The Isoperimetric Property and Lévy Processes: why are finite dimensional distributions useful?

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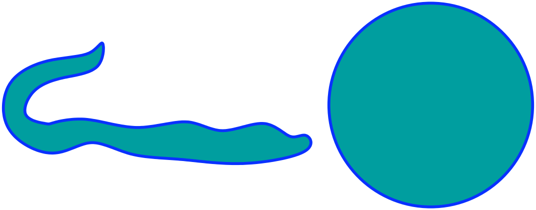
April 28, 2009

[1]Supported in part by NSF

The Classical Isoperimetric Problem

The Greek Philosopher, Proclus, wrote in the fifth century: “The circle (disk), is the first, the most simple, and the most perfect figure." The “perfect" symmetry of the disk justifies this statement as does the deep property discovered by **Queen Dido**, a Phoenician princess from the city of Tyre, shortly after her arrival in Africa in 900 B.C.

Dido’s property: Amongst all regions of fixed equal area, the disk has the smallest perimeter.



“Mathematically": $A(D) =$ Area, $L(\partial D) =$ perimeter. Then

$$A(D) \leq \frac{1}{4\pi} L^2(\partial D),$$

For disk,

$$\frac{1}{4\pi} = \frac{A(disk)}{L^2(circle)} = \frac{\pi r^2}{(2\pi r)^2}$$

The ratio of the area over the perimeter squared is maximized by a disk.

$$\frac{A(D)}{L^2(\partial D)} \leq \frac{A(disk)}{L^2(circle)}$$

If $D^* =$ **disk of same area as** D

$$L(\partial D^*) \leq L(\partial D)$$

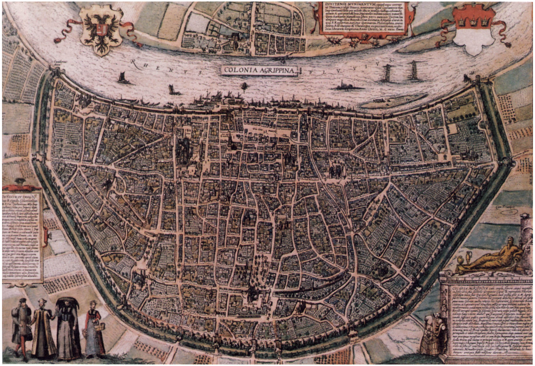
Equality if and only if $D = D^*$**.**

Dual formulation: Amongst all figures of equal perimeter, the circle encloses the largest area.

Dido upon arrival in Africa

Dido Purchased land from King Jarvas of Numidia. After some negotiations an agreement was reached. The Queen could only have as much land as she could enclose by the hide of an ox.
Dido had her people cut the hide of an ox into thin strips and had them enclosed a maximal region. In her case this would have been a semicircle as the city of Carthage was built on the shore.

Medieval map of Paris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Other Classical Isoperimetric Properties

- Amongst all “drums" of equal area the circular one has the smallest bass note:

$$\lambda_1(D^*) \leq \lambda_1(D) \quad \textbf{(The Faber-Krahn Theorem)}$$

- Amongst all sets of equal area, the Electrostatic Capacity is minimized by the disc:

$$Cap(A^*) \leq Cap(A) \quad \textbf{(Pólya–Szegö)}$$

- Exit times for Brown motion from regions of fixed area are maximized by the disc.

$$\tau_D \leq \tau_{D^*}$$

  That is,

$$P_x\{\tau_D > t\} \leq P_0\{\tau_{D^*} > t\}$$

- Integrals of heat kernels, Green functions, . . .

Rearrangements

For $A \subset \mathbb{R}^d$, A^* =ball centered at the origin and same volume as A. $\chi_A^* = \chi_{A^*}$
$f : \mathbb{R}^d \to \mathbb{R}$,

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$$

(Compare this with)

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)dt$$

Properties:

$$f^*(x) = f^*(y), \ \ |x| = |y|, \ \ f^*(x) \geq f^*(y), \ \ |x| \leq |y|$$

$$\{x : f^*(x) > t\} = \{x : |f(x)| > t\}^* \ \ \text{(same level sets)}$$

$$\Rightarrow m\{x : f^*(x) > \lambda\} = m\{x : |f(x)| > \lambda\}$$

The heart of the matter: finite dimensional distributions

Study (as a function of x and D)

$$\Phi_m(x, D) = P_x\{B_{t_1} \in D, B_{t_2} \in D, \ldots, B_{t_m} \in D\}$$

$B_t =$ Brownian motion (twice the speed) in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open connected $x \in D$,

$$0 < t_1 < t_2 \cdots < t_m$$

Same as studying Multiple Integrals:

$$\Phi_m(x, D) = \int_D \cdots \int_D \prod_{j=1}^{m} p_{t_j - t_{j-1}}(x_j - x_{j-1}) dx_1 \ldots dx_m,$$

$$x_0 = x \quad \text{and} \quad p_t(y) = \frac{1}{(4\pi t)^{d/2}} e^{-|y|^2/4t}$$

Theorem (Luttinger 1973)

Let $f_1, \ldots, f_m$ be nonnegative functions in $\mathbb{R}^d$ and let $f_1^*, \ldots, f_m^*$ be their symmetric decreasing rearrangement. Then for any $x_0 \in D$ we have

$$\int_{D^m} \prod_{j=1}^{m} f_j(x_j - x_{j-1}) dx_1 \cdots dx_m \leq \int_{\{D^*\}^m} f_1^*(x_1) \prod_{j=2}^{m} f_j^*(x_j - x_{j-1}) dx_1 \cdots dx_m.$$

D^*=ball center at zero and and same volume as D

Theorem (Brascamp–Lieb–Lutinger–1975, 1977)

$Q_j : \mathbb{R}^d \to [0, \infty)$ and $1 \leq j \leq r$. a_{jk}, $1 \leq j \leq r, 1 \leq k \leq m$ real numbers.

$$\int_{(\mathbb{R}^d)^m} \prod_{j=1}^{r} Q_j\Big(\sum_{k=1}^{m} a_{jk} z_k\Big) dz_1 \cdots dz_m \leq \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{r} Q_j^*\Big(\sum_{k=1}^{m} a_{jk} z_k\Big) dz_1 \cdots dz_m$$

Roots lie in inequalities of Hardy–Littlewood–Pólya–Riesz

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_1(x_1) H(x_2 - x_1) F_2(x_2) dx_1 dx_2 \leq *$$

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But why finite dimensional distributions?

Question

What is the smallest Dirichelt eigenvalue $\lambda_{1,\alpha}$ for the rotationally invariant stable processes of order $0 < \alpha < 2$ for the interval $(-1, 1)$?

Note: I learned this from Davar Khoshnevisan about 8 years ago.
Has been investigated by

- Investigated by: M.Kac-H. Pollar (1950). H. Widom (1961), J. Taylor (1967), B. Fristedt (1974), J. Bertoin (1996), Khosnevisan–Z. Shi (1998).

- I don't know the answer and, to be perfectly honest, don't care.
- In the process of investigating this “simple" question we "discovered" that little is known about the "fine" spectral theoretic properties of stables.
- More Exciting: The techniques give new Theorem for the Laplacian (BM).

Theorem (Chungs's LIL. Set $B_t^* = \sup_{0 \le s \le t} |B_s|$)

$$\liminf_{t\to\infty} \left(\frac{\log \log t}{t}\right)^{1/2} B_t^* = \frac{\pi}{2}, \quad a.s. \tag{1}$$

But, is $\frac{\pi}{2}$ really just our good-old-friend $\frac{\pi}{2}$ or is it something else?

(1) comes from Borel–Cantelli arguments and the “small balls" probability estimate.

$$P_0\left\{B_1^* < \varepsilon\right\} \approx e^{-\frac{\pi^2}{4\varepsilon^2}}, \ \varepsilon \to 0$$

$$P_0\left\{B_1^* < \varepsilon\right\} = P_0\left\{\frac{1}{\varepsilon} B_t^* < 1\right\} = P_0\left\{B^*_{\frac{1}{\varepsilon^2}} < 1\right\} = P_0\left\{\tau_{(-1,1)} > \frac{1}{\epsilon^2}\right\}$$

$$\tau_{(-1,1)} = \inf\{t > 0 : B_t \notin (-1,1)\} = \text{ first exit time from the interval}$$

In fact,

$$P_0\left\{\tau_{(-1,1)} > t\right\} \approx e^{-\lambda_1 t}\varphi_1(0)\int_1^1 \varphi_1(y)\,dy, \quad t \to \infty,$$

where λ_1 is the smallest eigenvalue for one half of the Laplacian in the interval $(-1,-1)$ with Dirichlet boundary conditions and φ_1 is the corresponding eigenfunction. That is, $\pi^2/4$ and the "sin" function.

For any $0 < \alpha < 2$, let X_t^α be the rotationally invariant stable process of order α. A similar statement holds for the "small ball" probabilities and there is

Theorem (J. Taylor 1967)

$$\liminf_{t\to\infty}\left(\frac{\log\log t}{t}\right)^{1/\alpha} X_t^* = (\lambda_{1,\alpha})^{1/\alpha}, \quad a.s. \tag{2}$$

For several other occurrences of the eigenvalue in "sample path behavior," see **Erkan Nane:** "Higher order PDE's and iterated Processes" and "Iterated Brownian motion in bounded domains in $\mathbb{R}^n$ "

Lévy Processes

Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: **de Finetti, Kolmogorov, Khintchine, Itô.**

- Rich stochastic processes, generalizing several basic processes in probability: Brownian motion, Poisson processes, stable processes, subordinators, . . .
- Regular enough for interesting analysis and applications. Their paths consist of continuous pieces intermingled with jump discontinuities at random times. Probabilistic and analytic properties studied by many.
- Many Developments in Recent Years:
  - **Applied:** Queueing Theory, Math Finance, Control Theory, Porous Media . . .
  - **Pure:** Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes

Definition

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments
- $X_0 = 0$ (with probability 1)
- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \dots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(dx) = (2\pi)^{d/2}\widehat{p}_t(\xi)$$

where p_t is the distribution of X_t. Notation (same with measures)

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx,\ \ f(x = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)d\xi$$

Theorem (The Lévy–Khintchine Formula)

The characteristic function has the form $\varphi_t(\xi) = e^{t\rho(\xi)}$*, where*

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \left(e^{i\xi\cdot x} - 1 - i\xi \cdot x1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$*, a non–negative definite symmetric* $n \times n$ *matrix* A *and a Borel measure* ν *on* $\mathbb{R}^d$ *with* $\nu\{0\} = 0$ *and*

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ *is called the* **symbol** *of the process or the* **characteristic exponent.** *The triple* (b, A, ν) *is called the* **characteristics of the process.**

Converse also true. Given such a triple we can construct a Lévy process.

Examples

1. **Standard Brownian motion:** With $(0, I, 0)$, I the identity matrix,

$$X_t = B_t, \qquad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**
$(0, A, 0)$, X_t is "generalized" Brownian motion, mean zero, covariance

$$E(X_s^j X_t^i) = a_{ij} \min(s, t)$$

X_t has the normal distribution (assume here that $det(A) > 0$)

$$\frac{1}{(2\pi t)^{d/2}\sqrt{\det(A)}} \exp\left(-\frac{1}{2t} x \cdot A^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, A, 0)$ get Brownian motion with a drift:

$$X_t = bt + B_t$$

Examples

4. **Poisson Process:** The Poisson Process $X_t = N_\lambda(t)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{N_\lambda(t) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \; m = 0, 2, \ldots$$

$N_\lambda(t)$ has continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : N_\lambda(t) = m\}$$

5. **Compound Poisson Process** obtained by summing iid random variables up to a Poisson Process.

6. **Relativistic Brownian motion** According to quantum mechanics, a particle of mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where c is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

Examples

7. **The zeta process:** Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}, \qquad z = x + iy \in \mathbb{C}$$

Khintchine: For every fix $x > 1$,

$$\rho_x(y) = \log\left(\frac{\zeta(x+iy)}{\zeta(y)}\right)$$

is the symbol of a Lévy process–in fact, limits of Poissons.
Biane–Pitman-Yor: "Probability laws related to the Jacobi theta and Riemann Zeta functions and Brownian excursions, Bull. Amer math. Soc., 2001.

M Yor: A note about Selberg's integrals with relation with the beta–gamma algebra, 2006.

Examples

8. **The rotationally invariant stable processes:** These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \le 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t^\alpha}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$ is **Brownian motion.** $\alpha = 1$ is the **Cauchy processes.**

$\alpha = 3/2$ is called the **Haltsmark distribution** used to model gravitational fields of stars. (See V.M. Zolotarev (1986) “One dimensional Stable Distributions".)

Transition probabilities:

$$P_x\{X_t^\alpha \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad \text{any Borel} \qquad A \subset \mathbb{R}^d$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}e^{-t|\xi|^\alpha}d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad \alpha = 2, \qquad \textbf{Brownian motion}$$

$$p_t^1(x) = \frac{C_d\, t}{(|x|^2 + t^2)^{\frac{d+1}{2}}}, \qquad \alpha = 1, \qquad \textbf{Cauchy Process}$$

For any $a > 0$, the two processes

$$\{\eta_{(at)}\,;\ t \geq 0\} \quad \text{and} \quad \{a^{1/\alpha}\eta_t\,;\ t \geq 0\},$$

have the same finite dimensional distributions **(self-similarity).**

In the same way, the transition probabilities scale similarly to those for BM:

$$p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha} x)$$

For the Lévy process $\{X(t); t \geq 0\}$, define

$$T_t f(x) = E[f(X(t))|X_0 = x] = E_0[f(X(t) + x)], \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

This is a Feller semigroup (takes $C_0(\mathbb{R}^d)$ into itself). Setting

$$p_t(A) = P_0\{X_t \in A\} \text{ (the distribution of } X_t)$$

we see that (by Fourier inversion formula)

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{t\rho(\xi)} \widehat{f}(\xi) d\xi$$

with generator

$$\begin{aligned} Af(x) &= \frac{\partial T_t f(x)}{\partial t}\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \rho(\xi) \hat{f}(\xi) d\xi = \text{a pseudo diff operator, in general} \end{aligned}$$

From the Lévy–Khintchine formula (and properties of the Foruier transform),

$$Af(x) = \sum_{i=1} b_i \partial_i f(x) \quad + \quad \frac{1}{2} \sum_{i,j} a_{i,j} \partial_i \partial_j f(x)$$
$$+ \quad \int \Big[f(x+y) - f(x) - y \cdot \nabla f(x) \chi_{\{|y|<1\}} \Big] \nu(dy)$$

Examples:

- Standard Brownian motion:
$$Af(x) = \frac{1}{2} \Delta f(x)$$
- Poisson Process of intensity λ:
$$Af(x) = \lambda \Big[f(x+1) - f(x) \Big]$$
- Rotationally Invariant Stable Processes of order $0 < \alpha < 2$, **Fractional Diffusions:**
$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &= A_{\alpha,d} \int \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy \end{aligned}$$

$X_t = X_t^\alpha$ is rotationally invariant stable with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

Let D be a bounded connected subset of $\mathbb{R}^d$. The first exit time of X_t^α from D is

$$\tau_D = \inf\{t > 0 : X_t^\alpha \notin D\}$$

Heat Semigroup in D is the self-adjoint operator

$$\begin{aligned} T_t^D f(x) &= E_x\Big[f(X_t^\alpha); \tau_D > t\Big], \qquad f \in L^2(D) \\ &= \int_D p_t^{D,\alpha}(x,y) f(y) dy, \end{aligned}$$

$$p_t^{D,\alpha}(x,y) = p_t^\alpha(x-y) - E^x(\tau_D < t; p_{t-\tau_D}^\alpha(X_{\tau_D}^\alpha, y)).$$

$p_t^{D,\alpha}(x,y)$ is called the **Heat Kernel for the stable process in D.**

$$
\begin{aligned}
p_t^{D,\alpha}(x,y) \le p_t^{\alpha}(x-y) \le p_1^{\alpha}(0)t^{-d/\alpha} &= \left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi\right) t^{-d/\alpha} \\
&= t^{-d/\alpha}\,\frac{\omega_d}{(2\pi)^d\alpha}\int_0^\infty e^{-s}s^{(\frac{n}{\alpha}-1)}ds \\
&= t^{-d/\alpha}\,\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}
\end{aligned}
$$

The general theory of heat semigroups gives an orthonormal basis of eigenfunctions

$$\{\varphi_{m,\alpha}\}_{m=1}^{\infty} \quad on \quad L^2(D)$$

with eigenvalues $\{\lambda_{m,\alpha}\}$ satisfying

$$0 < \lambda_{1,\alpha} < \lambda_{2,\alpha} \le \lambda_{3,\alpha} \le \cdots \to \infty$$

That is,

$$T_t^D\varphi_{m,\alpha}(x) = e^{-\lambda_{m,\alpha}t}\varphi_{m,\alpha}(x), \quad x \in D.$$

$$
\begin{aligned}
p_t^{D,\alpha}(x,y) &= \sum_{m=1}^{\infty} e^{-\lambda_{m,\alpha} t}\varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y) \\
&= e^{-\lambda_{1,\alpha} t}\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y) + \sum_{m=2}^{\infty} e^{-\lambda_{m,\alpha} t}\varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y)
\end{aligned}
$$

Theorem (“Intrinsic Ultracontractivity")

$$
e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t} \le \sup_{x,y\in D} \left| \frac{e^{\lambda_{1,\alpha} t} p_t^{D,\alpha}(x,y)}{\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y)} - 1 \right| \le C(D,\alpha) e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t}, \;\; t \ge 1.
$$

Apply the semigroup to the function $f(x) = 1,\ x \in D$

$$T_t^D f(x) = E_x[1_D(X_t^\alpha); \tau_D > t] = \int_D p_t^{D,\alpha}(x,y)dy$$

So that

$$P_x\{\tau_D > t\} = \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}} \varphi_{m,\alpha}(x) \int_D \varphi_{m,\alpha}(y)dy$$

$$= \quad e^{-t\lambda_{1,\alpha}} \varphi_{1,\alpha}(x) \int_D \varphi_{1,\alpha}(y)dy + \sum_{m=2}^{\infty} e^{-t\lambda_{m,\alpha}} \varphi_{m,\alpha}(x) \int_D \varphi_{m,\alpha}(y)dy$$

Theorem (By the Intrinsic Ultracontractivity)

$$\lim_{t\to\infty} e^{t\lambda_{1,\alpha}} P_x\{\tau_D > t\} = \varphi_{1,\alpha}(x) \int_D \varphi_{1,\alpha}(y)dy \tag{3}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log P_x\{\tau_D > t\} = -\lambda_{1,\alpha}, \tag{4}$$

uniformly for $x \in D$.

Conclusion: Finite dimensional distributions

If I want to study the eigenfunction $\varphi_{1,\alpha}$ and $\lambda_{1,\alpha}$ and how these are affected by the geometry of the domain D, I should (better, must, ...) study the distribution of the exit time τ_D of the process. That is, study

$$P_x\{\tau_D > t\}$$

as a function of D, $x \in D$, $t > 0$.

But this is the same as studying finite dimensional distributions:

$$\begin{aligned} P_x\{\tau_D > t\} &= P_x\{X_s^\alpha \in D;\ \forall s, 0 < s \leq t\} \\ &= \lim_{m\to\infty} P_x\{X_{jt/m}^\alpha \in D, j = 1, 2, \dots, m\} \\ &= \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}^\alpha(x - x_1)\cdots p_{t/m}^\alpha(x_m - x_{m-1})dx_1 \dots dx_m \end{aligned}$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}e^{-t|\xi|^\alpha}d\xi = \text{radial function}$$

Same for Heat Kernel

In the same way (integrating against a delta function at y)

$$p_t^{D,\alpha}(x,y) = \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}^{\alpha}(x-x_1)\cdots p_{t/m}^{\alpha}(y-x_{m-1})dx_1 \ldots dx_{m-1},$$

Alternatively, for Brownian motion, if $0 = t_0 < t_1 < \ldots < t_m < t$, then the conditional finite-dimensional distribution

$$P_{z_0}\{B_{t_1} \in dx_1, \ldots, B_{t_m} \in dx_m \,|\, B_t = y\},$$

is given by

$$\frac{p^2_{t-t_m}(z_n, y)}{p^2_t(z_0, y)} \prod_{i=1}^{m} p^2_{t_i - t_{i-1}}(z_i, z_{i-1}),$$

Theorem (Isoperimetric for F.D.D.)

$$\Phi_m(x, D) \leq \Phi_m(0, D^*)$$

Corollary (Exit times, eigenvalues, capacity, heat kernels,. . .)

$$P_x\{\tau_D^\alpha > t\} \leq P_0\{\tau_{D^*}^\alpha > t\}$$

$$\lambda_{1,\alpha}(D^*) \leq \lambda_{1,\alpha}(D) \quad \textbf{(The Faber-Krahn Theorem)}$$

$$Cap_\alpha(D) \geq Cap_\alpha(D^*),$$

(α-capacity version of a theorem of Pólya–Szegö. Proved by Watanabe 1984, conjectured by Mattila 1990, Proved by Betsakos 2003, P. Méndez 2006)

Corollary (Isoperimetric Inequality for “partition function")

$$\begin{aligned} Z_t^\alpha(D) &= \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D)} = \int_D p_t^{\alpha,D}(x,x)dx \\ &\leq \int_{D^*} p_t^{\alpha,D^*}(x,x)dx \leq \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D^*)} = Z_t^\alpha(D^*) \end{aligned}$$

Classical Isoperimetric Inequality

Amongst all regions of equal volume the ball minimizes surface area. It follows from "trace inequality" and

Theorem (M. Kac, “Can one hear the shape of a drum?")

With $\alpha = 2$, $|\partial D|$=surface area of boundary of D,

$$Z_t^2(D) \sim C_d t^{-d/2} vol(D) - C_d' t^{-(d-1)/2} |\partial D| + o(t^{-(d-1)/2}), \quad t \to 0$$

The first term is trivial from

$$\begin{aligned} P_t^{2,D}(x,y) &= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}} P_x\{\tau_D > t | B_t = y\} \\ &= \text{free motion times Brownian bridge in D} \end{aligned}$$

A detour into Weyl's asymptotics

$$\lim_{t\to 0} t^{\gamma} \int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A \Rightarrow \lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$$

Theorem (Weyl's Formula, $\alpha = 2$. $N_D(\lambda) = \#\{j \geq 1 : \lambda_j \leq \lambda\}$)

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2}, \quad \lambda \to \infty$$

More difficult (and no probabilistic treatment exists):

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2} - C'_d|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2})$$

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Theorem (R.B., T. Kulczycki B. Suideja (2009), $0 < \alpha \leq 2$)

$$\int_D p_t^{\alpha,D}(x,x)dx \sim C_{d,\alpha}t^{-d/\alpha}vol(D) - C_d' t^{-(d-1)/\alpha}|\partial D| + o(t^{-(d-1)/\alpha})$$

as $t \to 0$. This gives Weyl's version for all $0 < \alpha \leq 2$.

The $\$\$$ Question: Is there an α–version of the more general Weyl?

Question

Amongst all convex domains $D \subset \mathbb{R}^d$ of inradius 1, which one has the largest exit time for Brownian motion? Also, lowest eigenvalue? **Answer:** *The infinite strip:*

$$S = \mathbb{R}^{d-1} \times (-1, 1)$$

Theorem (For D convex with inradius 1.)

$$\Phi_m(x, D) \leq \Phi_m(0, S), \quad x \in D$$

R.B. Méndez-Latala (2001), $d = 2$ and (2003), $d \geq 3$. (Convexity is essential here!)

Corollary (For D convex with inradius 1 and $0 < \alpha \leq 2$.)

$$\begin{aligned} P_x\{\tau_D > t\} &\leq P_0\{\tau_S > t\} = P_0\{\tau_{(-1,1)} > t\} \\ \lambda_{1,\alpha}(-1, 1) &\leq \lambda_{1,\alpha}(D) \end{aligned}$$

Definition

$F : \mathbb{R}^d \to \mathbb{R}$ is said to be **log-concave** if

$$\log F(\beta x + (1-\beta)y) \geq \beta \log F(x) + (1-\beta) \log F(y), \quad x, y \in \mathbb{R}^d$$

or

$$F(\beta x + (1-\beta)y) \geq F(x)^{\beta} F(y)^{1-\beta}$$

Example

$F(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}$ and $F(x) = \chi_D(x)$, $D \subset \mathbb{R}^d$ is convex, are log–concave.

Theorem (Prékopa (1971))

Convolutions of log-concave functions are log-concave.

Corollary ($D \subset \mathbb{R}^d$ convex)

For Brownian motion, the function $\Phi_m(x, D)$ is log-concave.

Theorem (Lévy Isipoeimetric inequalities–R.B. P. Méndez, 2008)

Recall the Lévy–Khintchine $\varphi_t(\xi) = e^{t\rho(\xi)}$,

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \left(e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

Suppose ν *is continuous with respect to the Lebesgue measure with density* φ. *Let* X_t^* *be the Lévy process with characteristic triple* $(0, A^*, \varphi^*)$ *where* $A^* = (\det A)^{1/d}\, I_d$. *Let* $f_1, \ldots, f_m$, $m \geq 1$, *be nonnegative functions. Then for all* $z \in \mathbb{R}^d$,

$$E_z \left[\prod_{i=1}^{m} f_i(X_{t_i})\right] \leq E_0 \left[\prod_{i=1}^{m} f_i^*(X_{t_i}^*)\right],$$

for all $0 \leq t_1 \leq \ldots \leq t_m$.

With $f_i(x) = \chi_D(x)$, *this is about finite dimensional distributions:*

$$P_z\{X_{t_1} \in D, \ldots X_{t_m} \in D\} \leq P_0\{X_{t_1}^* \in D^*, \ldots X_{t_m}^* \in D^*\}$$

The maximum (and the minimum) of the “first" non-constant Neumann eigenfunction for bounded convex domains are attained on the boundary and only on the boundary of the domain.

Many partial results: R.B.-K.Burdzy (1999), D.Jerison-N.Darishavilli (2000), M. Pascu (2001), R. Bass–K. Burdzy (2000), R.B.-M. Pang (2003), R.B. M.Pang-Pascu (2004), R.Atar K.Burdzy (2005)

Counterexample: K. Burdzy-W. Werner (2000), K. Burdzy (2005)

Believed to be true for any simply connected domain, conjectured to be true for any convex domain.

Unknown even for an arbitrary triangle in the plane!

Conjecture: The maximum and minimum for the first nonconstant eigenfunction for the semigroup of Brownian motion conditioned to remain forever in a convex domain are attained on the boundary and only on the boundary of the domain.

That is, the function φ_2/φ_1 attains its maximum and minimum on the boundary and only on the boundary of D.

Theorem (R.B. Médez-Hernández, 2006)

The conditional "Hot Spots" conjecture is true for symmetric domains in the plane as those shown above. The maximum (and minimum) of the function

$$\Psi(z) = \frac{\varphi_2(z)}{\varphi_1(z)}$$

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“Hot–spots" Conjecture for conditioned Brownian motion

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Proof: Via finite dimensional distributions!

Theorem

Let D be a bounded domain in $\mathbb{R}^2$ *which is symmetric and convex with respect to both axes.*

(i) *If* $z_1 = (x, y_1) \in D^+$, $z_2 = (x, y_2) \in D^+$ *and* $y_1 < y_2$, *then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} < \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$. *In particular, the function*

$$\Psi(z,t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}},$$

for each $t > 0$ *arbitrarily fixed, cannot have a maximum at an interior point of* D^+.

(ii) *If* $z_1 = (x_1, y) \in D^+$ *and* $z_2 = (x_2, y) \in D^+$ *with* $|x_2| \le |x_1|$, *then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \le \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$.

Corollary

$D \subset \mathbb{R}^2$ as in Theorem φ_2 be such that its nodal line is the intersection of the x-axis with the domain. Without LOG, $\varphi_2 > 0$ in D^+ and $\varphi_2 < 0$ in D^-. Set $\Psi = \varphi_2/\varphi_1$.

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\Psi(z_1) < \Psi(z_2).$$

(ii) *If $z_1 = (x, y_1) \in D^-$ and $z_2 = (x, y_2) \in D^-$ with $y_2 < y_1$, then*

$$\Psi(z_1) < \Psi(z_2).$$

In particular, Ψ cannot attain a maximum nor a minimum in the interior of D.

(iii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then*

$$\Psi(z_1) \leq \Psi(z_2). \tag{5}$$

Corollary (Exact analogue of D. Jerison and N. Nadirashvili (2000) for classical "hot–spots")

Suppose $D \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary which is symmetric and convex with respect to both coordinate axes and that φ_2 is as in Theorem 1.2. Then strict inequality holds in (5) unless D is a rectangle. The maximum and minimum of Ψ on $\overline{D}$ are achieved at the points where the y-axis meets ∂D and, except for the rectangle, at no other points.