Bull. London Math. Soc. 41 (2009) 639–653
doi:10.1112/blms/bdp037

A direct proof of $\mathcal{Z}$-stability for approximately homogeneous C*-algebras of bounded topological dimension

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Abstract

We prove that a unital simple approximately homogeneous C*-algebra with no dimension growth absorbs the Jiang–Su algebra tensorially without appealing to the classification theory of these algebras. Our main result continues to hold under the slightly weaker hypothesis of exponentially slow dimension growth.

1. *Introduction*

The property of absorbing the Jiang–Su algebra $\mathcal{Z}$ tensorially, *$\mathcal{Z}$-stability*, briefly, is a powerful regularity property for separable amenable C*-algebras. It is a necessary condition for the confirmation of G. A. Elliott's K-theoretic rigidity conjecture, which predicts that Banach algebra K-theory and positive traces will form a complete invariant for simple separable amenable C*-algebras. We refer the reader to [**7**] for an up-to-date account of $\mathcal{Z}$-stability as it relates to Elliott's conjecture.

The necessity of $\mathcal{Z}$-stability for K-theoretic classification suggests a two-step approach to further positive classification results: first, establish broad classification theorems for $\mathcal{Z}$-stable C*-algebras; second, prove that natural examples of simple separable amenable C*-algebras are $\mathcal{Z}$-stable. Winter, in a series of papers, has made significant contributions to the first part of this program. For instance, he has shown that the C*-algebras associated to minimal uniquely ergodic diffeomorphisms satisfy Elliott's conjecture modulo $\mathcal{Z}$-stability. However, there has so far been no progress on the second part of the program. This is not to say that we do not have natural examples of $\mathcal{Z}$-stable C*-algebras. It is only that the $\mathcal{Z}$-stability of these examples is typically a consequence of having proved directly that the said examples satisfy Elliott's conjecture.

If we are to have any hope of carrying out the suggested two-step approach to Elliott's conjecture, then we must understand why already classified C*-algebras are $\mathcal{Z}$-stable *without* appealing to the heavy machinery of classification. The purpose of this article is to give a direct, read 'not passing through classification', proof that unital simple approximately homogeneous (AH) C*-algebras with no dimension growth are $\mathcal{Z}$-stable. The result that these C*-algebras satisfy Elliott's conjecture, due to various combinations of Elliott, Gong, and Li, is one of the most difficult theorems in the classification theory for separable stably finite C*-algebras, and is therefore an appropriate starting point for understanding $\mathcal{Z}$-stability. (See [**5**, **6**, **8**].)

Finally, we mention that Winter [**14**] has established $\mathcal{Z}$-stability for a class of simple C*-algebras that includes the unital simple AH algebras of no dimension growth, using techniques which differ substantially from ours. Our result, however, allows one to relax the

Received 5 September 2008; revised 4 February 2009; published online 12 June 2009.

2000 *Mathematics Subject Classification* 46L35 (primary), 46L80 (secondary).

Marius Dadarlat was partially supported by NSF grant DMS-0500693. N. Christopher Phillips was partially supported by NSF grant DMS-0701076, by the Fields Institute for Research in Mathematical Sciences, Toronto, Canada, and by an Elliott Distinguished Visitorship at the Fields Institute. Andrew S. Toms was partially supported by NSERC.

no dimension growth condition to a slightly weaker notion of 'exponentially slow dimension growth', and so is not subsumed by Winter's result.

2. *Preliminaries*

2.1. *Generalities*

We use M_n to denote the C*-algebra of $n \times n$ matrices with entries in $\mathbb{C}$. Let F and H be subsets of a metric space X (with metric d) and let $\epsilon > 0$ be given. We write $F \subseteq_\epsilon H$ if for each $f \in F$ there is some $h \in H$ such that $d(f, h) < \epsilon$. We write $F \approx_\epsilon H$ if there is a bijection $\eta : F \to H$ such that $d(f, \eta(f)) < \epsilon$. The primitive ideal space (or spectrum) of a C*-algebra A is denoted by $\mathrm{Spec}(A)$.

2.2. *AH algebras*

Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable infinite-dimensional Hilbert space. We call a C*-algebra *homogeneous* if it is isomorphic to $p(\mathrm{C}(X) \otimes \mathcal{K})p$ for a compact metric space X and a constant rank projection $p \in \mathrm{C}(X) \otimes \mathcal{K}$, and call a C*-algebra *semihomogeneous* if it is isomorphic to a finite direct sum of homogeneous C*-algebras. A unital AH C*-algebra (*AH algebra*) is the limit of an inductive sequence $(A_i, \phi_i)_{i=1}^{\infty}$ in which each $\phi_i : A_i \to A_{i+1}$ is unital and each A_i is semi-homogeneous, that is,

$$A_i = \bigoplus_{l=1}^{n_i} p_{i,l}(\mathrm{C}(X_{i,l}) \otimes \mathcal{K})p_{i,l},$$

where n_i is a natural number, $X_{i,l}$ is a compact metric space, and $p_{i,l}$ is a constant rank projection in $\mathrm{C}(X_{i,l}) \otimes \mathcal{K}$. It follows easily from an argument in the proof of [**9**, Proposition 3.4] that the spaces $X_{i,l}$ may always be assumed to be connected and have finite covering dimension. We refer to the sequence (A_i, ϕ_i) as an *AH sequence*.

Now let A be a unital AH algebra. If A is the limit of an AH sequence (A_i, ϕ_i) as above for which

$$\liminf_{i \to \infty} \max_{1 \leqslant l \leqslant n_i} \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} = 0,$$

then we say that A has *slow dimension growth*; if it is the limit of an AH sequence such that for some $M > 0$, we have $\dim(X_{i,l}) < M$ for all i and l, then we say that A has *no dimension growth*.

Given an AH sequence (A_i, ϕ_i) and $j > i$, we write $\phi_{i,j}$ for the composition $\phi_{j-1} \circ \ldots \circ \phi_i$ and $\phi_{i,\infty}$ for the canonical map from A_i into the limit algebra A. We define $\phi_{i,j}^{l,k} : A_{i,l} \to A_{j,k}$ and $\phi_{i,j}^{k} : A_i \to A_{j,k}$ to be the obvious restrictions of $\phi_{i,j}$. The $\phi_{i,j}^{l,k}$ are referred to as *partial maps*.

It is well known that an AH algebra $A = \varinjlim_{i \to \infty}(A_i, \phi_i)$ is simple if and only if for every $i \in \mathbb{N}$ and $a \in A_i \setminus \{0\}$, there is some $j \geqslant i$ such that $\phi_{i,j}(a)$ generates A_j as an ideal. This last condition is equivalent to $\phi_{i,j}(a)$ being nonzero at every $x \in X_{j,1} \cup \ldots \cup X_{j,n_j}$.

2.3. *Maps between homogeneous C*-algebras*

Let X and Y be compact metric spaces, and let $p \in \mathrm{C}(X) \otimes \mathcal{K}$ and $q \in \mathrm{C}(Y) \otimes \mathcal{K}$ be projections with constant rank. Let

$$\mathrm{ev}_x : p(\mathrm{C}(X) \otimes \mathcal{K})p \longrightarrow \mathrm{M}_{\mathrm{rank}(p)}$$

be given by $f \mapsto f(x)$; define ev_y for $y \in Y$ similarly. Let

$$\phi : p(\mathrm{C}(X) \otimes \mathcal{K})p \longrightarrow q(\mathrm{C}(Y) \otimes \mathcal{K})q$$

be a unital $*$-homomorphism. Set $N = \mathrm{rank}(q)/\mathrm{rank}(p)$. It is well known that for any $y \in Y$, the map $\mathrm{ev}_y \circ \phi$ has the following form, up to unitary equivalence:

$$\mathrm{ev}_y \circ \phi = \bigoplus_{j=1}^{N} \mathrm{ev}_{x_j},$$

where the x_j are points in X, not necessarily distinct. In other words, the x_j form an *N-multiset* $\{\{x_1, \ldots, x_N\}\}$ (the double brackets indicate that repetition counts), which we denote by $\mathrm{sp}_\phi(y)$. The set of all such multisets may be identified with the quotient of the Cartesian product X^N by the action of the symmetric group S_N on coordinates, and so inherits naturally a metric from X.

Suppose now that ϕ has finite-dimensional image. Let $R \subset X$ be a finite subset. We say for short that *ϕ has spectrum R* if for every $y \in Y$ the set of elements of the multiset $\mathrm{sp}_\phi(y)$ is exactly R. This terminology makes sense even if q does not have constant rank, since we can partition Y into finitely many closed sets on which q does have constant rank. (In the sense of this terminology, if Y is not connected, then the 'spectrum' of ϕ need not even exist. However, it captures the concept we actually need.)

2.4. *Semicontinuous projection-valued maps*

Let X be a topological space. By a lower semicontinuous function $f : X \to (\mathrm{M}_n)_+$ we mean a function such that for every vector $\xi \in \mathbb{C}^n$, the real-valued function $x \mapsto \langle f(x)\xi, \xi\rangle$ is lower semicontinuous (see [**1**, Section 3]). The following result from [**3**] will be used in the sequel.

PROPOSITION 2.1. *Let X be a compact metric space of dimension d, and let $P : X \to (\mathrm{M}_n)_+$ be a lower semicontinuous projection-valued map. Suppose that*

$$\mathrm{rank}(P(x)) > \tfrac{1}{2}(d+1) + k$$

for all $x \in X$. It follows that there is a continuous projection-valued map $R : X \to \mathrm{M}_n$ of constant rank equal to k such that

$$R(x) \leqslant P(x)$$

for all $x \in X$.

REMARK 2.2. By [**9**, Theorem 2.5(a)], if we replace $\frac{1}{2}(d+1)$ with $d+1$ in the hypotheses of Proposition 2.1, then we may assume that the projection-valued map R corresponds to a trivial complex vector bundle over X.

LEMMA 2.3. *Let X, Y, p, q, and $\phi : p(\mathrm{C}(X) \otimes \mathcal{K})p \to q(\mathrm{C}(Y) \otimes \mathcal{K})q \subseteq \mathrm{M}_m(\mathrm{C}(Y))$ be as in Subsection 2.3, except with p and q not necessarily of constant rank. Let U be an open subset of X, and let $r \in p(\mathrm{C}(X) \otimes \mathcal{K})p$ be a positive element that is equal to a projection at every $x \in U$. Define a projection-valued map $R : Y \to \mathrm{M}_m$ as follows: $R(y)$ is the image of r under the direct sum of those irreducible direct summands of $\mathrm{ev}_y \circ \phi$ that correspond to points in U. It follows that R is lower semicontinuous.*

Proof. We reduce to the case in which p and q have constant rank. For p, we partition Y into finitely many closed and open subsets $X_1, \ldots, X_l$ on which p has constant rank. Let $e_i \in p(\mathrm{C}(X) \otimes \mathcal{K})p$ be equal to p on X_i and equal to zero elsewhere. For each i, we then replace q by $\phi(e_i)$, obtaining lower semicontinuous functions R_i. It is easily checked that $R = R_1 + \ldots + R_l$ is then also lower semicontinuous. For q, we partition Y into finitely many closed and open subsets $Y_1, \ldots, Y_n$ on which q has constant rank, and consider the $*$-homomorphisms $a \mapsto \phi(a)|_{Y_j}$ for $j = 1, \ldots, n$. We find that R is lower semicontinuous on each Y_j, and hence on Y.

We now assume that p and q each have constant rank. For any $y \in Y$, let E_y denote the submultiset of $\mathrm{sp}_\phi(y)$ consisting of those points that lie in U. Fix $y_0 \in Y$, and let δ denote the smallest distance between a point in E_{y_0} and a point in the complement of U. The map $y \mapsto \mathrm{sp}_\phi(y)$ is continuous, from where there is an open neighborhood V of y_0 such that, for each $y \in V$, the submultiset F_y of $\mathrm{sp}_\phi(y)$ consisting of those points that are at distance at most $\delta/2$ from some point in E_{y_0} has the same cardinality as E_{y_0}, and moreover the map $y \mapsto F_y$ is continuous.

Define a continuous projection-valued map $\widetilde{R} : V \to \mathrm{M}_m$ as follows: $\widetilde{R}(y)$ is the image of r under the direct sum of the irreducible direct summands of $\mathrm{ev}_y \circ \phi$ that correspond to the elements of F_y. We have $\widetilde{R}(y) \leqslant R(y)$ for every $y \in V$, and $\widetilde{R}(y_0) = R(y_0)$. Let $z_n \to y_0$. For all n sufficiently large we have $z_n \in V$, from where, for each $\xi \in \mathbb{C}^{\mathrm{rank}(q)}$, we have $\langle R(z_n)\xi,\, \xi\rangle \geqslant \langle \widetilde{R}(z_n)\xi,\, \xi\rangle$. It follows that

$$\liminf_{n\to\infty} \langle R(z_n)\xi,\, \xi\rangle \geqslant \lim_{n\to\infty} \langle \widetilde{R}(z_n)\xi,\, \xi\rangle = \langle \widetilde{R}(y_0)\xi,\, \xi\rangle = \langle R(y_0)\xi,\, \xi\rangle,$$

and so R is lower semicontinuous. □

3. *A word on strategy*

Before plunging headlong into the technical details of our proof, we attempt to explain why a unital simple AH algebra with no dimension growth *ought* to absorb the Jiang–Su algebra tensorially.

Let $p, q \geqslant 2$ be relatively prime integers. Bearing in mind the isomorphism $\mathrm{M}_{pq} \cong \mathrm{M}_p \otimes \mathrm{M}_q$, one defines

$$\mathrm{I}_{p,q} = \big\{ f \in \mathrm{C}([0,1],\, \mathrm{M}_{pq}) : f(0) \in \mathbf{1}_p \otimes \mathrm{M}_q \text{ and } f(1) \in \mathrm{M}_p \otimes \mathbf{1}_q \big\}.$$

The algebra $\mathrm{I}_{p,q}$ is referred to as a *prime dimension drop algebra*, and the Jiang–Su algebra, denoted by $\mathcal{Z}$, is the unique unital simple inductive limit of prime dimension drop algebras with the same K-theory and tracial state space as the algebra of complex numbers (see [**10**]). In order to prove that a unital C*-algebra absorbs the Jiang–Su algebra tensorially, it suffices to prove that for each p, q as above, there is an approximately central sequence of unital $*$-homomorphisms $\gamma_n : \mathrm{I}_{p,q} \to A$ (see [**12**, Proposition 2.2]).

Let $A = \varinjlim_{i\to\infty} (A_i, \phi_i)$ be a unital simple AH algebra with no dimension growth, and assume for simplicity that each A_i is homogeneous with connected spectrum X_i. Fix a finite subset F of A_i. It is known that for any $\epsilon > 0$ there exists $j > i$ such that for every $y \in X_j$, the finite-dimensional representation $\mathrm{ev}_y \circ \phi_{i,j}$ of A_i has the following property: the multiset $\mathrm{sp}_{\phi_{i,j}}(y)$ can be partitioned into submultisets $S_1, \ldots, S_m$ such that all of the elements in a fixed S_t lie in a ball of radius at most ϵ, and such that each S_t has large cardinality relative to $\dim(X_j)$. Suppose that $S_t = \{\{x_1, \ldots, x_k\}\}$. (The notation is as in Subsection 2.3.) The projections $\mathrm{ev}_{x_1}(\mathbf{1}_{A_i}), \ldots, \mathrm{ev}_{x_k}(\mathbf{1}_{A_i})$ (whose sum is denoted by I_t) are pairwise orthogonal and Murray–von Neumann equivalent, and so they and the partial isometries implementing the said equivalences generate a copy of M_k, which almost commutes with the image of F under the map $I_t(\mathrm{ev}_y \circ \phi_{i,j})I_t$. If k is large enough, then there is a unital $*$-homomorphism from $\mathrm{I}_{p,q}$ into M_k, which almost commutes with the image of F. Repeating this procedure for each of $S_1, \ldots, S_m$, we obtain a unital $*$-homomorphism from $\mathrm{I}_{p,q}$ into the fiber $\mathrm{M}_{\mathrm{rank}(\mathbf{1}_{A_j})}$ of A_j over $y \in X_j$, which almost commutes with the image of F. By the semiprojectivity of $\mathrm{I}_{p,q}$, this $*$-homomorphism can be extended to have codomain equal to the restriction of A_j to a closed neighborhood of y. Thus, it is straightforward to see the existence of the required maps γ_n in a 'local' sense. This article handles the passage from local to global. What makes this possible is the fact [**4**] that the homotopy groups of the space of k-dimensional representations of $\mathrm{I}_{p,q}$ vanish in low dimensions.

4. *Excising point evaluations*

Let A be a unital simple AH algebra with slow dimension growth. We say that an AH sequence (A_i, ϕ_i) with limit A *realizes slow dimension growth* if

$$\lim_{i\to\infty} \max_{1\leqslant l\leqslant n_i} \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} = 0$$

and $\dim(X_{i,l})$ is finite for all i and l. (The second condition can always be arranged by dropping terms of the sequence.) Assume that (A_i, ϕ_i) is such a sequence. Our aim in this section is to prove that for each finite subset F of A_i, there is some $j > i$ with the property that the bonding map $\phi_{i,j}$ is 'almost' a direct sum of a suitably dense family of irreducible representations of A_i together with a second map $\overline{\phi_{i,j}}$.

NOTATION 4.1. Let X be a compact metric space and $N \geqslant 1$ an integer. Let $U_1, \ldots, U_m$ be open subsets of X whose closures are pairwise disjoint. The C*-subalgebra of $A = \mathrm{M}_N(\mathrm{C}(X))$ consisting of those functions $f : X \to \mathrm{M}_N$ that are constant on each U_s is denoted by $A_{\{U_1,\ldots,U_m\}}$. It is easily verified that $A_{\{U_1,\ldots,U_m\}} \cong \mathrm{M}_N(\mathrm{C}(X'))$, where X' is the quotient of X obtained by shrinking each set $\overline{U_s}$ to a distinct point w_s for $s = 1, \ldots, m$.

If $\rho : A \to B$ is a $*$-homomorphism, then we write $L \cdot \rho$ for a $*$-homomorphism $A \to \mathrm{M}_L(B)$ that is unitarily equivalent to the direct sum of L copies of ρ.

LEMMA 4.2. *Let X and Y be compact metric spaces and let*

$$\gamma : A = \mathrm{M}_N(\mathrm{C}(X)) \longrightarrow q(\mathrm{C}(Y) \otimes \mathcal{K})q$$

be a unital $*$-homomorphism. Let $U_1, \ldots, U_m$ be open subsets of X whose closures are pairwise disjoint, and let $w_1, \ldots, w_m$ be as in Notation 4.1. Suppose that for all $s = 1, \ldots, m$ and all $y \in Y$,

$$\mathrm{card}\big(\mathrm{sp}_\gamma(y) \cap U_s\big) \geqslant (K + 2)\dim(Y).$$

Then the restriction of γ to $A_{\{U_1,\ldots,U_m\}}$ decomposes as a direct sum $\overline{\gamma} \oplus L \cdot \rho$, where $L \geqslant K\dim(Y)$ and ρ is a $*$-homomorphism with finite-dimensional image and spectrum equal to $\{w_1, \ldots, w_m\}$ in the sense of Subsection 2.3.

Proof. We may clearly assume that $\dim(Y) \neq 0$. We may also assume that $q \in \mathrm{M}_R(\mathrm{C}(Y))$ for some $R \geqslant 1$. Fix a system of matrix units $(p_{cd})_{c,d=1}^N$ for M_N. For each $y \in Y$, let $q_{cd}^{(s)}(y)$ be the image of p_{cd} under the direct sum of all the irreducible direct summands of $\mathrm{ev}_y \circ \gamma$ that correspond to points in U_s. Using Lemma 2.3, we see that $q_{11}^{(s)}$ is a lower semicontinuous projection-valued map on Y whose rank is at least $(K+2)\dim(Y)$ at every point.

Apply Proposition 2.1 and Remark 2.2 to find a continuous constant rank subprojection $r_{11}^{(s)} : Y \to \mathrm{M}_R$ of $q_{11}^{(s)}$ whose rank L is at least $K\dim(Y)$ and which corresponds to a trivial vector bundle over Y. Since $r_{11}^{(s)}(y) \leqslant q(y)$ for all $y \in Y$, it follows that $r_{11}^{(s)} \in q(\mathrm{C}(Y) \otimes \mathcal{K})q$. Set $r_{cd}^{(s)} = q_{c1}^{(s)} r_{11}^{(s)} q_{1d}^{(s)} = \gamma(p_{c1}) r_{11}^{(s)} \gamma(p_{1d})$. It is straightforward to check that $\big(r_{cd}^{(s)}\big)_{c,d=1}^N$ is a system of matrix units in $q(\mathrm{C}(Y) \otimes \mathcal{K})q$. Let I_s denote the unit of the subalgebra of $q(\mathrm{C}(Y) \otimes \mathcal{K})q$ generated by the $r_{cd}^{(s)}$.

To complete the proof of the lemma, it will suffice to show that I_s commutes with the image of $\gamma|_{A_{\{U_1,\ldots,U_m\}}}$ and that, up to unitary equivalence,

$$I_s(\gamma|_{A_{\{U_1,\ldots,U_m\}}})I_s = \bigoplus_{t=1}^{L} \mathrm{ev}_{w_s} = L \cdot \mathrm{ev}_{w_s}.$$

Fix $y \in Y$. Observe that the irreducible direct summands of $\mathrm{ev}_y \circ \gamma$, which correspond to points in U_s are, upon restricting γ to $A_{\{U_1,\ldots,U_s\}}$, replaced by irreducible representations of $A_{\{U_1,\ldots,U_s\}}$ corresponding to the point $w_s \in \mathrm{Spec}\big(A_{\{U_1,\ldots,U_s\}}\big)$. In particular, the image of any $a \in A_{\{U_1,\ldots,U_s\}}$ under these irreducible representations is contained in the linear span of the $q_{cd}^{(s)}(y)$. An easy exercise using the definition of the $r_{cd}^{(s)}$ shows that $I_s q_{cd}^{(s)} = q_{cd}^{(s)} I_s = r_{cd}^{(s)}$. Thus I_s commutes with the image of $\gamma|_{A_{\{U_1,\ldots,U_m\}}}$.

The map $I_s(\gamma|_{A_{\{U_1,\ldots,U_m\}}})I_s$ factors through the evaluation of $A_{\{U_1,\ldots,U_s\}}$ at w_s, and has multiplicity L. To see that this finite-dimensional representation of $A_{\{U_1,\ldots,U_s\}}$ decomposes as the direct sum of L representations of multiplicity one, we observe that $r_{11}^{(s)}$ can be decomposed into the direct sum of L equivalent rank one projections by virtue of its triviality. Let ξ be one such projection. We can form matrix units $\xi_{cd} = q_{c1}^{(s)} \xi q_{1d}^{(s)}$ to obtain an irreducible subrepresentation of $I_s(\gamma|_{A_{\{U_1,\ldots,U_m\}}})I_s$ of multiplicity one. There are L such subrepresentations, and they are mutually orthogonal. This completes the proof of the lemma. □

LEMMA 4.3. *Let A be an infinite-dimensional unital simple AH algebra with slow dimension growth, and let (A_j, ϕ_j) be an AH sequence which realizes the slow dimension growth of A. Suppose that $A_i = \mathrm{M}_N(\mathrm{C}(X_i))$ for some i and let there be given $F \subseteq A_i$ finite, a tolerance $\epsilon > 0$, a natural number K, and distinct points $x_1, \ldots, x_m \in X_i$.*

It follows that there are $j > i$, open neighborhoods U_s of x_s in X_i for $s = 1, \ldots, m$, with pairwise disjoint closures, and a finite set $F' \subseteq A_i' = (A_i)_{\{U_1,\ldots,U_m\}}$ with the following properties for each $k \in \{1, \ldots, n_j\}$.

(i) *We have $F' \approx_\epsilon F$.*

(ii) *Using the notation of Subsection 2.2, the map $\gamma_{j,k} : A_i' \to A_{j,k}$ obtained by restricting $\phi_{i,j}^k$ to A_i' is, up to unitary equivalence inside its codomain, of the form $\overline{\gamma} \oplus L \cdot \rho$, where $L \in \mathbb{N}$, ρ is a $*$-homomorphism with finite-dimensional image and spectrum (in the sense of Subsection 2.3) consisting of the points $w_1, \ldots, w_m$ corresponding to the images of the sets $\overline{U_1}, \ldots, \overline{U_m}$ in the quotient space of X_i representing the spectrum of A_i'.*

(iii) *We have $L \geqslant K \dim(X_{j,k})$.*

Proof. A standard approximation argument shows that part (i) of the conclusion of the lemma is satisfied with an appropriate choice of $U_1, \ldots, U_m$. We show that this choice suffices for the other two parts of the conclusion of the lemma. Set $\gamma_{j,k} = \phi_{i,j}^k$. For $j \geqslant i$, $y \in X_{j,k}$, $k \in \{1, \ldots, n_j\}$, and $s \in \{1, \ldots, m\}$, let $L(j,k,s)(y)$ denote the number of irreducible direct summands that correspond to points in U_s of the finite-dimensional representation $\mathrm{ev}_y \circ \gamma_{j,k}$ of A_i. Thus $L(j,k,s)(y) = \mathrm{card}\big(\mathrm{sp}_{\gamma_{j,k}}(y) \cap U_s\big)$.

Choose positive scalar-valued functions $a_1, \ldots, a_m \in A_i$ such that

$$\{x \in X : a_s(x) \neq 0\} = U_s.$$

By the simplicity of A, there exist $j_0 > i$ and $M \geqslant 1$ such that for each $j \geqslant j_0$, $s \in \{1, \ldots, m\}$, and $k \in \{1, \ldots, n_j\}$, there are elements $b_1, \ldots, b_M$ in $A_{j,k}$ such that $\sum_{t=1}^{M} b_t \gamma_{j,k}(a_s) b_t^* = p_{j,k}$. It follows that for each $y \in X_{j,k}$, we have

$$M \cdot N \cdot L(j,k,s)(y) = M \cdot \mathrm{rank}(\gamma_{j,k}(a_s)(y)) \geqslant \mathrm{rank}(p_{j,k}(y))$$

and hence

$$\frac{\dim(X_{j,k})}{L(j,k,s)(y)} \leqslant \frac{\dim(X_{j,k})}{\operatorname{rank}(p_{j,k})} \cdot M \cdot N.$$

By the slow dimension growth condition, if j is large enough, then

$$L(j,k,s)(y) \geqslant (K+2)\dim(X_{j,k})$$

for all s and k, and all $y \in X_{j,k}$. Properties (ii) and (iii) in the conclusion of the present lemma now follow from an application of Lemma 4.2. ☐

5. *Approximate relative commutants*

The main result of this section is Proposition 5.7.

Definition 5.1. Let X be a compact metric space. Let $\epsilon > 0$ and a finite set $F \subseteq \mathrm{C}(X)$ be given. Let R be a finite subset of X. Let B be a unital separable homogeneous C*-algebra (as in Subsection 2.2) with spectrum Y. Let $\gamma : \mathrm{C}(X) \to B$ be a unital $*$-homomorphism. Given an integer $K \geqslant 1$, a *K-large system of compatible local finite-dimensional approximations* for γ with respect to the data ϵ, F, and R consists of two finite closed covers $(W_s)_{s=1}^{M}$ and $(V_s)_{s=1}^{M}$ of Y with $W_s \subseteq \overset{\circ}{V_s}$ such that for each s there is a partition of unity of B over V_s into projections $e(s,i) \in B|_{V_s}$,

$$\sum_{i=1}^{n(s)} e(s,i) = 1_B|_{V_s}$$

with the following properties.

(i) The restrictions of $e(s_1,i_1),\ldots,e(s_m,i_m)$ to any nonempty intersection $V_{s_1} \cap \ldots \cap V_{s_m} \neq \varnothing$ mutually commute and the rank of the product $e(s_1,i_1)\cdot\ldots\cdot e(s_m,i_m)$ is either 0 or at least K at all points of $V_{s_1} \cap \ldots \cap V_{s_m}$.

(ii) For any s there are points $x_1,\ldots,x_{n(s)}$ in R such that for all $f \in F$, with $\|\cdot\|_{V_s}$ denoting the supremum norm of the restriction to V_s, we have

$$\left\| \gamma(f) - \sum_{i=1}^{n(s)} f(x_i) e(s,i) \right\|_{V_s} < \epsilon.$$

This definition has two useful consequences.

Lemma 5.2. *Let the notation be as in Definition* 5.1, *and suppose that a K-large system of compatible local finite-dimensional approximations as there are given. Let $B^\sharp$ be the C*-algebra consisting of those elements $g \in B$ such that, for $s = 1,\ldots,M$, for $i = 1,\ldots,n(s)$, and for every $y \in W_s$, the projection $e(s,i)(y)$ commutes with $g(y)$. Then if $g \in B^\sharp$ and $\|g\| \leqslant 1$, then we have $\|[g,\gamma(f)]\| < 2\epsilon$ for all $f \in F$.*

Proof. Let $f \in F$, let $g \in B^\sharp$ satisfy $\|g\| \leqslant 1$, and let $x \in X$. Choose s such that $x \in W_s$. Let $x_1,\ldots,x_{n(s)} \in R$ be as in Definition 5.1(ii). Then $g(x)$ commutes with $\sum_{i=1}^{n(s)} f(x_i)e(s,i)$ and

$$\left\| \gamma(f) - \sum_{i=1}^{n(s)} f(x_i) e(s,i) \right\|_{V_s} < \epsilon,$$

so $\|[\gamma(f)(x),\, g(x)]\| < 2\epsilon$. Take the supremum over $x \in X$, remembering that X is compact. ☐

Lemma 5.3. *Let the notation be as in Definition* 5.1, *and suppose that a K-large system of compatible local finite-dimensional approximations as there are given. The subalgebra $B^\sharp$ of Lemma* 5.2 *is the section algebra of a continuous field of* C*-*algebras over Y, which we also denote by $B^\sharp$, such that for every $y \in Y$, every irreducible representation of the fiber $B^\sharp(y)$ has dimension at least K.*

Proof. We note that $B^\sharp$ is a $\mathrm{C}(Y)$-subalgebra of the section algebra B of a continuous field, also called B, over Y. Hence $B^\sharp$ is itself the section algebra of a continuous field over Y. For $y \in Y$, let $B(y)$ be the fiber of B over y, and let $\pi_y : B \to B(y)$ be the evaluation map.

Fix $y \in Y$. Let $s_1, \ldots, s_m$ be the indices s such that $y \in W_s$. Then y has a neighborhood $V \subseteq V_{s_1} \cap \ldots \cap V_{s_m}$ such that $V \cap W_s = \varnothing$ for all $s \notin \{s_1, \ldots, s_m\}$. Let S be the set of all elements $a = \{(s_1, i_1), \ldots, (s_m, i_m)\}$ with $1 \leqslant i_l \leqslant n(s_l)$ for $l = 1, \ldots, m$, and for which the projection $e(a)$, defined on $V_{s_1} \cap \ldots \cap V_{s_m}$ by $e(a)(z) = e(s_1, i_1)(z) \cdot \ldots \cdot e(s_m, i_m)(z)$, is nonzero at y. For $a \in S$, Definition 5.1(i) implies that the rank $r(a)$ of $e(a)(y)$ is always at least K. Also, $\sum_{a \in S} e(a)(z) = 1$ for all $z \in V_{s_1} \cap \ldots \cap V_{s_m}$.

Let

$$C = \bigoplus_{a \in S} e(a)(y)B(y)e(a)(y).$$

We claim that $B^\sharp(y) = C$. Since $C \cong \bigoplus_{a \in S} \mathrm{M}_{r(a)}$, this will prove the lemma. It is clear from the commutation relations in the definition of $B^\sharp$ that $B^\sharp \subseteq C$. For the reverse inclusion, let $c \in C$. Choose $b_0 \in B$ such that $\pi_y(b_0) = c$. Choose $f \in \mathrm{C}(Y)$ such that $f(y) = 1$ and $\mathrm{supp}(f) \subseteq V$. Define

$$b(z) = \sum_{a \in S} f(z)e(a)(z)b_0(z)e(a)(z)$$

for $z \in Y$. Then $b \in B^\sharp$ and $\pi_y(b) = c$. □

Definition 5.4. Let X be a compact metric space. For $\delta > 0$ we denote by $r(X, \delta)$ the smallest number r with the property that for every finite set $G \subseteq X$ there are open subsets $U_1, \ldots, U_r$ of X of diameter less than δ whose union contains G and such that their closures are mutually disjoint.

Remark 5.5. Using compactness, the finiteness of $r(X, \delta)$ in Definition 5.4 is unchanged if the metric is replaced by an equivalent metric. One can therefore see that $r(X, \delta) < \infty$ by embedding X homeomorphically in the Hilbert cube and choosing the U_s to be rectangles of the form

$$X \cap \left(\prod_{i \leqslant N} (a_i, b_i) \times \prod_{i > N} [0, 1] \right).$$

If $F \subseteq p\mathrm{M}_N(\mathrm{C}(X))p \subseteq \mathrm{M}_N(\mathrm{C}(X))$ is a subset, we denote by $\omega_X(F, \delta)$ the δ-oscillation of the family F:

$$\omega_X(F, \delta) = \sup\left(\left\{\|f(x) - f(x')\|_{\mathrm{M}_N} : d(x, x') < \delta\right\}\right).$$

We omit X from the notation when no confusion can arise.

Proposition 5.6. *Let X be a compact metric space and let $\gamma : \mathrm{C}(X) \to B$ be a unital $*$-homomorphism to a separable homogeneous C*-algebra B (as in Subsection* 2.2) *with spectrum of dimension d. Let $\delta > 0$ and suppose that γ admits a direct sum decomposition* *of the form $\gamma = \phi \oplus L \cdot \rho$, where ρ is a $*$-homomorphism with finite-dimensional image that has spectrum R (in the sense of Subsection 2.3) which is δ-dense in X, and $L \geqslant \big((r(X,\delta)+1)^{d+1}-1\big)K$. If $F \subseteq \mathrm{C}(X)$ is a finite set, then γ admits a K-large system of compatible local finite-dimensional approximations with respect to the data $\epsilon = 2\omega(F, 3\delta)$, F, and R.*

Proof. Suppose that γ, ϕ, ρ, and L are as in the statement. The cover $(V_s)_{s=1}^M$ and the corresponding partitions of unity are constructed as follows. Set $r = r(X,\delta)$. By the compactness of Y there is a finite open cover $\mathcal{V} = \{V_1, \ldots, V_M\}$ of the spectrum Y of B such that for each s there is a family $U_{(s,1)}, \ldots, U_{(s,r)}$ of open subsets of X of diameter less than δ (some possibly empty), whose union contains $\mathrm{sp}_\phi(y)$ for all $y \in \overline{V_s}$, and such that $\overline{U_{(s,i)}} \cap \overline{U_{(s,j)}} = \varnothing$ for $i \neq j$. By [**11**, Proposition 1.6] and since $\dim(Y) = d$, after replacing $\mathcal{V}$ by a refinement, we may assume that $\{V_1, \ldots, V_M\}$ can be colored with $d+1$ colors such that sets of the same color have disjoint closures. In other words we can write $\{V_1, \ldots, V_M\}$ as a disjoint union $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{d+1}$ such that if $V_s, V_t \in \mathcal{V}_i$ for some $1 \leqslant i \leqslant d+1$, and $s \neq t$, then $\overline{V_s} \cap \overline{V_t} = \varnothing$. For each s, we further replace V_s by $\{y \in Y : \mathrm{dist}(y, V_s) \leqslant \alpha\}$ for a suitable $\alpha > 0$. The sets V_s are now closed, but if α is sufficiently small, then all the other properties of our cover remain valid. In addition, there are now closed sets $W_s \subseteq \overset{\circ}{V_s}$ such that $(W_s)_{s=1}^M$ is a cover of Y. These sets V_s and W_s will be the ones called for in Definition 5.1.

We need to work with the coloring map $s \mapsto \overline{s} : \{1, \ldots, M\} \to \{1, \ldots, d+1\}$, where $\overline{s}$ is defined by the condition that V_s has color $\overline{s}$, that is, V_s is an element of the family $\mathcal{V}_{\overline{s}}$. For $a = \{(s_1, i_1), \ldots, (s_m, i_m)\}$ with $s_1, \ldots, s_m \in \{1, \ldots, M\}$ distinct and $1 \leqslant i_1, \ldots, i_m \leqslant r$, set

$$U_a = U_{(s_1,i_1)} \cap \ldots \cap U_{(s_m,i_m)}.$$

Let S be the set of all a as above such that

$$V_{s_1} \cap \ldots \cap V_{s_m} \neq \varnothing \quad \text{and} \quad U_a \neq \varnothing.$$

(Note that necessarily $m \leqslant d+1$, since distinct sets V_s of the same color are disjoint.) For $a = \{(s_1, i_1), \ldots, (s_m, i_m)\}$ as above, extend the notation $s \mapsto \overline{s}$ by setting

$$\overline{a} = \{(\overline{s}_1, i_1), \ldots, (\overline{s}_m, i_m)\}.$$

Consider also the set $\widehat{S}$ consisting of all sets of the form $\{(\overline{s}_1, i_1), \ldots, (\overline{s}_m, i_m)\}$, where $m \leqslant d+1$, $\overline{s}_1, \ldots, \overline{s}_m$ are mutually distinct elements (colors) in the set $\{1, \ldots, d+1\}$, and $1 \leqslant i_1, \ldots, i_m \leqslant r$. Then $\widehat{S} \supseteq \{\overline{a} : a \in S\}$, but in general the containment is proper. Note that

$$\mathrm{card}\big(\widehat{S}\big) = (d+1)r + \left(\frac{(d+1)d}{2}\right) r^2 + \ldots + r^{d+1} = (1+r)^{d+1} - 1.$$

Set $L_0 = \big((1+r)^{d+1} - 1\big)K$. Replacing ϕ by $\phi \oplus (L - L_0) \cdot \rho$, we may assume that $L = L_0$. Let $q = \rho(1)$. Then we can factor $L \cdot \rho$ as a composite

$$\mathrm{C}(X) \xrightarrow{\sigma} qBq \otimes \mathcal{L}(\ell 2(\widehat{S})) \otimes \mathrm{M}_K \xrightarrow{\mu} B,$$

in which σ is the unital $*$-homomorphism given by

$$\sigma(f) = \rho(f) \otimes 1 \otimes \mathbf{1}_K$$

for $f \in \mathrm{C}(X)$, and μ is an isomorphism of $qBq \otimes \mathcal{L}(\ell^2(\widehat{S})) \otimes \mathrm{M}_K$ with $(1 - \phi(1))B(1 - \phi(1))$. In particular, $\mu(1) = 1 - \phi(1)$.

By assumption, there are a positive integer c, mutually orthogonal nonzero projections $q_1, \ldots, q_c$ such that $q_1 + \ldots + q_c = q$, and $x_1, \ldots, x_c \in X$, such that $\rho : \mathrm{C}(X) \to qBq$ has the form

$$\rho(f) = \sum_{k=1}^{c} f(x_k) q_k$$

for all $f \in \mathrm{C}(X)$, and such that for any $x \in X$ there is k with $d(x, x_k) < \delta$. Therefore there is a map $\kappa : S \to \{1, \ldots, c\}$ with the property that

$$\operatorname{dist}(x_{\kappa(a)}, U_a) < \delta. \tag{1}$$

For each fixed s we are going to define a partition of $1_B|_{V_s}$. For $i = 1, \ldots, r$, let $h_{(s,i)}$ be an element of $\mathrm{C}(X)$ such that $h_{(s,i)}(x) = 1$ for all $x \in U_{(s,i)}$ and such that $h_{(s,i)}(x) = 0$ on $\bigcup_{j \neq i} U_{(s,j)}$. Define projections $p(s,i)$ on V_s by $p(s,i) = \phi\big(h_{(s,i)}\big)|_{V_s}$ for $i = 1, \ldots, r$. For each (s,i) let $S(s,i)$ be the subset of those elements $a \in S$ with the property that $(s,i) \in a$. Let $T = \{1, \ldots, c\} \times \widehat{S}$ and set

$$T(s,i) = \{(\kappa(a), \overline{a}) : a \in S(s,i)\} = \{(\kappa(a), \overline{a}) : (s,i) \in a\} \subseteq T.$$

Let $\xi(s, r+1), \ldots, \xi(s, n(s))$ be an enumeration of the complement of $\bigcup_{i=1}^{r} T(s,i)$ in T. If this complement is nonempty, then $n(s) > r$; otherwise set $n(s) = r$. Set $T(s,i) = \{\xi(s,i)\}$ for $i = r+1, \ldots, n(s)$ and we observe that for each s the family $(T(s,i))_{i=1}^{n(s)}$ forms a partition of T. Indeed, for $1 \leqslant i \neq j \leqslant r$, we have $T(s,i) \cap T(s,j) = \varnothing$ since if $a = \{(s_1, i_1), \ldots, (s_m, i_m)\} \in S$, then $V_{s_1} \cap \ldots \cap V_{s_m} \neq \varnothing$, and therefore the colors $\overline{s}_1, \ldots, \overline{s}_r$ are mutually distinct.

After this preparation, and recalling that $q_1, \ldots, q_c$ are the spectral projections of ρ, for each $(s,i) \in T$ we define a projection

$$q(s,i) = \sum_{(k,b) \in T(s,i)} \mu\big(q_k \otimes \chi_{\{b\}} \otimes \mathbf{1}_K\big) \in (1 - \phi(1))B(1 - \phi(1)).$$

We also define a family of projections $\big(e(s,i)\big)_{i=1}^{n(s)}$ on V_s by

$$e(s,i) = \begin{cases} p(s,i) + q(s,i)|_{V_s} & \text{if } 1 \leqslant i \leqslant r, \\ q(s,i)|_{V_s} & \text{if } r < i \leqslant n(s). \end{cases}$$

Then we have

$$\sum_{i=1}^{n(s)} e(s,i) = 1_B|_{V_s}$$

is a partition of unity on V_s. Indeed $\sum_{i=1}^{r} p(s,i) = \sum_{i=1}^{r} \phi(h_{(s,i)}) = \phi(1)|_{V_s}$ and $\sum_{i=1}^{n(s)} q(s,i) = \mu(1) = 1 - \phi(1)$ since $(T(s,i))_{i=1}^{n(s)}$ is a partition of T. Note that if $U_{(s,i)} = \varnothing$, then $S(s,i) = T(s,i) = \varnothing$ and $p(s,i) = q(s,i) = 0$.

It remains to verify the properties (i) and (ii) of Definition 5.1.

We begin with condition (i). For convenience, define $p(s,i) = 0$ whenever $i > r$. Thus, $e(s,i) = p(s,i) + q(s,i)$ for all s and i. Also,

$$q(s,i) \leqslant \mu(1) \quad \text{and} \quad p(s,i) \leqslant \phi(1)|_{V_s} = (1 - \mu(1))|_{V_s} \tag{2}$$

for all s and i. Fix $s_1, \ldots, s_m$ such that $V_{s_1} \cap \ldots \cap V_{s_m} \neq \varnothing$. By construction, the projections $e(s_1, i_1), \ldots, e(s_m, i_m)$ commute on $V_{s_1} \cap \ldots \cap V_{s_m}$. It follows from equation (2) that

$$e(s_1, i_1) \cdot \ldots \cdot e(s_m, i_m) = p(s_1, i_1) \cdot \ldots \cdot p(s_m, i_m) + q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m).$$

We now claim that, first, $q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m)$ is either zero or has rank at least K, and, second, if $p(s_1, i_1) \cdot \ldots \cdot p(s_m, i_m) \neq 0$, then $q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m) \neq 0$. Condition (i) of Definition 5.1 will follow immediately.

For the first part of the claim, suppose that $q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m) \neq 0$. Then there are k and b such that $(k, b) \in T(s_1, i_1) \cap \ldots \cap T(s_m, i_m)$. Therefore

$$q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m) \geqslant \mu(q_k \otimes \chi_{\{b\}} \otimes \mathbf{1}_K),$$

and the right-hand side has rank at least K. For the second part of the claim, suppose that $p(s_1, i_1) \cdot \ldots \cdot p(s_m, i_m) \neq 0$. Then, in particular, $U_{(s_1, i_1)} \cap \ldots \cap U_{(s_m, i_m)} \neq \varnothing$. Thus

$a = \{(s_1, i_1), \ldots, (s_m, i_m)\} \in S$. Set $k = \kappa(a)$ and $b = \overline{a}$. Then for $l = 1, \ldots, m$, we have

$$q(s_l, i_l) \geqslant \mu(q_k \otimes \chi_{\{b\}} \otimes \mathbf{1}_K),$$

hence $q(s_1, i_1) \cdot \ldots \cdot q(s_m, i_m) \neq 0$.

Let us now verify condition (ii) of Definition 5.1 for a fixed s. The number $\kappa(\{(s,i)\})$, which we write hence forth as $\kappa(s,i)$, was defined whenever $\{(s,i)\} \in S$. It is convenient to extend this notation as follows. If $1 \leqslant i \leqslant r$ but $\{(s,i)\} \notin S$ (this happens if $U_{(s,i)} = \varnothing$), then set $\kappa(s,i) = 1$. If $i > r$, then we let $\kappa(s,i)$ denote the first coordinate of $\xi(s,i)$, which is in $\{1, \ldots, c\}$. We are going to show that if $f \in F$, then

$$\left\| \gamma(f) - \sum_{i=1}^{n(s)} f(x_{\kappa(s,i)}) e(s,i) \right\|_{V_s} \leqslant \frac{\epsilon}{2}. \tag{3}$$

Define $\phi'_s(f)(y) = \sum_{i=1}^{r} f(x_{\kappa(s,i)}) p(s,i)(y)$ for $y \in V_s$ and

$$\sigma'_s(f) = \sum_{i=1}^{r} f(x_{\kappa(s,i)}) q(s,i) + \sum_{i=r+1}^{n(s)} f(x_{\kappa(s,i)}) e(s,i).$$

For $y \in V_s$, recall that $\phi(f)(y)$ depends only on the restriction of f to $\bigcup_{i=1}^{r} U_{(s,i)}$. Since $\operatorname{dist}\bigl(x_{\kappa(s,i)}, U_{(s,i)}\bigr) < \delta$ for $1 \leqslant i \leqslant r$ and $U_{(s,i)} \neq \varnothing$ (and $q(s,i) = 0$ when $U_{(s,i)} = \varnothing$), and since $|f(x) - f(x')| \leqslant \epsilon/2$ if $d(x, x') < 3\delta$ and $f \in F$, it follows that

$$\|\phi(f) - \phi'_s(f)\|_{V_s} \leqslant \epsilon/2$$

for all $f \in F$. Since T is partitioned into the sets $T(s,i)$ for $i = 1, \ldots, r$ and $\{\xi(s,i)\}$ for $i = r+1, \ldots, n(s)$, we can write

$$\mu \circ \sigma(f) = \sum_{i=1}^{r} \sum_{(k,b) \in T(s,i)} f(x_k) \mu\bigl(q_k \otimes \chi_{\{b\}} \otimes \mathbf{1}_K\bigr) + \sum_{i=r+1}^{n(s)} f(x_{\kappa(s,i)}) e(s,i). \tag{4}$$

Note that if $(k,b) \in T(s,i)$ for $1 \leqslant i \leqslant r$, then $(k,b) = (\kappa(a), \overline{a})$ for some $a \in S(s,i)$. On the other hand if $a \in S(s,i)$, then we see that $d(x_{\kappa(a)}, x_{\kappa(s,i)}) < 3\delta$ using the inequality (1), the inclusion $U_a \subseteq U_{(s,i)}$, and the fact that $U_{(s,i)}$ has diameter less than δ. Since $q(s,i) = \sum_{(k,b) \in T(s,i)} \mu\bigl(q_k \otimes \chi_{\{b\}} \otimes \mathbf{1}_K\bigr)$, equation (4) leads to

$$\|\mu \circ \sigma(f) - \sigma'_s(f)\| \leqslant \epsilon/2$$

for all $f \in F$. We set $\gamma'_s = \phi'_s \oplus \sigma'_s$. Thus $\gamma'_s(f) = \sum_{i=1}^{n(s)} f(x_{\kappa(s,i)}) e(s,i)$. Recalling that $\gamma = \phi \oplus \mu \circ \sigma$, we then obtain

$$\|\gamma(f) - \gamma'_s(f)\|_{V_s} \leqslant \epsilon/2$$

for all $f \in F$. This completes the proof of the inequality (3). □

Proposition 5.7. *Given relatively prime positive integers p_1 and p_2, there is an integer $\ell \geqslant 1$ with the following property. Let X be a compact metric space and let $\gamma : \mathrm{M}_N(\mathrm{C}(X)) \to B$ be a unital $*$-homomorphism to a separable homogeneous C^*-algebra B (as in Subsection 2.2) with spectrum of dimension $d < \infty$. Let $\delta > 0$ and suppose that γ decomposes as a direct sum $\gamma = \phi \oplus L^{d+2} \cdot \rho$, where ρ is a $*$-homomorphism with finite-dimensional image that has spectrum R in the sense of Subsection 2.3, which is δ-dense in X and such that $L \geqslant r(X, \delta) + 1 + \ell$. Let $F \subseteq \mathrm{M}_N(\mathrm{C}(X))$ be finite. Then there is a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to B$ such that $\|[\eta(g), \gamma(f)]\| \leqslant 4N^2 \omega(F, 3\delta)$ for all $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$ and for all $f \in F$.*

Proof. In the first part of the proof, we consider the case $N = 1$. By [**4**, Theorem 6.2] there is ℓ depending only on p_1 and p_2 such that whenever D is a separable recursive subhomogeneous algebra of topological dimension d and minimum matrix size at least $\ell(d+1)$, then there is a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to D$.

Given $\delta > 0$, set $r = r(X, \delta)$, and suppose that γ is as in the statement with $L \geqslant r + 1 + \ell$. Set $K = \ell(d+1)$. Then we have

$$L^{d+2} \geqslant (d+2)(r+1)^{d+1}\ell \geqslant \big((r+1)^{d+1} - 1\big)K.$$

By Proposition 5.6, γ admits a K-large system of compatible local finite-dimensional approximations with respect to the data $2\omega(F, 3\delta)$, F, and R. Let $B^\sharp$ be the corresponding commutant C*-algebra as in Lemma 5.2. Lemma 5.3 implies that $B^\sharp$ is the section algebra of a unital separable continuous field with fibers finite-dimensional C*-algebras whose direct summands all have matrix size at least K. It follows from [**2**, Theorem 4.6] that there is a recursive subhomogeneous algebra D of topological dimension d and minimum matrix size at least $K = \ell(d+1)$ such that $D \subseteq B^\sharp$. By the choice of ℓ using [**4**, Theorem 6.2] there is a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to D$. By Lemma 5.2 we conclude that $\|[\eta(g), \gamma(f)]\| \leqslant 4\omega(F, 3\delta)$ for all $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$ and for all $f \in F$.

Consider now the general case with $\gamma : \mathrm{M}_N(\mathrm{C}(X)) \to B$. Let ℓ be as above and let $\delta > 0$. Let F_0 be the set of matrix entries of elements of F. Then F_0 is finite and $\omega(F_0, 3\delta) \leqslant \omega(F, 3\delta)$. Let D be the commutant of $\gamma(\mathrm{M}_N)$ in B. Then D is a homogeneous C*-algebra with spectrum Y and we can identify γ with

$$\mathrm{id}_N \otimes \gamma_0 : \mathrm{M}_N \otimes \mathrm{C}(X) \longrightarrow \mathrm{M}_N \otimes D$$

for some unital $*$-homomorphism $\gamma_0 : \mathrm{C}(X) \to D$. Moreover γ_0 can be written as a direct sum $\gamma_0 = \phi_0 \oplus L^{d+2} \cdot \rho_0$, where ρ_0 is a $*$-homomorphism with finite-dimensional image and spectrum R. By the first part of the proof there is a unital $*$-homomorphism $\eta_0 : \mathrm{I}_{p_1,p_2} \to D$ such that $\|[\eta_0(g), \gamma_0(f)]\| \leqslant 4\omega(F_0, 3\delta)$ for all $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$ and for all $f \in F_0$. Set $\eta = \mathbf{1}_N \otimes \eta_0$. Now let $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$ and let $f \in F$. Write $f = (f_{jk})_{j,k=1}^N$ with $f_{jk} \in F_0$. Then $\gamma(f) = \big(\gamma_0(f_{jk})\big)_{j,k=1}^N$ and

$$\begin{aligned}\|[\eta(g),\, \gamma(f)]\| &= \big\|\big([\eta_0(g),\, \gamma_0(f_{jk})]\big)_{j,k=1}^N\big\| \leqslant \sum_{j,k=1}^N \|[\eta_0(g),\, \gamma_0(f_{jk})]\| \\ &\leqslant N^2 \cdot 4\omega(F_0, 3\delta) \leqslant 4N^2\omega(F, 3\delta).\end{aligned}$$

This completes the proof. □

Let (X, d) be a compact metric space and let $V_1, \ldots, V_m$ be disjoint closed subsets of X. Let (X', d') be the compact metric space obtained by shrinking each V_s to a point w_s. Let $\pi : X \to X'$ be the quotient map. The induced metric d' on X' is given by

$$d'(\pi(x), \pi(y)) = \inf(\{d(x_1, y_1) + \ldots + d(x_n, y_n)\}),$$

where the infimum is taken over all finite sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ with $\pi(x) = \pi(x_1), \pi(y) = \pi(y_n)$, and $\pi(y_i) = \pi(x_{i+1})$ for $i = 1, \ldots, n-1$.

Lemma 5.8. *Let $A = \varinjlim_{i\to\infty}(A_i, \phi_i)$, where each A_i is semihomogeneous* (*as in Subsection* 2.2) *and assume that there is $d \geqslant 0$ such that $\dim(\mathrm{Spec}(A_i)) \leqslant d$ for every $i \in \mathbb{N}$. Assume further that $A_1 = \mathrm{M}_N(\mathrm{C}(X))$. For any finite subset $F \subseteq A_1$, any $\epsilon > 0$, and any relatively prime integers $p_1, p_2 \geqslant 2$, there are $j > 1$ and a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to A_j$ such that $\|[\eta(g),\, \phi_{1,j}(f)]\| \leqslant \epsilon$ for all $f \in F$ and $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$.*

Proof. Given F and ϵ, set $\epsilon_0 = \epsilon/(18N^2)$, and choose and fix $\delta > 0$ small enough so that $\omega_X(F, 4\delta) < \epsilon_0$. Let $\{x_1, \ldots, x_m\}$ be a δ-dense subset of X and let $U_1, \ldots, U_m$ be open sets in

X with disjoint closures and such that $x_s \in U_s$ for $s = 1, \ldots, m$. We may assume that these sets are sufficiently small that there is a finite subset F' of $\mathrm{M}_N(\mathrm{C}(X))$ such that each $f \in F'$ is constant on each U_s and $F' \approx_{\epsilon_0} F$. Moreover, by replacing the sets U_s by even smaller sets, we may arrange that if (X', d') denotes the metric space obtained by shrinking each $\overline{U_s}$ to a point w_s, and $\pi : X \to X'$ denotes the quotient map, then $d(x, y) < 4\delta$ whenever $x, y \in X$ satisfy $d'(\pi(x), \pi(y)) < 3\delta$. Therefore

$$\omega_{X'}(F', 3\delta) \leqslant \omega_X(F', 4\delta) \leqslant \omega_X(F, 4\delta) + 2\epsilon_0 < 3\epsilon_0.$$

Let ℓ be given by Proposition 5.7. Set $L = r(X', \delta) + 1 + \ell$. Apply Lemma 4.3 with $K = L^{d+2}$, and use the notation of Subsection 2.2. We find $j > 1$ such that for each k there are $*$-homomorphisms $\overline{\gamma}, \rho : \mathrm{M}_N(\mathrm{C}(X')) \to A_{j,k}$ such that ρ has finite-dimensional image and spectrum (in the sense of Subsection 2.3) the set $\{w_1, \ldots, w_m\}$, and such that the map $\gamma : \mathrm{M}_N(\mathrm{C}(X')) \to A_{j,k}$ obtained by restricting $\phi_{1,j}^k$ to $\mathrm{M}_N(\mathrm{C}(X'))$ decomposes as $\overline{\gamma} \oplus L^{d+2} \cdot \rho$.

It suffices to construct for each k separately a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to A_{j,k}$ such that $\big\| \big[\eta(g),\, \phi_{1,j}^k(f)\big] \big\| \leqslant \epsilon$ for all $f \in F$ and $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$. The set $\{w_1, \ldots, w_m\}$ is δ-dense in X'. Applying Proposition 5.7 we obtain a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to A_j \subseteq A$ such that $g \in \mathrm{I}_{p_1,p_2}$ with $\|g\| \leqslant 1$ and for all $f' \in F'$, we have

$$\big\| \big[\eta(g),\, \phi_{1,j}^k(f')\big] \big\| \leqslant 4N^2 \omega_{X'}(F', 3\delta) < 12N^2\epsilon_0 = 2\epsilon/3.$$

Since $F' \approx_{\epsilon_0} F$ and $\epsilon_0 \leqslant \epsilon/6$, we conclude that this η satisfies the conclusion of the lemma. $\square$

6. *The main result*

Recall that $\mathcal{Z}$ denotes the Jiang–Su algebra.

THEOREM 6.1. *If A is an infinite-dimensional unital simple AH algebra with no dimension growth (as in Subsection* 2.2*), then $A \cong A \otimes \mathcal{Z}$.*

Proof. To prove $\mathcal{Z}$-stability for A, by [**12**, Proposition 2.2] it suffices to prove that for each pair of relatively prime positive integers $p_1, p_2 \geqslant 2$ there is an approximately central sequence of unital $*$-homomorphisms $\gamma_n : \mathrm{I}_{p_1,p_2} \to A$. In other words, for every $\epsilon > 0$, finite subset F of A, integers p_1, p_2 as above, and finite generating set G for I_{p_1,p_2} consisting of elements of norm at most one, it will suffice to find a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to A$ such that $\|[\eta(g), f]\| \leqslant \epsilon$ for all $g \in G$ and $f \in F$.

By assumption, $A = \varinjlim_{i\to\infty}(A_i, \phi_i)$, where each A_i is semihomogeneous. There is moreover a $d \geqslant 0$ such that $\dim(\mathrm{Spec}(A_i)) \leqslant d$ for every $i \in \mathbb{N}$. Since $\bigcup_i \phi_{i,\infty}(A_i)$ is dense in A, we may assume that F is the image of a finite subset of some A_i; relabeling, we simply assume that $F \subseteq A_1$. Let us observe that A_1 is of the form $p_0 \mathrm{M}_N(\mathrm{C}(X))p_0$ with X not necessarily connected and for some projection p_0 such that $p_0(x) \neq 0$ for all $x \in X$.

To prove the theorem, it will suffice to find $j > 1$ and a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to A_j$ such that $\|[\eta(g), \phi_{1,j}(f)]\| \leqslant \epsilon$ for every $g \in G$ and $f \in F$. We may assume that $\|f\| \leqslant 1$ for all $f \in F$. Since I_{p_1,p_2} is semiprojective (see [**10**]), there is $\epsilon_0 > 0$ smaller than $\epsilon/3$ such that for any unital linear map $\mu : \mathrm{I}_{p_1,p_2} \to B$, which satisfies $\|\mu(gh) - \mu(g)\mu(h)\| \leqslant \epsilon_0$ for all $g, h \in G$, there is a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to B$ with $\|\eta(g) - \mu(g)\| \leqslant \epsilon/3$ for all $g \in G$. Set $F_0 = F \cup \{p_0\} \subseteq \mathrm{M}_N(\mathrm{C}(X))$.

Set $\gamma = \phi_{1,2} : p_0 \mathrm{M}_N(\mathrm{C}(X))p_0 = A_1 \to A_2$. We are going to show that there are m and a commutative diagram

$$\begin{tikzcd} p_0\mathrm{M}_N(\mathrm{C}(X))p_0 \arrow[r, "\gamma"] \arrow[d] & A_2 \arrow[d] \\ \mathrm{M}_N(\mathrm{C}(X)) \arrow[r, "\gamma_0"] & q\mathrm{M}_m(A_2)q\,, \end{tikzcd}$$

where γ_0 is a unital $*$-homomorphism and the vertical arrows are inclusions of full corners. Set $D = \mathrm{M}_N(\mathrm{C}(X))$. Choose m so large that there is a partial isometry $w \in \mathrm{M}_m(D)$ such that

$$w^* w = \mathrm{diag}(1_D - p_0, 0, \ldots, 0) \quad \text{and} \quad ww^* \leqslant \mathrm{diag}(0, p_0, \ldots, p_0).$$

Set $p = \mathrm{diag}(p_0, p_0, \ldots, p_0) \in \mathrm{M}_m(D)$. Then $v = \mathrm{diag}(p_0, 0, \ldots, 0) + w \in \mathrm{M}_m(D)$ is a partial isometry such that

$$v^* v = \mathrm{diag}(1_D, 0, \ldots, 0) \quad \text{and} \quad vv^* \leqslant p.$$

Define $\iota : D \to p\mathrm{M}_m(D)p = \mathrm{M}_m(p_0 D p_0)$ by $\iota(a) = vav^*$. Then $\mathrm{id}_m \otimes \gamma : \mathrm{M}_m(p_0 D p_0) \to \mathrm{M}_m(A_2)$ has the property that $(\mathrm{id}_m \otimes \gamma) \circ \iota : D \to \mathrm{M}_m(A_2)$ satisfies

$$(\mathrm{id}_m \otimes \gamma) \circ \iota(p_0 a p_0) = \mathrm{diag}(\gamma(p_0 a p_0), 0, \ldots, 0).$$

We set $q = (\mathrm{id}_m \otimes \gamma) \circ \iota(1_D)$ and define $\gamma_0 = (\mathrm{id}_m \otimes \gamma) \circ \iota : D \to q\mathrm{M}_m(A_2)q$.

For $j \geqslant 2$ set $q_j = (\mathrm{id}_m \otimes \phi_{2,j})(q)$. Then define $\psi_1 = \gamma_0$ and $\psi_j = (\mathrm{id}_m \otimes \phi_j)|_{q_j \mathrm{M}_m(A_j) q_j}$ for $j \geqslant 2$. We apply Lemma 5.8 to the AH sequence

$$\mathrm{M}_N(\mathrm{C}(X)) \xrightarrow{\psi_1} q\mathrm{M}_m(A_2)q \xrightarrow{\psi_2} q_3\mathrm{M}_m(A_3)q_3 \xrightarrow{\psi_3} q_4\mathrm{M}_m(A_4)q_4 \longrightarrow \cdots.$$

We obtain $j \geqslant 2$ and a unital $*$-homomorphism $\eta_0 : \mathrm{I}_{p_1,p_2} \to q_j \mathrm{M}_m(A_j) q_j$ such that (using the obvious analog of the notation of Subsection 2.2) we have $\|[\eta_0(g),\, \psi_{1,j}(f)]\| \leqslant \epsilon_0$ for all $g \in G$ and $f \in F \cup \{p\}$. Set $e = \psi_{1,j}(p) = 1_{A_j}$ and observe that the map $e\eta_0(\cdot)e$ is ϵ_0-multiplicative on G. Therefore, by the choice of ϵ_0 using the semiprojectivity of I_{p_1,p_2}, there is a unital $*$-homomorphism $\eta : \mathrm{I}_{p_1,p_2} \to e\mathrm{M}_m(A_j)e = A_j$ such that $\|\eta(g) - e\eta_0(g)e\| \leqslant \epsilon/3$ for all $g \in G$. Since $\epsilon_0 < \epsilon/3$, since F is normalized, and since for $f \in F$ we have $\phi_{1,j}(f) = \psi_{1,j}(f) = e\psi_{1,j}(f)e$, it follows that

$$\|[\eta(g),\, \phi_{1,j}(f)]\| \leqslant \|[\eta(g) - e\eta_0(g)e,\, \phi_{1,j}(f)]\| + \|[e\eta_0(g)e,\, \psi_{1,j}(f)]\| \leqslant 2\epsilon/3 + \epsilon_0 \leqslant \epsilon$$

for all $g \in G$ and $f \in F$. □

Remark 6.2. The no dimension growth hypothesis of Theorem 6.1 can be weakened somewhat. We say that a unital simple AH algebra A has *exponentially slow dimension growth* if for any constant $L > 1$ there is an AH sequence (A_i, ϕ_i) with limit A satisfying

$$\liminf_{j \to \infty} \max_{1 \leqslant t \leqslant n_j} \frac{L^{\dim(X_{j,k})}}{\mathrm{rank}(\mathbf{1}_{A_{j,t}})} = 0.$$

If one replaces the slow dimension growth hypothesis of Lemma 4.3 with the stronger condition of exponentially slow dimension growth, then one can replace the quantity $K\dim(X_{j,k})$ in conclusion (3) with $K(L')^{\dim(X_{j,k})+2}$ for any constant $L' > 1$. (In the proof, one replaces the numerators equal to $\dim(X_{j,k})$ with $(L')^{\dim(X_{j,k})+2}$.) One can then use exponentially slow dimension growth instead of slow dimension growth in Lemma 5.8; the latter hypothesis is only required for an application of Lemma 4.3. The proof of Theorem 6.1 then goes through as written, with the weakened assumption of exponentially slow dimension growth for A.

There are examples of unital simple AH algebras that have exponentially slow dimension growth but for which it is not known how to prove bounded dimension growth without the classification theory of AH algebras: the proof of [**13**, Proposition 5.2] shows that the so-called Villadsen algebras of the first type have exponentially slow dimension growth whenever they have slow dimension growth.

Acknowledgement. This work was carried out during the Fields Institute Thematic Program on Operator Algebras in the fall of 2007. The authors are grateful to that institution for its support.

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