Martingales and the Beurling–Ahlfors operator, an overview*

Rodrigo Bañuelos†
Department of Mathematics
banuelos@math.purdue.edu

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PURDUE UNIVERSITY

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The Problem: Conjecture of Tadeusz Iwaniec, 1982

$$Bf(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\, dA(w) \quad \Rightarrow \|Bf\|_p \le (p^*-1)\|f\|_p,$$

$$p^* - 1 = \begin{cases} p-1, & 2 \le p < \infty \\ \\ \frac{1}{p-1}, & 1 < p \le 2 \end{cases}$$

$$\widehat{Bf}(\xi) = \frac{\bar{\xi}^2}{|\xi|^2}\hat{f}(\xi), \quad \text{and} \quad B(\overline{\partial} f) = \partial f$$

$$\partial f = \left(\frac{\partial f}{\partial x_1} - i\,\frac{\partial f}{\partial x_2}\right), \quad \overline{\partial} f = (\frac{\partial f}{\partial x_1} + i\,\frac{\partial f}{\partial x_2}\right)$$

Equivalent to: $\|\partial f\|_p \le (p^*-1)\|\overline{\partial} f\|_p, \quad \forall \quad f \in C_0^\infty(\mathbb{C})$

Outline

- **Burkholder's Inequalities—The Heart of the Matter**

- **Inequalities for Stochastic Integrals**

- **The Hilbert, Riesz, and Beurling–Ahlfors Singular Integrals, what is known, what is not known, what will be nice to know**

- **Stochastic Integrals, Singular Integrals, and their marriage**

- **Why should we care? Copious references (216) given at end of "talk" and also available as a separate download.**

Martingales: A sequence of functions $\{f_n\}_0^\infty$ together with the sequence of σ–algebras $\{\mathcal{F}_0 \subset \mathcal{F}_1, \cdots\}$ on the Probability Space $(\Omega, \mathcal{F}_\infty, P)$

$$E(f_{n+1}|\mathcal{F}_n) = f_n$$

$$f_n = \sum_{k=1}^{n} d_k, \qquad d_k = f_k - f_{k-1},$$

Martingale Transform: $g_n = \sum_{k=1}^{n} v_k d_k \qquad v_k \in \mathcal{F}_{k-1}$ (predictable) $|v_k(\omega)| \leq 1.$

(1) $$\|g_n\|_p \leq C_p \|f_n\|_p,$$ **(Burkholder 1966)**

For Haar $$\Rightarrow \Big\| \sum_{k=1}^{n} \varepsilon_k a_k h_k \Big\|_p \leq C_p \Big\| \sum_{k=1}^{n} a_k h_k \Big\|_p$$ (R.E.A.C. Paley 1932)

(2) $$\|g_n\|_p \leq (p^* - 1)\|f_n\|_p,$$ (Burkholder 1984)

$$V(x, y) = |y|^p - (p^* - 1)^p |x|^p$$

Want:

$$E(V(f_n, g_n)) \leq 0.$$

Burkholder: There is a function $U(x,y)$ such that

$$V(x,y) \leq U(x,y)$$

and

$$E(U(f_n,g_n)) \leq \ldots \leq E(U(f_0,g_0)) \leq 0.$$

$\{e_k\}$, $\{d_k\}$ **Hilbert space** $\mathbb{H}$ valued martingale difference sequences with

$$\|e_k(\omega)\|_{\mathbb{H}} \leq \|d_k(\omega)\|_{\mathbb{H}}, \qquad \forall \quad \omega \in \Omega, k \geq 0.$$

$$g_n = \sum_{k=1}^{n} e_k, \qquad f_n = \sum_{k}^{n} d_k$$

(3) $$\|g_n\|_p \leq (p^* - 1)\|f_n\|_p, \qquad \textbf{(Burkholder 1988)}$$

For Haar $$\Rightarrow \Big\| \sum_{k=1}^{n} e^{i\theta_k} a_k h_k \Big\|_p \leq (p^* - 1) \Big\| \sum_{k=1}^{n} a_k h_k \Big\|_p$$

$$U(x,y) = \alpha_p \left(|y| - (p^*-1)|x|\right)\left(|y|+|x|\right)^{p-1}, \qquad \alpha_p = p\left(1 - \frac{1}{p^*}\right)^{p-1}$$

Stochastic Integrals

$$X_t = (X_t^1, \ldots, X_t^m), \qquad Y_t = (Y_t^1, \ldots, Y_t^m)$$

$$X_t^j = \int_0^t H_s^j \cdot dB_s, \qquad Y_t^j = \int_0^t K_s^j \cdot dB_s,$$

$$\langle X^j \rangle_t = \int_0^t |H_s^j|^2 ds, \qquad \langle X^j, Y^j \rangle_t = \int_0^t H_s^j \cdot K_s^j ds$$

Subordination Inequality (B–Wang 1995): Assume $(Y << X)$

$$\sum_{j=1}^m |K_s^j(\omega)|^2 \le \sum_{j=1}^m |H_s^j(\omega)|^2, \quad \forall\ \omega \in \Omega,\ \forall\ s > 0.$$

Then

$$\Big\| \left(\sum_{j=1}^m |Y^j|^2 \right)^{1/2} \Big\|_p \le (p^* - 1) \Big\| \left(\sum_{j=1}^m |X^j|^2 \right)^{1/2} \Big\|_p$$

and $p^* - 1$ is best possible.

Orthogonality: Take $m = 1$. The martingales X_t and Y_t are orthogonal, $X \perp Y$, if $\langle X, Y\rangle_t = 0 \; \forall \; t$.

(B–Wang 1995, 1996, 2000): Assume $X \perp Y$ and $Y << X$.

$$\|Y\|_p \leq \cot\left(\frac{\pi}{2p^*}\right)\|X\|_p, \quad \textbf{(Pichorides–type)}$$

$$\|\sqrt{Y^2 + X^2}\|_p \leq \csc\left(\frac{\pi}{2p^*}\right)\|X\|_p, \quad \textbf{(Essén–type)},$$

$$P\{|Y| > 1\} \leq K_1\|X\|_1, \qquad \textbf{(Davis–type)}$$

$$K_1 = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2}} = \frac{3\zeta(2)}{4\beta(2)} = \frac{\pi^2}{8\beta(2)} \approx 1.328434313301$$

$\beta(2) \approx .9159655$ is the so called **"Catalan constant"**

$$\cot\left(\frac{\pi}{2p^*}\right) \approx \frac{2}{\pi}(p^* - 1) \text{ as } p \to 1 \text{ or } \infty.$$

Weak (p,p)

(P. Janakiraman, 2004): Assume $X \perp Y$ and $Y << X$.

$$P\{|Y| > 1\} \le K_p \|X\|_p \qquad 1 \le p \le 2,$$

$$K_p = \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi} \log |t|\right|^p}{t^2+1} dt \right)^{-1}$$

UNKNOWN FOR $2 < p < \infty$.

Under Subordination only $(Y << X)$: $\quad P\{|Y| > 1\} \le \gamma_p \|X\|_p$

$$\gamma_p = \frac{2}{\Gamma(p+1)}, \qquad 1 \le p \le 2 \qquad \textbf{(Burkholder 1994)}$$

$$\gamma_p = \frac{p^{p-1}}{2}, \qquad 2 < p < \infty \qquad \textbf{(Jiyeong Suh 2004)}$$

Hilbert Transform

$$Hf(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy,$$

$$\|Hf\|_p \leq \cot\left(\frac{\pi}{2p^*}\right)\|f\|_p$$ **(Pichorides 1972)**

$$\|\sqrt{|Hf|^2+|f|^2}\|_p \leq \csc\left(\frac{\pi}{2p^*}\right)\|f\|_p,$$ **(Essén 1984)**

$$m\{x \in \mathbb{R} : |Hf(x)| > 1\} \leq K_1\|f\|_1$$ **(Davis 1974)**

$$m\{x \in \mathbb{R} : |Hf(x)| > 1\} \leq K_p\|f\|_p$$ **(Janakiraman, 2004)**

$1 \leq p \leq 2.$

OPEN FOR $2 < p < \infty.$

Riesz Transforms in $\mathbb{R}^n$, $n \geq 2$:

$$R_j f(x) = C_n \int_{\mathbb{R}^n} \frac{(x_j - y_j)}{|x-y|^{n+1}} f(y)\, dy, \qquad \widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|}\widehat{f}(\xi)$$

$$R_j f(x) = \int_0^\infty P_t \star \frac{\partial f}{\partial x_j}(x) dt = \int_0^\infty P_t(\frac{\partial f}{\partial x_j})(x) dt$$

$$\|R_j f\|_p \quad \leq \quad \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p,$$ **(Iwaniec–Martin 1993)**

$$\|\sqrt{|R_i f|^2 + |f|^2}\|_p \quad \leq \quad \csc\left(\frac{\pi}{2p^*}\right) \|f\|_p$$ **(B–Wang 1994)**

Other Applications of the subordination and orthogonality Martingale Inequalities

- **N. Arcozzi:** Riesz Transforms on Compcat Lie Groups, Spheres and Gaussian Space, Arkiv (1998)
- **N. Arcozzi and X. Li:** Riesz transforms on spheres. Math. Res. Lett. (1997),
- **L. Larsson–Cohn:** On the constant in the Meyer Inequality, Monatshefte f'ür Mathematik (2002)

Weak–type Inequalities for Riesz Transforms

- **Question 1:** E. M. Stein, 1986 Berkeley ICM lecture, Page 203: Do the Riesz Transforms in $\mathbb{R}^n$ have a have weak–type constant independent of the dimension n?

- **Question 2:** From work of P.A. Meyer (1984): Do the Riesz Transforms on Wiener Space have a weak–type inequality?

Open, Except

1. **P. Janakiraman, Indiana Math. J. (2004):**

$$m\{x \in \mathbb{R}^n : |R_j f(x)| > 1\} \leq C \log n \, \|f\|_1,$$

   C does not depend on the dimension n.

2. **Fabes–Gutiérrez–Scotto, Revista Iberoamericana (1994):** gives weak-type estimates for the Riesz transforms associated with the gaussian measure in $\mathbb{R}^n$ but with constants depending on dimension n.

Second Order Riesz Transforms

$$\widehat{R_j^2 f}(\xi) = \frac{-\xi_j^2}{|\xi|^2}\widehat{f}(\xi) = \frac{1}{2}\int_0^\infty e^{-2\pi^2 t|\xi|^2}(-4\pi^2\xi_j^2)\hat{f}(\xi)\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-2\pi^2 t|\xi|^2}\widehat{\frac{\partial^2 f}{\partial x_j^2}}(\xi)\,dt$$

$$= \frac{1}{2}\int_0^\infty \widehat{T_t(\frac{\partial^2 f}{\partial x_j^2})}(\xi)\,dt,$$

$$T_t(g)(x) = \frac{1}{(2\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{2t}} g(y)\,dy = \int_{\mathbb{R}^n} H_t(x-y)g(y)\,dy$$

$$R_j(f)(x) = \int_0^\infty P_t\left(\frac{\partial f}{\partial x_j}\right)(x)dt = -\left(\frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f\right),$$ Poisson Semigroup

$$R_j^2(f)(x) = \frac{1}{2}\int_0^\infty T_t\left(\frac{\partial^2 f}{\partial x_j^2}\right)(x)\,dt = -\left(\frac{\partial^2}{\partial x_j^2}(-\Delta)^{-1} f\right),$$ Heat Semigroup

$$R_j R_k(f)(x) = \frac{1}{2}\int_0^\infty T_t\left(\frac{\partial^2 f}{\partial x_j \partial x_k}\right)(x)\,dt = -\left(\frac{\partial^2}{\partial x_j \partial x_k}(-\Delta)^{-1} f\right)$$

Beurling–Ahlfors

$$\widehat{Bf}(\xi) = \frac{\bar{\xi}^2}{|\xi|^2}\widehat{f}(\xi) \quad \Rightarrow B = R_2^2 - R_1^2 + 2iR_2R_1,$$

$$B(f)(z) = \frac{1}{2}\int_0^\infty T_t\left(\frac{\partial^2 f}{\partial z^2}\right)(z)\,dt = -\left(\frac{\partial^2}{\partial z^2}(-\Delta)^{-1}f\right)$$

Problem: Find best constants ($1 < p < \infty$) in the inequalities:

$$\|R_j^2 f\|_p \leq C_p^1\|f\|_p, \qquad j = 1, \ldots, n$$

$$\|R_jR_kf\|_p \leq C_p^2\|f\|_p, \qquad j \neq k.$$

B–Wang, 1995:

$$\Big\|\sum_{j=1}^n a_jR_j^2f\Big\|_p \leq 2(p^*-1)\Big\|f\Big\|_p, \qquad a_j \in \{-1, 0, 1\}$$

$$\Big\|R_jR_kf\Big\|_p \leq (p^*-1)\Big\|f\Big\|_p$$

$$\Longrightarrow \qquad \Big\|Bf\Big\|_p \leq 4(p^*-1)\Big\|f\Big\|_p$$

Nazarov–Volberg 2003, (Dragičević–Volberg, 2003, 2003): For all $f, g \in C_0^\infty$, $2 \le p < \infty$,

$$2\int_0^\infty \int_{\mathbb{R}^2} \Big|\frac{\partial U_f(x,t)}{\partial x_1}\Big|\Big|\frac{\partial U_g(x,t)}{\partial x_1}\Big|\,dx\,dt \quad + \quad 2\int_0^\infty \int_{\mathbb{R}^2} \Big|\frac{\partial U_f(x,t)}{\partial x_2}\Big|\Big|\frac{\partial U_g(x,t)}{\partial x_2}\Big|\,dx\,dt$$

$$\le \quad (p^* - 1)\|f\|_p \|g\|_q,$$

$$U_f(x,t) = T_t f(x)$$

Proved using Green theorem applied to

$$b(x,t) = B\left(|U_f(x,t)|^p, |U_g(x,t)|^p, U_f(x,t), U_g(x,t)\right)$$

where $B(X, Y, \xi, \eta)$ is a "Bellman" function for (Burkholder's inequalities used to find it!)

$$D_p = \{(X, Y, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 : \|\xi\|^p < X, \|\eta\|^p < Y\}$$

The Littlewood–Paley Inequality above and

$$\int_{\mathbb{R}^2} R_1^2 f \cdot g dx = -2\int_0^\infty \int_{\mathbb{R}^2} \frac{\partial U_f(x,t)}{\partial x_1}\frac{\partial U_g(x,t)}{\partial x_1}\,dx\,dt$$

$$\Longrightarrow \ \Big\|R_1^2 - R_2^2\Big\|_p \le (p^* - 1), \qquad \Big\|R_1 R_2 f\Big\|_p \le \frac{1}{2}(p^* - 1)$$
$$\Longrightarrow \ \Big\|Bf\Big\|_p \le 2(p^* - 1)\Big\|f\Big\|_p$$

B–Méndez 2003:

$$\Big\| \sum_{j=1}^{n} a_j R_j^2 f \Big\|_p \leq (p^* - 1)\Big\| f \Big\|_p, \qquad a_j \in \{-1, 0, 1\}$$

$$\Big\| R_j R_k f \Big\|_p \leq \frac{1}{2}(p^* - 1)\Big\| f \Big\|_p, \qquad j \neq k,$$

$$\Longrightarrow \qquad \Big\| Bf \Big\|_p \leq 2(p^* - 1)\Big\| f \Big\|_p$$

Space–Time Brownian motion: Let Z_t be 2–dimensional Brownian motion with initial distribution the Lebesgue measure. Fix $T > 0$. **Space–Time BM is:**

$$B_t = (Z_t, T - t)\,, \quad 0 \leq t \leq T$$

B_t starts on hyperplane $\mathbb{R}^2 \times T$ with the Lebesgue measure. If $f \in C_0^\infty(\mathbb{R}^2)$,

$$E^T\left[f(B_T)\right] = \int_{\mathbb{R}^2} E_z\left[f(B_T)\right] dz = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_T(z - w) f(w) dw dz = \int_{\mathbb{R}^2} f(w) dw$$

$$\begin{cases} \frac{\partial U_f}{\partial t}(z,t) = \frac{1}{2}\Delta U_f(z,t), & (z,t) \in \mathbb{R}^3_+ \\ \\ U_f(z,0) = f(z), & z \in \mathbb{R}^2 \end{cases}$$

$\Longrightarrow$

$$U_f(B_t) = U_f(Z_t, T-t), \qquad t < T, \qquad \textbf{is a martingale}$$

Itô $\Longrightarrow$

$$\begin{aligned} f(Z_T) = U_f(B_T) &= U_f(B_0) + \int_0^T \nabla_z U_f(B_t) \cdot dZ_t \\ &= U_f(Z_0, T) + \int_0^T \nabla_z U_f(B_t) \cdot dZ_t \end{aligned}$$

For any 2×2 matrix A define the martingale transform

$$A * f = \int_0^T A \nabla_z U_f(B_t) \cdot dZ_t$$

Subordination for Stochastic Integrals Gives

$$\begin{aligned}\Big\| \int_0^T A\nabla_z U_f(B_t)\cdot dZ_t \Big\|_{L^p(\Omega)} &\leq \|A\|(p^*-1)\Big\| \int_0^T \nabla_z U_f(B_t)\cdot dZ_t \Big\|_{L^p(\Omega)} \\ &= \|A\|(p^*-1)\Big\| f(Z_T) - U_f(B_0,T) \Big\|_{L^p(\Omega)},\end{aligned}$$

$$\|A\| = \sup\Big\{ \|A(z,w)\| : z,w\in\mathbb{C},\quad \|z\|^2+\|w\|^2\leq 1 \Big\}$$

The "projection" operator

$$S_A^T f(z) = E^T\Big[\int_0^T A\nabla_z U_f(B_t)\cdot dZ_t \Big| B_T = (z,0) \Big]$$

on $\mathbb{R}^2$ has

$$\Big\| S_A^T f \Big\|_{L^p(\mathbb{R}^2)} \leq \|A\|(p^*-1)\Big\| f(Z_T) - U_f(B_0,T) \Big\|_{L^p(\Omega)}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Longrightarrow$$

$$S_A^T f(z) \quad = \quad \frac{1}{2}\int_0^{2T} T_t\left(\frac{\partial^2 f}{\partial x_2^2}\right)(z)\,dt - \frac{1}{2}\int_0^{2T} T_t\left(\frac{\partial^2 f}{\partial x_1^2}\right)(z)\,dt$$

$$\longrightarrow \quad R_2^2 f(z) - R_1^2 f(z), \quad \text{as} \quad T \to \infty$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow$$

$$S_A^T f(z) = \int_0^{2T} T_t\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)(z)\,dt \quad \longrightarrow \quad 2R_1R_2 f(z)$$

$\|A\| = 1$ **in both cases, given the bounds above.**

Other Related Inequalities: The subordination inequalities in B–Wang 1995 give many more similar bounds. We use $f = f_1 + if_2$ and

$$T_1 = R_2^2 - R_1^2, \qquad T_2 = 2R_1R_2$$

so that

$$B = T_1 + iT_2 \qquad \text{and} \qquad \overline{B} = T_1 - iT_2.$$

Then

(4) $$\left\| \sqrt{|Bf|^2 + |\overline{B}f|^2} \right\|_p \leq 2(p^* - 1)\|f\|_p$$

(5) $$\left\| \sqrt{|T_1 f|^2 + |T_2 f|^2} \right\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p$$

(6) $$\left\| \sqrt{|T_1 f_1|^2 + |T_2 f_2|^2} \right\|_p \leq (p^* - 1)\|f\|_p$$

(7) $$\left\| \Re(Bf) \right\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p$$

The last inequality leads to (**Dragičević–Volberg 2003**)

$$\|Bf\|_p \leq \tau(p)\sqrt{2}(p^* - 1)\|f\|_p, \qquad \tau(p) \to 1 \text{ as } p \to \infty.$$

Idea of Proof: Subordination Inequalities

$$Y_t = \int_0^t K_s \cdot dB_s, \qquad X_t = \int_0^t H_s \cdot dB_s$$

$$\|K_s(\omega)\| \leq \|H_s(\omega)\|.$$

$$V(x,y) = |y|^p - (p^* - 1)^p |x|^p.$$

Want $E(V(X_t, Y_t)) \leq 0$.

Burkholder's function:

$$U(x,y) = p(1 - \frac{1}{p^*})^{p-1}(|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1}$$

(a) $V(x,y) \leq U(x,y)$

(b) $U_{xx}(x,y)|H|^2 + 2U_{xy} H \cdot K + U_{yy}|K|^2 \leq -C_p A, \quad C_p > 0,$

$$A = (|H|^2 - |K|^2)(|x| + |y|)^{p-2}.$$

“Apply” Itô:

$$\begin{aligned} U(X_t,Y_t) &= M_t + \frac{1}{2}\Big[\int_0^t U_{xx}(X_s,Y_s)d\langle X_s\rangle \\ &+ \int_0^t U_{xy}(X_s,Y_s)d\langle X_s,Y_s\rangle + \int_0^t U_{yy}(X_s,Y_s)d\langle Y_s\rangle\Big] \\ &= M_t + I_t, \qquad I_t \leq 0. \end{aligned}$$

Idea of Proof: Orthogonality Inequalities

With Orthogonality we use "essentially" functions of Pichorides and Essén :

$$V(x,y) = |y|^p - \cot^p\left(\frac{\pi}{2p^*}\right)|x|^p, \quad 1 < p \le 2$$

$$U(x,y) = \frac{-\sin^{p-1}\left(\frac{\pi}{2p}\right) R^p \cos p\theta}{\cos\left(\frac{\pi}{2p}\right)}$$

$$|x| = R\cos\theta, \quad y = R\sin\theta, \quad -\pi/2 \le \theta \le \pi/2$$

(a) $V(x,y) \le U(x,y)$

(b) $U_{xx}|H|^2 + U_{yy}|K|^2 \le -C(x,y)(|H|^2 - |K|^2), \quad C(x,y) \ge 0.$

As above, apply Itô.

The k–form in $\mathbb{R}^n$, $k = 1, 2, \dots, n$,

$$\omega(x) = \sum_I \omega_I dx_I, \quad dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

is in $L^p(\mathbb{R}^n, \wedge^k)$ if

$$\|\omega\|_{L^p(\mathbb{R}^n, \wedge^k)} = \left\| \left(\sum_I |\omega_I|^2 \right)^{1/2} \right\|_{L^p} < \infty.$$

$$L^p(\mathbb{R}^n, \wedge) = \bigoplus_{k=0}^{n} L^p(\mathbb{R}^n, \wedge^k)$$

Donaldson–Sullivan 1989: Define

$$S_n \omega = (d\delta - \delta d) \circ \Delta^{-1} \omega.$$

$d =$ exterior derivative, $\delta =$ its adjoint (Hodge operator),

Then

$$S_n : L^p(\mathbb{R}^n, \wedge) \to L^p(\mathbb{R}^n, \wedge), \qquad 1 < p < \infty.$$

Iwaniec–Martin 1993:

$$\|S_n\|_p \le (n+1)\|B\|_p \le C(n+1)p^2$$

C independent of n and p.

Via "method of rotations" in $\mathbb{C}^n$: $\Omega : \mathbb{C}^n \to \mathbb{C}$, s.t. $\forall\ (\lambda, \xi) \in \mathbb{C} \times S^{2n-1}$,

$$\Omega(\lambda\xi) = \frac{\lambda^2 \Omega(\xi)}{|\lambda|^{2n+2}},$$

$$Tf(z) = \int_{\mathbb{C}^n} \Omega(w) f(z-w)dw = \frac{\omega_{2n-1}}{2} \int_{\mathbb{C}P^{n-1}} \Omega(\xi) B_\xi f(z) d\mu(\xi)$$

$$\|T\|_p \le \left(\frac{\omega_{2n-1}}{2} \int_{\mathbb{C}P^{n-1}} |\Omega(\xi)| d\mu(\xi) \right) \|B\|_p.$$

Conjecture: Iwaniec–Martin 1993:

$$\|S_n\|_p = (p^* - 1).$$

B–Lindeman (1997):

$$\|S_n\|_p \leq \begin{cases} (n+2)\,(p^*-1), & 2 \leq n \leq 14 \text{ and even} \\ (n+1)(p^*-1), & 3 \leq n \leq 13 \text{ and odd} \\ \left(\frac{4n}{3}-2\right)(p^*-1), & \text{otherwise.} \end{cases}$$

Using the heat martingale techniques of B–Mendez (2003) some improvements on these bounds are possible but the following problem is of much more interest:

Problem (more modest than the Iwaniec–Martin conjecture): Prove that $\|S_n\|_p$ has a bound independent of dimension.

QUESTION: Is there a Beurling–Ahlfors operator in "Truly" infinite dimensions? That is, a Beurling–Ahlfors operator on Wiener space?

Quasiconvexity, Rank–one Convexity, Related Matters

$$I(f) = \int_\Omega F\left(\frac{\partial f_i}{\partial x_j}(x)\right) dx, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, \; f \in W^{1,p}(\Omega, \mathbb{R}^m).$$

Morrey (1952): "Quasi–convexity and lower semicontinuity of multiple integrals."

- I is weakly lower semicontinuous $\iff$ F quasiconvex
- The Euler equations $I'(f) = 0$ are elliptic $\iff$ F is rank–one convex
- **Quasiconvexity:** $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is for each $A \in \mathbb{R}^{m\times n}$, each bounded $D \subset \mathbb{R}^n$, each compactly supported Lipschitz function $f\colon D \to \mathbb{R}^m$,

$$F(A) \leq \frac{1}{|D|} \int_D F(A + \frac{\partial f_i}{\partial x_j})$$

- **Rank one convexity:** $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, $A,\; B \in \mathbb{R}^{m\times n}$, rank $B = 1$,

$$h(t) = F(A + tB) \quad \text{is convex}$$

- $n = 1$ or $m = 1$, quasiconvex or rank one convex $\iff$ convex.
- If $n \geq 2$ and $m \geq 2$, convexity $\implies$ quasiconvexity $\implies$ rank one convexity.
- **Morrey 1952:** Conjectured that Rank one convexity $\not\Rightarrow$ quasiconvexity.
- Šverak 1992: Conjecture correct if $m \geq 3$. Case $m = 2,\ n \geq 2$ open.

Enter Burkholder's function: $\forall z,\ w,\ h,\ k \in \mathbb{C}$, $|k| \leq |h|$,

$$h(t) = -U(z + th,\ w + tk) \qquad \text{is convex.}$$

Define

$$\Gamma\colon \mathbb{R}^{2\times 2} \to \mathbb{C} \times \mathbb{C}$$

$$\Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (z, w),$$

$$z = (a + d) + i(c - b), \qquad w = (a - d) + i(c + b)$$

$$F_U = -U \circ \Gamma.$$

F_U is rank–one convex. **(B–Lindeman 1997).**

$$f : \mathbb{C} \to \mathbb{C}, \qquad \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \; f = u + iv \in C_0^\infty(\mathbb{C})$$

$$F_U \left(\frac{\partial f_i}{\partial x_j}\right) = -U\left(\overline{\partial} f, \; \partial f\right).$$

Quasiconvexity of F_U at $0 \in \mathbb{R}^{2\times 2} \iff$

$$-\iint_{\text{supp } f} U\left(\frac{\partial f}{\partial \overline{z}}, \frac{\partial f}{\partial z}\right) \geq 0.$$

Question (raised in B. Wang 1995): Is F_U quasiconvex?

- **If true:** Iwaniec's conjecture true.
- **If false:** Morrey's conjecture true for $n = 2 = m$.

Either way, you find gold!

Test with “extremals:” ($0 < \theta < 1$)

$$f_\theta(z) = \begin{cases} z|z|^{-\frac{2\theta}{p}} & |z| < 1 \\ \overline{z}^{\;-1} & |z| \geq 1 \end{cases}$$

$\Longrightarrow$

$$\iint_{\mathbb{C}} U\left(\overline{\partial} f_\theta,\ \partial f_\theta\right) dA \equiv 0.$$

Least majorant of V with “desired concavity properties is:”

$$\tilde{U}(x,y) = \begin{cases} V(x,y); & |y| \leq (p^* - 1)|x| \\ U(x,y); & |y| > (p^* - 1)|x|, \end{cases}$$

for $2 \leq p < \infty$, and U and V interchange for $1 < p \leq 2$. For $\tilde{U}$,

$$\iint\limits_{\mathbb{C}} \tilde{U}(\overline{\partial} f_\theta,\ \partial f_\theta) dA = \pi\left[p\left(1 - \frac{1}{p}\right)^{p-1} - (p-1)^{p-1}\right] < 0$$

Conjecture of A. Baernstein and S. Montgomery–Smith, 1999

If

$$L(x,y) = \begin{cases} |y|^2 - |x|^2; & |x| + |y| \le 1 \\ 1 - 2|x|; & |x| + |y| > 1. \end{cases}$$

Then

$$\iint_{\mathbb{C}} L(\overline{\partial} f,\ \partial f)\, dA \le 0, \qquad f \in C_0^\infty(\mathbb{C}).$$

Relationship between L and U: For $1 < p < 2$

$$\int_0^\infty s^{p-1} L\left(\frac{x}{s}, \frac{y}{s}\right) ds \quad = \quad \beta_p U(x,y)$$

$$\beta_p \quad = \quad \left(\frac{1}{2}\, p(p-2)\alpha_p\right)^{-1}$$

For $2 < p < \infty$,

$$\begin{aligned} M(x,y) &= L(x,y) + (|x|^2 - |y|^2) \\ &= (1 - 2|x|) + (|x|^2 - |y|^2) \\ &= -(|y|^2 - (|x| - 1)^2)\chi_{\{|x|+|y|>1\}} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \int_0^\infty s^{p-1} M\left(\frac{x}{s}, \frac{y}{s}\right) ds &= \gamma_p U(x,y) \\ \gamma_p &= [(p-1)(p-2)\alpha_p]^{-1}. \end{aligned}$$

The Baernstein–Montgomery-Smith Conjecture $\Longrightarrow$ Inequality with Burkholder's function $\Longrightarrow$ Iwaniec's conjecture.

Verified for $f\colon \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = g(r)e^{i\theta},\ g \geq 0,\ g(0) = g(0+) = 0.$$

These all give $= 0$ for the above conjectures.

FELICIDADES, Albert Baernstein II

May you live long and may we all live to see the resolution of the Baernstein–Montgomery-Smith conjecture.

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