Four unknown constants*

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1 Four inequalities

Let $\Omega \subset \mathbb{R}^2$ be an arbitrary simply connected domain in the plane. We define $R_\Omega = \sup_{z\in\Omega} d_\Omega(z)$ (the inradius of the domain) where $d_\Omega(z)$ is the distance from z to the boundary of Ω. Let $\sigma_\Omega(z)$ be the density of the hyperbolic metric in Ω and let $\sigma_\Omega = \inf_{z\in\Omega} \sigma_\Omega(z)$. Finally, denote by λ_1 the lowest eigenvalue for the Dirichlet Laplacian in Ω and denote by τ_Ω the first exit time of Brownian motion from Ω. The following four inequalities hold.

1. There exists a positive constant C_1, independent of the domain, such that for all functions $u \in C_0^\infty(\Omega)$

$$\int_\Omega \frac{|u|^2}{d_\Omega^2} \leq C_1 \int_\Omega |\nabla u|^2. \tag{1.1}$$

   This inequality is known as the "Hardy" inequality in the literature. It holds for domains which are more general than simply connected but does not hold for all domains, see [2]. The survey paper [11] contains a detailed account of this inequality as of around 1998. For some recent work, please see [1], [12], [13], [15], [17], [19] and references therein. In the setting of simply connected domains the inequality can be easily reduced to that of the unit disc or half–space with the aid of the Koebe $\frac{1}{4}$-theorem. In fact, the Koebe $\frac{1}{4}$-theorem proof gives the inequality with $C_1 = 16$, (see [2]).

2. There exists a positive constant C_2, independent of the domain, such that

$$\frac{C_2}{R_\Omega^2} \leq \lambda_1 \leq \frac{j_0^2}{R_\Omega^2}. \tag{1.2}$$

   The right hand side inequality is trivial by domain monotonicity of the eigenvalue–the larger the domain the smaller the eigenvalue. The constant j_0 is the smallest positive zero of the first Bessel function J_0. Of course, the right hand side inequality is sharp. The left hand side inequality follows from the variational characterization of

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the eigenvalue and the Hardy inequality (1.1). As above, the left hand side inequality holds for more general domains than just simply connected domains but not all. (Adding points to a domain has no affect on the eigenvalue but it can have a drastic affect on the inradius.) This inequality also has a long and interesting history, see [3] and [4].

3. There exists a positive constant C_3, independent of the domain, such that

$$\frac{1}{2}R_\Omega^2 \leq \sup_{z\in\Omega} E_z(\tau_\Omega) \leq C_3 R_\Omega^2. \tag{1.3}$$

   Here we use E_z to denote the expectation with respect to the Brownian motion starting at the point $z \in \Omega$. Again, the lower bound is trivial by domain monotonicity (the larger the domain the larger the lifetime). A necessary and sufficient condition (which includes all simply connected domains in $\mathbb{R}^2$) for a domain in $\mathbb{R}^d$ to have (1.3) is given in [8]. Again, since Brownian motion does not "see" points in two dimensions, the right hand side inequality cannot hold for all domains.

4. There exist a positive constant C_4, independent of the domain, such that

$$\frac{C_4}{R_\Omega} \leq \sigma_\Omega \leq \frac{1}{R_\Omega}. \tag{1.4}$$

   As above, the upper bound is obtained by domain monotonicity and the existence of the constant C_4 follows at once from the Koebe $\frac{1}{4}$-theorem since $\sigma_\Omega(z) = \frac{1}{|F'(0)|}$, where F is the conformal mapping from the unit disc onto the domain Ω with $F(0) = z$.

Problem 1 *Identify the extremal constants C_1, C_2, C_3, C_4 in the above inequalities and the geometry of the "extremal" domains (whenever they exist).*

2 Convex domains

In the case of convex domains, all constants are known:

1. $C_1(convex) = 4$ which is the constant for the half space (or oven the one dimensional half-line). For a proof of this, see Davies [11]. There are also other sharper generalizations such as the one given in [1]. (Please also consult references given in [1] for more on these kind of extensions.) These results hold for convex domains in $\mathbb{R}^d$.

2. $C_2(convex) = \pi^2/4$ and the extremal domain is an infinite strip. The same constant works also for any convex domain in $\mathbb{R}^d$. There are several proofs of this result including the original one given by J. Hersh in [14]. (See also [1] for a proof based on the Hardy inequality and other references.)

3. $C_3(convex) = 1$ (see R. Sperb in [18]). Again, the extremal is given by an infinite strip (which reduces the problem to an interval). Here again, there is a more general inequality which asserts that for any convex domain in $\mathbb{R}^d$ of inradius R_Ω,

$$P_z\{\tau_\Omega > t\} \le P_0\{\tau_{(-R_\Omega, R_\Omega)} > t\}, \tag{2.1}$$

where $\tau_{(-R_\Omega,R_\Omega)}$ is the exit time from the interval $(-R_\Omega, R_\Omega)$ on the real line. (For this, see [6] and [7].) The inequality (2.1) together with the well-known classical characterization of the the eigenvalue as

$$-\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log P_z\{\tau_\Omega > t\}$$

gives a different proof that $C_2(convex) = \pi^2/4$. Again, the same results holds in all dimensions where the extremal domain is the infinite slab.

4. $C_4(convex) = \pi/4$. This result was proved by Szegö in 1923 (see [3] for exact reference). Again, the extremal domain is the infinite strip.

3 Arbitrary simply connected domains

The following estimates for the optimal constants C_1, C_2, C_3, C_4 are known.

$$\begin{aligned}
4 \le\ & C_1 \le 16 && (3.1)\\
0.6194 <\ & C_2 < 2.095 && (3.2)\\
1.584 <\ & C_3 < 3.228 && (3.3)\\
0.57088 <\ & C_4 < 0.6563937 && (3.4)
\end{aligned}$$

For the estimates for C_2 and C_3, and some history on these constants, we refer the reader to [3] and [9]. The paper [3] also contains some examples of simply connected domains which we conjecture are very close to the extremals for these four problems. The problem of determining the best constant C_4 (known as the Schlicht Bloch-Landau constant) has a long history in function theory. For the above estimates on C_4 we refer the reader to [16] and [10] and [9]. (The reference [10] contains many references to the literature on the Schlicht Bloch-Landau constant.) The upper estimate for C_3 follow from the lower estimate on C_4 and inequality (3.5) below. From the upper estimate on C_3 we get a lower estimate on C_2 using (3.6). The lower estimate for C_3 and upper estimate on C_2 follow from the example in [3], (see Theorems 2 and 3) and the calculations in [9]. For an approach using a Hardy-type inequality with σ_Ω replacing the distance function, see [5].

Theorem 3.1 ([3]) *For any simply connected domain $\Omega \subset \mathbb{R}^2$, we have*

$$\frac{1}{2\sigma_\Omega^2} \le \sup_{z\in\Omega} E_z(\tau_\Omega) \le \frac{7\zeta(3)}{8\sigma_\Omega^2} \tag{3.5}$$

and

$$\frac{2}{\sup_{z\in\Omega} E_z(\tau_\Omega)} \le \lambda_\Omega \le \frac{7\zeta(3) j_0^2}{8 \sup_{z\in\Omega} E_z(\tau_\Omega)}, \tag{3.6}$$

where $7\zeta(3)/8 = \sum_{n=0}^\infty (2n+1)^{-3}$.

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