n}, \quad \forall j > i.$$

Since $N_{i,j}/M_{i,j}$ increases with increasing i, we may assume that i is large enough to permit the construction of the element

$$f \cdot \Pi^*\left(\bigoplus_{i=1}^{3} \xi^{\otimes 3n}\right)$$

—this is just the element P of Eq. (2), with the number of direct summands altered. This element will be our b_1. (The maps ϕ_i are injective, so we identify forward images in the inductive sequence.) For $b_2, \dots, b_n$ we simply take mutually orthogonal copies of b_1. Let a be Θ_2, chosen orthogonal to each of $b_1, \dots, b_n$. We have by construction that

$$d_\tau(a) < d_\tau(b_s), \quad 1 \leqslant s \leqslant n,$$

whenever $\mu_\tau(B) > 0$ (recall that $B \subseteq X_i$ is the support of P and Θ_2—it is an open set). If $f : X_i \to [0,1]$ has support equal to B, then we may write

$$\mu_\tau(B) = d_\tau(f \cdot 1_{A_i}).$$

Now for $\tau \in \mathrm{T}(A)$ we have

$$d_\tau\big(\phi_{i\infty}(f \cdot 1_{A_i})\big) = d_{\phi_{i\infty}^\sharp(\tau)}(f \cdot 1_{A_i}). \tag{5}$$

Since A is simple and $\phi_{i\infty}(f \cdot 1_{A_i}) \neq 0$ we have

$$0 < \tau\big(\phi_{i\infty}(f \cdot 1_{A_i})\big) < \lim_{n\to\infty} \tau\big(\phi_{i\infty}(f \cdot 1_{A_i})^{1/n}\big) = d_\tau\big(\phi_{i\infty}(f \cdot 1_{A_i})\big).$$

Combining this with (5) above we see that

$$\begin{aligned} d_\tau\big(\phi_{i\infty}(a)\big) &= d_{\phi_{i\infty}^\sharp(\tau)}(a) \\ &< d_{\phi_{i\infty}^\sharp(\tau)}(b) \\ &= d_\tau(\phi_{i\infty}(b) \end{aligned}$$

for every $\tau \in \mathrm{T}(A)$.

It remains to prove that

$$\langle a\rangle \not\leqslant \langle b_1\rangle + \cdots + \langle b_n\rangle = \langle b_1 + \cdots + b_n\rangle.$$

Notice that $b_1 + \cdots + b_n$, viewed as an element of $\mathrm{M}_\infty(A_i)$, is simply the element P of Eq. (2) with parameter $l = 3n$. Thus, with this choice of l, we are in fact trying to prove that

$$\big\langle \phi_{i\infty}(\Theta_2)\big\rangle \not\leqslant \big\langle \phi_{i\infty}(P)\big\rangle.$$

It will suffice to prove that for each $j > i$ and $v \in \mathrm{M}_\infty(A_j)$

$$\big\| v\phi_{i,j}(P)v^* - \phi_{i,j}(\Theta_2)\big\| \geqslant 1/2.$$

Let S be the set of eigenvalue maps of $\phi_{i,j}$. S is the disjoint union of the set S_1 of eigenvalue maps which are co-ordinate projections and the set S_2 of eigenvalue maps which are point evaluations. (The fact that $\dim(X_i) \neq 0$ ensures that $S_1 \cap S_2 = \emptyset$.) Note that $|S_1| = N_{i,j}$. For $\lambda \in S_1$, let $m(\lambda)$ denote the number of times that λ occurs as an eigenvalue map of some $\phi_{i,j}$.

Write $\phi_{i,j} = \gamma_1 \oplus \gamma_2$, where γ_1 is a $\mathcal{V}$I map corresponding to the eigenvalue maps of $\phi_{i,j}$ contained in S_1, and γ_2 corresponds similarly to the elements of S_2. By construction, $\gamma_2(P)$ is a constant positive matrix-valued function over X_j. Put $\tilde{P} = \gamma_1(\mathbf{1}_{\mathrm{M}_{m_i}(\mathrm{C}(X_i))}) \oplus \gamma_2(P)^{1/2}$, and $q = \lim_{n\to\infty} \gamma_2(P)^{1/n}$. It follows that

$$\phi_{i,j}(P) = \gamma_1(P) \oplus \gamma_2(P) = \tilde{P}\big(\gamma_1(P) \oplus q\big)\tilde{P},$$

and that the projection q corresponds to a trivial vector bundle.

Suppose that there exists $v \in \mathrm{M}_\infty(A_j)$ such that

$$\big\| v\phi_{i,j}(P)v^* - \phi_{i,j}(\Theta_2)\big\| < 1/2.$$

Then,

$$\big\| v\tilde{P}\big(\gamma_1(P) \oplus q\big)\tilde{P}v^* - \big(\gamma_1(\Theta_2) \oplus \gamma_2(\Theta_2)\big)\big\| < 1/2.$$

Cutting down by $\gamma_1(\mathbf{1}_{A_i})$ and setting $w = \gamma_1(\mathbf{1}_{A_i})v\tilde{P}$ we have

$$\big\| w\big(\gamma_1(P) \oplus q\big)w^* - \gamma_1(\Theta_2)\big\| < 1/2, \tag{6}$$

and this estimate holds *a fortiori* over any closed subset of X_j.

Fix a point $x_0 \in X_i$ and let C be the closed subset of $X_j = X_i^{\times n_j/n_i}$ consisting of those (n_j/n_i)-tuples which are equal to x_0 in those co-ordinates which are not the range of an element of S_1, and whose remaining co-ordinates belong to $A \subseteq X_i$. Notice that

$$C \cong A^{\times l N_{i,j}} \cong \big(\mathrm{S}^2\big)^{\times l N_{i,j}}.$$

We have

$$\gamma_1(P)|_C \cong \bigoplus_{\lambda \in S_1} \bigoplus_{m=1}^{l m(\lambda)} \lambda^*\big(\xi^{\otimes l}\big),$$
$$\gamma_2(P)|_C \cong \theta_{lr},$$

and

$$\gamma_1(\Theta_2)|_C \cong \theta_{2\,\mathrm{mult}(\gamma_1)},$$

where $r \leqslant \mathrm{mult}(\gamma_2)$. [26, Lemma 2.1] and (6) together imply that

$$\theta_{2\,\mathrm{mult}(\gamma_1)} \precsim \left(\bigoplus_{\lambda \in S_1} \bigoplus_{m=1}^{l m(\lambda)} \lambda^*\big(\xi^{\otimes l}\big)\right) \oplus \theta_{lr}$$

in the sense of Murray and von Neumann. In other words, there is a $t \in \mathbb{N}$ and a complex vector bundle ω over C of fibre dimension $(l-2)\,\mathrm{mult}(\gamma_1) + lr$ such that

$$\theta_{2\,\mathrm{mult}(\gamma_1)+t} \oplus \omega \cong \left(\bigoplus_{\lambda \in S_1} \bigoplus_{m=1}^{lm(\lambda)} \lambda^*\big(\xi^{\otimes l}\big) \right) \oplus \theta_{lr+t}.$$

Applying the total Chern class to this equation yields

$$\begin{aligned} c(\omega) &= c\left(\left(\bigoplus_{\lambda \in S_1} \bigoplus_{m=1}^{lm(\lambda)} \lambda^*\big(\xi^{\otimes l}\big) \right) \oplus \theta_{lr+t} \right) \\ &= \prod_{\lambda \in S_1} c\big(\lambda^*\big(\xi^{\otimes l}\big)\big)^{lm(\lambda)} \\ &= \prod_{\lambda \in S_1} \big[\lambda^*\big(c\big(\xi^{\otimes l}\big)\big)\big]^{lm(\lambda)}. \end{aligned}$$

Let us take the elements of S_1 to be numbered $\lambda_1, \ldots, \lambda_{N_{i,j}}$, so that

$$c(\omega) = \prod_{k=1}^{N_{i,j}} (1 + e_{1,k} + \cdots + e_{l,k})^{lm(\lambda)}.$$

Recall our description of the ring structure of $H^*((\mathrm{S}^2)^{\times l})^{\otimes N_{i,j}}$ from the end of Section 4.1. The class $c_{lN_{i,j}}(\omega)$ is the sum of all possible products of $lN_{i,j}$ elements of the form $e_{s,k}$s or 1. Since $H^*((\mathrm{S}^2)^{\times l})^{\otimes N_{i,j}}$ is torsion free and the products in question generate independent copies of $\mathbb{Z}$ whenever the products themselves are distinct, we see that $c_{lN_{i,j}}(\omega) \neq 0$ if even one of the products in question is not zero. Since the only relation on the generators of $H^*((\mathrm{S}^2)^{\times l})^{\otimes N_{i,j}}$ is that $e_{s,k}^2 = 0$, we see that

$$\prod_{k=1}^{N_{i,j}} \prod_{s=1}^{l} e_{s,k} \neq 0.$$

Thus, $c_{lN_{i,j}}(\omega) \neq 0$. This in turn necessitates $\mathrm{rank}(\omega) \geqslant lN_{i,j}$—the nth Chern class of a vector bundle of dimension $< n$ is always zero. We conclude that

$$\begin{aligned} lN_{i,j} &\leqslant (l-2)\,\mathrm{mult}(\gamma_1) + lr \\ &\leqslant (l-2)\,\mathrm{mult}(\gamma_1) + l\,\mathrm{mult}(\gamma_2) \\ &\leqslant (l-2)M_{i,j} + 2(M_{i,j} - N_{i,j}). \end{aligned}$$

Dividing the last inequality above by $lM_{i,j}$ we get

$$\frac{N_{i,j}}{M_{i,j}} \leqslant \frac{l-2}{l} + \frac{2}{l}\left(1 - \frac{N_{i,j}}{M_{i,j}}\right).$$

Using the assumption

$$\frac{N_{i,j}}{M_{i,j}} \geqslant \frac{6n-1}{6n} = \frac{2l-1}{2l}$$

we have

$$\frac{2l-1}{2l} \leqslant \frac{N_{i,j}}{M_{i,j}} \leqslant \frac{l-2}{l} + \frac{2}{l^2} < \frac{l-1}{l},$$

a contradiction. □

5. Strict comparison implies bounded dimension growth

The next lemma says that if a simple $\mathcal{V}$I algebra has strict comparison of positive elements, then it not only has slow dimension growth, but even has slow dimension growth *as a* $\mathcal{V}$I *algebra*.

Lemma 5.1. *Let A be a simple $\mathcal{V}$I algebra*; *suppose that A admits a standard decomposition (A_i, ϕ_i) with seed space a CW-complex X. If A has strict comparison of positive elements, then* $\dim(X) = 0$ (*in which case A is AF*), *or for every $i \in \mathbb{N}$,*

$$\frac{N_{i,j}}{M_{i,j}} \xrightarrow{j\to\infty} 0.$$

If A does not have strict comparison of positive elements, then

$$\lim_{i\to\infty} \lim_{j\to\infty} \frac{N_{i,j}}{M_{i,j}} = 1. \tag{7}$$

Proof. Suppose, for a contradiction, that A satisfies the hypotheses of the lemma, $\dim(X) \geqslant 1$, and there is some $i_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\frac{N_{i_0,j}}{M_{i_0,j}} \geqslant \delta, \quad \forall j > i_0.$$

We must show that A does not have strict comparison of positive elements and that (7) holds.

A is simple with $\dim(X) = 1$, so $M_{i,j} \to \infty$ as $j \to \infty$ for any fixed $i \in \mathbb{N}$. This forces $N_{i,j} \to \infty$, too, so that $\dim(X_i) \to \infty$ as $i \to \infty$.

For any $j > m > i_0$ we have

$$\delta \leqslant \frac{N_{i_0,j}}{M_{i_0,j}} = \frac{N_{i_0,m}}{M_{i_0,m}} \cdot \frac{N_{m,j}}{M_{m,j}}.$$

The sequence $(\frac{N_{i_0,j}}{M_{i_0,j}})_{j>i_0}$ is decreasing, so its limit exists and is larger than or equal to δ. It follows that $N_{m,j}/M_{m,j}$ approaches 1 as $m, j \to \infty$, whence (7) holds. Now apply Lemma 4.1 to see that A must fail to have strict comparison of positive elements. □

Proposition 5.2. *Let A be a simple $\mathcal{V}$I algebra with strict comparison of positive elements, and suppose that A admits a standard decomposition (A_i, ϕ_i) with seed space a finite-dimensional CW-complex X. Then, A has bounded dimension growth.*

Proof. A satisfies the hypotheses of Lemma 5.1. If A is AF, then it has bounded dimension growth, so we may assume that $\dim(X) \geqslant 1$. The conclusion of Lemma 5.1 then implies that

$$\frac{N_{i,j}}{M_{i,j}} \xrightarrow{j\to\infty} 0 \tag{8}$$

for every $i \in \mathbb{N}$. We will prove that A has very slow dimension growth in the sense of Gong; bounded dimension growth follows by the reduction theorem of [15] or, alternatively, the classification theorem of [20].

Let there be given a positive tolerance $\epsilon > 0$. For natural numbers $j > i$ let $\pi_1, \dots, \pi_{N_{i,j}}$ be the co-ordinate projection maps from $X_j = X_i^{n_j/n_i}$ to X_i appearing as eigenvalue maps of $\phi_{i,j}$, and let $l_{i,j}$ be the number of eigenvalue maps of $\phi_{i,j}$ which are point evaluations. Since A is simple and $\dim(X) \geqslant 1$, $l_{i,j_0} > 0$ for some $j_0 > i$. Straightforward calculation then shows that there are at least

$$l_{i,j_0} \cdot M_{j_0,j} = M_{i,j} \cdot \frac{l_{i,j_0}}{M_{i,j_0}}$$

point evaluations in the map $\phi_{i,j}$. Combining this with (8) yields

$$L_j := \left\lfloor \frac{M_{i,j}}{N_{i,j}} \cdot \frac{l_{i,j_0}}{M_{i,j_0}} \right\rfloor \xrightarrow{j\to\infty} \infty.$$

In other words, if one wants to partition the eigenvalue maps of $\phi_{i,j}$ which are point evaluations into $N_{i,j}$ roughly equally sized multisets, then these multisets become arbitrarily large as $j \to \infty$. (We say multisets as some point evaluations may well be repeated.) Assume that we have specified such a partition, let $S_1, \dots, S_{N_{i,j}}$ denote the multisets in the partition, and assume that $\dim(X_i)^3/L_j < \epsilon$. Each $f \in S_l$, $1 \leqslant l \leqslant N_{i,j}$, factors through any of the co-ordinate projections $\pi_1, \dots, \pi_{N_{i,j}}$. Factor $f \in S_l$ as $f = \tilde{f} \circ \pi_l$, where $\tilde{f} : X_i \to X_i$ has range equal to a point; let R_l be the multiset of all maps from X_i to itself obtained in this manner. Let $t(l)$ be the number of copies of π_l appearing among the eigenvalue maps of $\phi_{i,j}$. Put

$$B_i^{(l)} = \mathrm{M}_{m_i(t(l)+|R_l|)} \otimes \mathrm{C}(X_i),$$

and observe that

$$\operatorname{rank}(\mathbf{1}_{B_i^{(l)}}) = m_i\bigl(t(l) + |R_l|\bigr) \geqslant |R_l| = |S_l| \geqslant L_j.$$

Define a map $\psi^{(l)} : A_i \to B_i^{(l)}$ by

$$\psi^{(l)}(a) = \left(\bigoplus_{m=1}^{t(l)} a\right) \oplus \left(\bigoplus_{\tilde{f}\in R_l} a \circ \tilde{f}\right).$$

Put $B_i = B_i^{(1)} \oplus \cdots \oplus B_i^{(N_{i,j})}$, and let $\psi : A_i \to B_i$ be the direct sum of the $\psi^{(l)}$, $1 \leqslant l \leqslant N_{i,j}$. For each $1 \leqslant l \leqslant N_{i,j}$, let $P_l \in A_j$ be the projection which is the sum of the images of the unit of A_i under all of the copies of π_l in $\phi_{i,j}$ and all of the point evaluations in S_l. Let $\gamma_l : B_i^{(l)} \to P_l A_j P_l$ be induced by π_l, and let $\gamma : B_i \to A_i$ be the direct sum of the γ_l. We now have the factorisation

$$A_i \xrightarrow{\psi} B_i \xrightarrow{\gamma} A_j,$$

and each direct summand $B_i^{(l)}$ has the property that

$$\frac{\dim(X_i)^3}{\operatorname{rank}(\mathbf{1}_{B_i^{(l)}})} \leqslant \frac{\dim(X_i)^3}{L_j} < \epsilon.$$

Both i and ϵ were arbitrary, so A has very slow dimension growth. □

Proposition 5.2 establishes the implications (ii) ⇒ (iv) and (ii) ⇒ (v) of Theorem 3.4. (The first observation of the proof is that the hypotheses guarantee that A has slow dimension growth as a $\mathcal{V}$I algebra, and so *a fortiori* as an AH algebra.)

6. Finite decomposition rank

In the present section we prove the remaining implication of Theorem 3.4, namely that finite decomposition rank implies strict comparison of positive elements in $\mathcal{V}$I algebras. The technical key step is Lemma 6.1 below; under the additional assumption of real rank zero, a related result was already observed in [36, Proposition 3.7]. Our proof is inspired by that argument.

Lemma 6.1. *Let A be a simple, separable and unital C^*-algebra with* $\operatorname{dr} A = n < \infty$. *If $a, d^{(0)}, \ldots, d^{(n)} \in A_+$ satisfy*

$$d_\tau(a) < d_\tau\big(d^{(i)}\big)$$

for $i = 0, \ldots, n$ and every $\tau \in T(A)$, then

$$\langle a \rangle \leqslant \big\langle d^{(0)} \big\rangle + \cdots + \big\langle d^{(n)} \big\rangle.$$

Proof. It will be convenient to set up some notation: Given $0 \leqslant \alpha < \beta \leqslant 1$, define functions on the real line by

$$g_\beta(t) := \begin{cases} 0, & t < \beta, \\ 1, & t \geqslant \beta, \end{cases} \qquad g_{\alpha,\beta}(t) := \begin{cases} 0, & t \leqslant \alpha, \\ 1, & t \geqslant \beta, \\ (t-\alpha)/(\beta-\alpha) & \text{else} \end{cases}$$

and

$$f_{\alpha,\beta}(t) := \begin{cases} 0, & t \leqslant \alpha, \\ t, & t \geqslant \beta, \\ \beta(t-\alpha)/(\beta-\alpha) & \text{else.} \end{cases}$$

Before turning to the proof, observe first that our hypotheses imply that a is not invertible in A: indeed, if a was invertible, we had $a^{1/n} \to \mathbf{1}_A$ as $n \to \infty$, so

$$1 = \lim_{i\to\infty} \tau\big(a^{1/n}\big) = d_\tau(a) < d_\tau\big(d^{(i)}\big) \leqslant 1,$$

a contradiction, whence $0 \in \sigma(a)$.

By passing to $M_{n+1}(A)$ (which again has decomposition rank n by [18, Corollary 3.9]), replacing each $d^{(i)} \in A \cong e_{00}M_{n+1}(A)e_{00}$ by a Cuntz equivalent element in the corner $e_{ii}M_{n+1}(A)e_{ii}$ for $i = 1, \dots, n$, and observing that $d_\tau(a) < d_\tau(d^{(i)})$ also holds for every $\tau \in T(M_{n+1}(A))$, we may as well assume that the $d^{(i)}$ themselves are already pairwise orthogonal.

For the actual proof of the lemma, we distinguish two cases. Suppose first that $0 \in \sigma(a)$ is an isolated point. Then, there is $\theta > 0$ such that

$$p := g_\theta(a) \in A$$

is a projection satisfying $\langle p\rangle = \langle a\rangle$, whence $d_\tau(a) = d_\tau(p) = \tau(p)$ for all $\tau \in T(A)$. Furthermore, for any $\tau \in T(A)$ and $i = 0, \dots, n$, we have

$$d_\tau\big(d^{(i)}\big) = \lim_{\delta\searrow 0} \tau\big(g_{\delta/2,\delta}\big(d^{(i)}\big)\big), \tag{9}$$

so there are $\delta_\tau > 0$ and $\eta_\tau > 0$ such that

$$\tau(p) = d_\tau(a) < d_\tau\big(d^{(i)}\big) - \eta_\tau < \tau\big(g_{\delta_\tau/2,\delta_\tau}\big(d^{(i)}\big)\big)$$

for all i. Since the elements of A are continuous when regarded as functions on $T(A)$, each τ has an open neighborhood $U_\tau \subset T(A)$ such that

$$\tau'(p) < \tau'\big(g_{\delta_\tau/2,\delta_\tau}\big(d^{(i)}\big)\big)$$

for $i = 0, \dots, n$ and $\tau' \in U_\tau$. Now by compactness of $T(A)$ (and since, for any positive h,

$$g_{\delta'/2,\delta'}(h) \leqslant g_{\delta/2,\delta}(h) \tag{10}$$

if only $\delta \leqslant \delta'$) it is straightforward to find $\delta_1 > 0$ such that

$$\tau(p) < \tau\big(g_{\delta_1/2,\delta_1}\big(d^{(i)}\big)\big)$$

for all $i = 0, \dots, n$ and $\tau \in T(A)$. Now by [36, Proposition 3.7], we have

$$p \precsim g_{\delta_1/2,\delta_1}\big(d^{(0)}\big) + \cdots + g_{\delta_1/2,\delta_1}\big(d^{(n)}\big),$$

whence

$$\langle a\rangle = \langle p\rangle \leqslant \big\langle g_{\delta_1/2,\delta_1}\big(d^{(0)}\big)\big\rangle + \cdots + \big\langle g_{\delta_1/2,\delta_1}\big(d^{(n)}\big)\big\rangle \leqslant \big\langle d^{(0)}\big\rangle + \cdots + \big\langle d^{(n)}\big\rangle.$$

Next, suppose 0 is a limit point of $\sigma(a)$. The proof in this case is similar to that of [36, Proposition 3.7], but we have to deal with some extra technical difficulties. Since $a = \lim_{\epsilon \searrow 0} f_{\epsilon/2,\epsilon}(a)$ and since the $d^{(i)}$ are pairwise orthogonal, it will be enough to show that

$$f_{\epsilon/2,\epsilon}(a) \precsim d^{(0)} + \cdots + d^{(n)}$$

for all $\epsilon > 0$. So, given some $\epsilon > 0$, we set

$$b := f_{\epsilon/2,\epsilon}(a) \quad \text{and} \quad c := (g_{0,\epsilon/4} - g_{\epsilon/4,\epsilon/2})(a),$$

then

$$c \perp b \quad \text{and} \quad b + c \precsim a.$$

Since 0 is a limit point of $\sigma(a)$, we have $c \neq 0$, hence (each $\tau \in T(A)$ is faithful by simplicity of A, c is continuous as a function on $T(A)$ and $T(A)$ is compact)

$$\alpha := \min\{\tau(c) \mid \tau \in T(A)\} > 0.$$

Using that $c \leqslant \mathbf{1}_A$ and that $c \perp b$, we obtain for all $\tau \in T(A)$

$$\begin{aligned} d_\tau(b) + \alpha &\leqslant d_\tau(b) + \tau(c) \\ &\leqslant d_\tau(b) + d_\tau(c) \\ &= d_\tau(b + c) \\ &\leqslant d_\tau(a) \end{aligned}$$

and, by hypothesis,

$$d_\tau(b) < d_\tau\big(d^{(i)}\big) - \alpha \tag{11}$$

for $i = 0, \ldots, n$ and $\tau \in T(A)$.

Again, to show that $b \precsim d^{(0)} + \cdots + d^{(n)}$ it will suffice to prove that

$$f_{\eta,2\eta}(b) \precsim d^{(0)} + \cdots + d^{(n)}$$

for any given $\eta > 0$. To this end, we set

$$\bar{b} := g_{\eta/2,\eta}(b) \tag{12}$$

and choose $0 < \delta_2 < \alpha/4$ such that

$$\tau(\bar{b}) < \tau\big(\bar{d}^{(i)}\big) - \frac{3\alpha}{4} \tag{13}$$

for $i = 0, \ldots, n$ and $\tau \in T(A)$, where

$$\bar{d}^{(i)} := g_{\delta_2/2,\delta_2}\big(d^{(i)}\big).$$

The number δ_2 is obtained in a similar way as δ_1 in the first part of the proof, using compactness of $T(A)$: From (9) and (11) we see that for each $\tau \in T(A)$ there is $\delta_\tau > 0$ such that for $i = 0, \ldots, n$

$$\tau(\bar{b}) \leqslant d_\tau(b) \overset{(11)}{<} d_\tau\big(d^{(i)}\big) - \alpha \overset{(9)}{\leqslant} \tau\big(g_{\delta_\tau/2,\delta_\tau}\big(d^{(i)}\big)\big) - \frac{3\alpha}{4}.$$

Each τ has an open neighborhood U_τ such that

$$\tau'(\bar{b}) < \tau'\big(g_{\delta_\tau/2,\delta_\tau}\big(d^{(i)}\big)\big) - \frac{3\alpha}{4}$$

for $i = 0, \ldots, n$ and $\tau' \in U_\tau$. Similar as in the first part of the proof, compactness of $T(A)$ and (10) now yield $\delta_2 > 0$ such that (13) holds.

Since $\operatorname{dr} A = n$, by [18, Proposition 5.1], there is a system $(F_k, \psi_k, \varphi_k)_{k\in\mathbb{N}}$ of c.p. approximations for A such that the φ_k are n-decomposable and the ψ_k are approximately multiplicative. In other words, for each $k \in \mathbb{N}$ there are finite-dimensional C^*-algebras F_k and c.p.c. maps

$$A \xrightarrow{\psi_k} F_k \xrightarrow{\varphi_k} A$$

such that

(i) $\varphi_k\psi_k(a) \to a$ for each $a \in A$ as $k \to \infty$,

(ii) F_k admits a decomposition $F_k = \bigoplus_{i=0}^n F_k^{(i)}$ such that

$$\varphi_k^{(i)} := \varphi_k|_{F_k^{(i)}}$$

preserves orthogonality (i.e., has order zero in the sense of [35, Definition 2.1(b)]) for each $i = 0, \ldots, n$ and $k \in \mathbb{N}$,

(iii) $\|\psi_k(aa') - \psi_k(a)\psi_k(a')\| \to 0$ for any $a, a' \in A$ as $k \to \infty$.

We set

$$\psi_k^{(i)}(\cdot) := \mathbf{1}_{F_k^{(i)}}\psi_k(\cdot)$$

for each i and k and note that the $\psi_k^{(i)}$ are also approximately multiplicative for each i since the $\mathbf{1}_{F_k^{(i)}}$ are central projections in F_k. As in [18, Remark 5.2(ii)], we may (and will) assume that the ψ_k are unital.

Recall from [36, 1.2], that each of the order zero maps $\varphi_k^{(i)}$ has a supporting $*$-homomorphism

$$\sigma_k^{(i)} : F_k^{(i)} \to A'';$$

this a $*$-homomorphism satisfying

$$\varphi_k^{(i)}(x) = \sigma_k^{(i)}(x)\varphi_k^{(i)}(\mathbf{1}_{F_k^{(i)}}) = \varphi_k^{(i)}(\mathbf{1}_{F_k^{(i)}})\sigma_k^{(i)}(x) \in A$$

for all $x \in F_k^{(i)}$.

We proceed to show that there is $K \in \mathbb{N}$ such that

$$\tau\big(g_{1-\alpha/4}\big(\psi_k^{(i)}(\bar{b})\big)\big) < \tau\big(g_{\alpha/4}\big(\psi_k^{(i)}(\bar{d}^{(i)})\big)\big) \tag{14}$$

for all $i = 0, \dots, n$, $\tau \in T(F_k^{(i)})$ and $k \geqslant K$. If this was not the case, there would be a strictly increasing sequence $(k_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that, for some fixed $i_0 \in \{0, \dots, n\}$, there are $\tau_l \in T(F_{k_l}^{(i_0)})$ satisfying

$$\tau_l\big(g_{1-\alpha/4}\big(\psi_{k_l}^{(i_0)}(\bar{b})\big)\big) \geqslant \tau_l\big(g_{\alpha/4}\big(\psi_{k_l}^{(i_0)}(\bar{d}^{(i_0)})\big)\big) \tag{15}$$

for all $l \in \mathbb{N}$. But then

$$\begin{aligned} \tau_l\big(\psi_{k_l}^{(i_0)}(\bar{b})\big) &\geqslant \tau_l\big(g_{1-\alpha/4}\big(\psi_{k_l}^{(i_0)}(\bar{b})\big)\big) - \frac{\alpha}{4} \\ &\overset{(15)}{\geqslant} \tau_l\big(g_{\alpha/4}\big(\psi_{k_l}^{(i_0)}(\bar{d}^{(i_0)})\big)\big) - \frac{\alpha}{4} \\ &\geqslant \tau_l\big(\psi_{k_l}^{(i_0)}(\bar{d}^{(i_0)})\big) - 2 \cdot \frac{\alpha}{4} \end{aligned} \tag{16}$$

for all $l \in \mathbb{N}$. Now fix some free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, then

$$\bar{\tau}(\cdot) := \lim_{\omega} \tau_l \psi_{k_l}^{(i_0)}(\cdot)$$

obviously is a well-defined positive functional on A. It is tracial, since the τ_l are traces and the $\psi_{k_l}^{(i_0)}$ are approximately multiplicative. It is a state, since $\bar{\tau}(\mathbf{1}_A) = 1$ (the τ_l are states and the $\psi_{k_l}^{(i_0)}$ are unital). We have now constructed a tracial state $\bar{\tau}$ on A satisfying

$$\bar{\tau}(\bar{b}) \geqslant \bar{\tau}\big(\bar{d}^{(i_0)}\big) - \frac{\alpha}{2},$$

a contradiction to (13). Therefore, there is $K \in \mathbb{N}$ such that (14) holds for all $i = 0, \dots, n$, $\tau \in T(F_k^{(i)})$ and $k \geqslant K$.

As a consequence, for $i = 0, \dots, n$ and $k \geqslant K$ there exist partial isometries $v_k^{(i)} \in F_k^{(i)}$ such that

$$\big(v_k^{(i)}\big)^* v_k^{(i)} = g_{1-\alpha/4}\big(\psi_k^{(i)}(\bar{b})\big)\big(\geqslant g_{1-\alpha/4,1}\big(\psi_k^{(i)}(\bar{b})\big)\big) \tag{17}$$

and

$$v_k^{(i)}\big(v_k^{(i)}\big)^* \leqslant g_{\alpha/4}\big(\psi_k^{(i)}\big(\bar{d}^{(i)}\big)\big)$$

(the $g_{1-\alpha/4}(\psi_k^{(i)}(\bar{b}))$ and $g_{\alpha/4}(\psi_k^{(i)}(\bar{d}^{(i)}))$ are projections in $F_k^{(i)}$—which in turn are finite-dimensional algebras, hence satisfy the comparison property).

Note that

$$\begin{aligned} v_k^{(i)}\big(v_k^{(i)}\big)^* &\leqslant g_{\alpha/4}\big(\psi_k^{(i)}\big(\bar{d}^{(i)}\big)\big) \\ &\leqslant g_{\alpha/4}\big(\psi_k\big(\bar{d}^{(i)}\big)\big) \\ &\leqslant g_{0,\delta_2}\big(\psi_k\big(\bar{d}^{(i)}\big)\big) \\ &\leqslant \mathbf{1}_{F_k} \end{aligned}$$

for $i = 0, \dots, n$ and $k \geqslant K$; since the $v_k^{(i)}(v_k^{(i)})^*$ are projections, from this one easily concludes that

$$g_{0,\delta_2}\big(\psi_k\big(\bar{d}^{(i)}\big)\big)v_k^{(i)}\big(v_k^{(i)}\big)^* = v_k^{(i)}\big(v_k^{(i)}\big)^*.$$

Because the ψ_k are approximately multiplicative, and using that the $\bar{d}^{(i)}$ are mutually orthogonal, we also have

$$\left\| g_{0,\delta_2}\left(\psi_k\left(\sum_{j=0}^n \bar{d}^{(j)}\right)\right) - \psi_k\left(\sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\right)\right\| \xrightarrow{k\to\infty} 0$$

and

$$\big\| g_{0,\delta_2}\big(\psi_k\big(\bar{d}^{(i')}\big)\big)v_k^{(i)}\big(v_k^{(i)}\big)^* - \delta_{i',i}\cdot v_k^{(i)}\big(v_k^{(i)}\big)^*\big\| \xrightarrow{k\to\infty} 0$$

for $i, i' \in \{0, \dots, n\}$ ($\delta_{i,i'}$ denotes the Kronecker delta of i and i'). Moreover, we have

$$\left\| \sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\varphi_k\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big) - \varphi_k\left(\psi_k\left(\sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\right)v_k^{(i)}\big(v_k^{(i)}\big)^*\right)\right\| \leqslant 2\mu_k^{\frac{1}{2}}$$

by [18, Lemma 3.5] (an easy consequence of Stinespring's theorem), where

$$\mu_k := \max\left\{\left\|(\varphi_k\psi_k - \mathrm{id})\left(\sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\right)\right\|, \left\|(\varphi_k\psi_k - \mathrm{id})\left(\left(\sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\right)^2\right)\right\|\right\}.$$

Observing that $\mu_k \to 0$ as $k \to \infty$ and combining all these facts we obtain

$$\left\| \sum_{j=0}^n g_{0,\delta_2}\big(\bar{d}^{(j)}\big)\varphi_k\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big) - \varphi_k\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big)\right\| \xrightarrow{k\to\infty} 0$$

and

$$\big\| g_{0,\delta_2}\big(\bar{d}^{(i')}\big)\varphi_k\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big) - \delta_{i,i'}\cdot\varphi_k\big(v_k^{(i)}(v_k^{(i)})^*\big)\big\| \xrightarrow{k\to\infty} 0$$

for $i, i' = 0, \dots, n$. Using that

$$\varphi_k\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big) = \varphi_k^{(i)}\big(v_k^{(i)}\big(v_k^{(i)}\big)^*\big) = \varphi_k^{(i)}(\mathbf{1}_{F_k^{(i)}})\sigma_k^{(i)}\big(v_k^{(i)}\big)\sigma_k^{(i)}\big(v_k^{(i)}\big)^*$$

it follows easily that

$$\begin{aligned}\Bigg\| \sigma_k^{(i')}\big(v_k^{(i')}\big)^* \varphi_k^{(i')}(\mathbf{1}_{F_k^{(i')}})^{\frac{1}{2}} \sum_{j=0}^{n} g_{0,\delta_2}\big(\bar{d}^{(j)}\big) \varphi_k^{(i)}(\mathbf{1}_{F_k^{(i)}})^{\frac{1}{2}} \sigma_k^{(i)}\big(v_k^{(i)}\big) \\ - \delta_{i',i} \cdot \varphi_k^{(i)}\big(\big(v_k^{(i)}\big)^* v_k^{(i)}\big) \Bigg\| \xrightarrow{k\to\infty} 0\end{aligned}$$

for $i, i' = 0, \ldots, n$, whence

$$\left\| s_k^* \sum_{j=0}^{n} g_{0,\delta_2/2}\big(\bar{d}^{(j)}\big) s_k - f_{\eta,2\eta}(b)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \varphi_k^{(i)}\big(\big(v_k^{(i)}\big)^* v_k^{(i)}\big) \right) f_{\eta,2\eta}(b)^{\frac{1}{2}} \right\| \xrightarrow{k\to\infty} 0,$$

where

$$s_k := \sum_{i=0}^{n} \varphi_k^{(i)}(\mathbf{1}_{F_k^{(i)}})^{\frac{1}{2}} \sigma_k^{(i)}\big(v_k^{(i)}\big) f_{\eta,2\eta}(b)^{\frac{1}{2}}$$

for $k \geqslant K$. We now have

$$\begin{aligned}
&\lim_{k\to\infty} \left\| f_{\eta,2\eta}(b) - s_k^* \sum_{j=0}^{n} g_{0,\delta_2}\big(\bar{d}^{(j)}\big) s_k \right\| \\
&\quad = \lim_{k\to\infty} \left\| f_{\eta,2\eta}(b) - f_{\eta,2\eta}(b)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \varphi_k^{(i)}\big(\big(v_k^{(i)}\big)^* v_k^{(i)}\big) \right) f_{\eta,2\eta}(b)^{\frac{1}{2}} \right\| \\
&\quad \overset{(17)}{\leqslant} \lim_{k\to\infty} \left\| f_{\eta,2\eta}(b) - f_{\eta,2\eta}(b)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \varphi_k^{(i)}\big(g_{1-\alpha/4,1}\big(\psi_k^{(i)}(\bar{b})\big)\big) \right) f_{\eta,2\eta}(b)^{\frac{1}{2}} \right\| \\
&\quad \overset{\text{(iii)}}{=} \lim_{k\to\infty} \left\| f_{\eta,2\eta}(b) - f_{\eta,2\eta}(b)^{\frac{1}{2}} \left(\sum_{i=0}^{n} \varphi_k^{(i)} \psi_k^{(i)}\big(g_{1-\alpha/4,1}(\bar{b})\big) \right) f_{\eta,2\eta}(b)^{\frac{1}{2}} \right\| \\
&\quad \overset{\text{(i)}}{=} \lim_{k\to\infty} \big\| f_{\eta,2\eta}(b) - f_{\eta,2\eta}(b)^{\frac{1}{2}} g_{1-\alpha/4,1}(\bar{b}) f_{\eta,2\eta}(b)^{\frac{1}{2}} \big\| \\
&\quad \overset{(12)}{=} 0.
\end{aligned}$$

This shows that

$$f_{\eta,2\eta}(b) \precsim \sum_{j=0}^{n} g_{0,\delta_2}\big(\bar{d}^{(j)}\big).$$

Since η was arbitrary, and because

$$\sum_{j=0}^{n} g_{0,\delta_2}\big(\bar{d}^{(j)}\big) \precsim \sum_{j=0}^{n} \bar{d}^{(j)},$$

it follows that

$$b \precsim \sum_{j=0}^{n} d^{(j)},$$

as desired. □

Corollary 6.2. *Let A be as in the hypotheses of Theorem* 3.4. *If A has finite decomposition rank, then A has strict comparison of positive elements.*

Proof. We prove the contrapositive. Suppose that A does not have strict comparison of positive elements and fix a standard decomposition as in 3.4. Then, condition (7) of Lemma 5.1 holds. It follows that A satisfies the hypotheses of Lemma 4.1, and so strict comparison fails in the manner prescribed the in the conclusion of that lemma. In light of Lemma 6.1, this failure excludes the possibility that A has finite decomposition rank. □

The preceding corollary establishes the implication (iii) ⇒ (ii) of Theorem 3.4.

7. Real rank zero

In this section we prove that an algebra of real rank zero which also satisfies the hypotheses of Theorem 3.4 must then satisfy conditions (i)–(vi) of the same theorem, thus completing the proof of our main result. The result is a special case of a theorem of Toan Ho and the first named author which will appear in Toan Ho's PhD thesis. As no preprint of this result was available at the time of writing, we give a proof here which applies only to $\mathcal{V}$I algebras.

Let X be a compact connected Hausdorff space and a a self-adjoint element of $\mathrm{M}_n(\mathrm{C}(X))$. For each $x \in X$, form an n-tuple consisting of the eigenvalues of a listed in decreasing order. For each $m \in \{1, \ldots, n\}$ let $\lambda_m : X \to \mathbb{R}$ be the function whose value at x is the mth entry of the eigenvalue n-tuple for x. The *variation of the normalised trace* of a (v. [1]), denoted $TV(a)$, is defined as

$$\sup\left\{\left|\frac{1}{n}\sum_{m=1}^{n}\bigl(\lambda_m(x) - \lambda_m(y)\bigr)\right|:\ x, y \in X\right\}.$$

Suppose that $A = \lim_{i\to\infty}(\mathrm{M}_{m_i}(\mathrm{C}(X_i), \phi_i)$ is of real rank zero, and let a be a self-adjoint element of some $\mathrm{M}_{m_i}(\mathrm{C}(X_i))$. Then, by Theorem 1.3 of [1], the variation of the normalised trace tends to zero as $j \to \infty$ for each direct summand of $\phi_{i,j}(a)$ corresponding to a connected component of X_j.

Proposition 7.1. *Let $A = \lim_{i\to\infty}(A_i, \phi_i)$ be a simple $\mathcal{V}$I algebra with seed space a finite-dimensional CW-complex. If A has real rank zero, then A has bounded dimension growth.*

Proof. If A is AF, then there is nothing to prove. If A is not AF, then all but finitely many of the ϕ_is contain at least one co-ordinate projection as an eigenvalue map, and each X_i has dimension strictly greater than zero. It will be enough to prove that for each $i \in \mathbb{N}$,

$$\frac{N_{i,j}}{M_{i,j}} \xrightarrow{j\to\infty} 0.$$

The proof of Proposition 5.2 then shows that A has bounded dimension growth.

Let $\epsilon > 0$ be given, and suppose for a contradiction that for some $i \in \mathbb{N}$ and $c > 0$ we have

$$\frac{N_{i,j}}{M_{i,j}} \xrightarrow{j\to\infty} c.$$

By increasing i if necessary (and following the lines of the proof of Lemma 5.1) we may assume that $c > 7/8$. Choose a continuous function $f : X_i \to [0,1]$ such that for some points x_0, x_1 in the same connected component of X_i we have $f(x_0) = 0$ and $f(x_1) = 1$; put $a := f \cdot \mathbf{1}_{A_i}$.

For any $j > i$ we have

$$\phi_{i,j}(a)(x) = \operatorname{diag}\big(a(\gamma_1(x)), \ldots, a\big(\gamma_{M_{i,j}}(x)\big)\big), \quad \forall x \in X_j,$$

where the γ_ls are the eigenvalue maps of $\phi_{i,j}$. Let $\pi_1, \ldots, \pi_{N_{i,j}} : X_j \to X_i$ be the distinct co-ordinate projections appearing among the γ_ls. Since $TV(\phi_{i,j}(a))$ is unaffected by unitary conjugation in A_j, we may assume that

$$\phi_{i,j}(a)(x) = \operatorname{diag}\big(a\big(\pi_1(x)\big), \ldots, a\big(\pi_{N_{i,j}}(x)\big), \ldots, a\big(\gamma_{M_{i,j}}(x)\big)\big), \quad \forall x \in X_j.$$

Fix a point $y_0 \in X_j$ which when viewed as an element of a Cartesian power of X_i has the value x_0 in each co-ordinate; define y_1 similarly with respect to x_1, and notice that y_0 and y_1 are in the same connected component of X_j. Then, the eigenvalue list of $\phi_{i,j}(a)(y_0)$ contains at least $m_i N_{i,j}$ 0s, while the list for $\phi_{i,j}(a)(y_1)$ contains at least $m_i N_{i,j}$ 1s. By the pigeonhole principle, at least $m_i[M_{i,j} - 2(M_{i,j} - N_{i,j})]$ of the eigenfunctions λ_m corresponding to $\phi_{i,j}(a)$ have the value 0 at y_0 and 1 at y_1, while the remaining $2m_i(M_{i,j} - N_{i,j})$ eigenfunctions satisfy $\lambda_m(y_1) - \lambda_m(y_0) \geqslant -1$. Now,

$$\begin{aligned} TV\big(\phi_{i,j}(a)\big) &\geqslant \left| \frac{1}{m_i M_{i,j}} \sum_{m=1}^{m_j} \big(\lambda_m(y_1) - \lambda_m(y_0)\big) \right| \\ &\geqslant \frac{M_{i,j} - 4(M_{i,j} - N_{i,j})}{M_{i,j}} \\ &= \frac{4N_{i,j}}{M_{i,j}} - 3 \\ &> \frac{1}{2} \end{aligned}$$

since $N_{i,j}/M_{i,j} > c > 7/8$. This contradicts our real rank zero assumption for A, completing the proof. □

8. Non-$\mathcal{Z}$-stable $\mathcal{V}$I algebras

We now give examples of non-isomorphic $\mathcal{V}$I-algebras which cannot be distinguished using topological K-theory and traces. These are not the first such—examples are already given in [26]—but the results of the present paper allow us to construct a large class of examples with relatively little further effort. They will also demonstrate the variety of tracial state spaces which can occur in a simple nuclear C^*-algebra of infinite decomposition rank.

Any subgroup G of $\mathbb{Q}$ corresponds to a list of prime powers $P_G = \{p_1^{n_1}, p_2^{n_2}, \ldots\}$, $n_i \in \mathbb{Z}^+ \cup \{\infty\}$, in the following sense: the elements of G are those rationals which, when in loweset terms, have denominators of the form $p_1^{r_1} p_2^{r_2} \ldots$, where $r_i < n_i$ for all i and $r_i = 0$ for all but finitely many i. If $n_i = \infty$ for some i then we will say that G is of *infinite type*. Let p be a prime. If $p^\infty \in P_G$ and H is the subgroup of $\mathbb{Q}$ with $P_H = \{p^\infty\}$, then $H \otimes G \cong G$.

Let X be a contractible and finite-dimensional CW-complex. Construct a $\mathcal{V}$I algebra $A_X = \lim_{i\to\infty}(A_i, \phi_i)$ satisfying:

(i) The ratio $N_{1,j}/M_{1,j}$ does not vanish;
(ii) A_X is simple by virtue of a judicious inclusion of point evaluations as eigenvalue maps of the ϕ_i;
(iii) The K_0-group of A_X (necessarily a subgroup of $\mathbb{Q}$ by the contractibility of $X_1 = X$ and each X_i) is of infinite type.

Inspection of Villadsen's construction in [33] shows that for a fixed X, one can arrange for $\mathrm{K}_0(A_X)$ to be an arbitrary infinite type subgroup of $\mathbb{Q}$. There are uncountably many such subgroups, and hence, for a fixed X, uncountably many non-isomorphic algebras A_X satisfying (i)–(iii). Condition (i) ensures that A_X does not have strict comparison of positive elements (use Lemmas 4.1 and 5.1).

Fix an algebra A_X as above. Let p be a prime such that $p^\infty \in P_{\mathrm{K}_0(A_X)}$, and let $\mathfrak{U}$ be a UHF algebra with $P_{\mathrm{K}_0(\mathfrak{U})} = \{p^\infty\}$. We claim that the tensor product $A_X \otimes \mathfrak{U}$ has the same topological K-theory and tracial state space as A. At the level of K-theory this statement follows from the Künneth theorem, the triviality of the K_1-groups of both $\mathfrak{U}$ and A_X (in the case of A_X this is due to the contractibility of X_i), and the isomorphism

$$\mathrm{K}_0(A_X) \otimes \mathrm{K}_0(\mathfrak{U}) \cong \mathrm{K}_0(A_X).$$

At the level of tracial state spaces the statement is due to the fact that $\mathfrak{U}$ admits a unique tracial state. There is only one possible pairing of traces with K_0 in each of A_X and $A_X \otimes \mathfrak{U}$, as their K_0-groups are subgroups of the rationals. As noted above, A_X does not have strict comparison of positive elements, but $A_X \otimes \mathfrak{U}$ does by virtue of [22, Lemma 5.1]. Thus, A_X and $A_X \otimes \mathfrak{U}$ are not isomorphic, and by varying X and $\mathrm{K}_0(A_X)$ independently we obtain a large class of examples of the desired variety.

Straightforward but laborious calculation shows that the tracial state space of A_X as above is a Bauer simplex with extreme boundary homeomorphic to $X^{\times\infty}$. (The details of this calculation are more or less contained in the proof of [30, Theorem 4.1]—we will not reproduce them here.) A_X also has infinite decomposition rank by Theorem 3.4, and this does not depend on X being contractible. Thus, a large variety of structure can occur in the tracial state space of a simple nuclear C^*-algebra with infinite decomposition rank.

Finally, we remark that infinite decomposition rank can also occur in the case of a simple AH algebra with *unique* tracial state, as observed in [36, Example 6.6(i)], using the examples of [34].

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