SHARP INEQUALITIES FOR THE BEURLING-AHLFORS TRANSFORM ON RADIAL FUNCTIONS

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ABSTRACT. For $1 \leq p \leq 2$, we prove sharp weak-type (p, p) estimates for the Beurling-Ahlfors operator acting on the radial function subspaces of $L^p(\mathbb{C})$. A similar sharp L^p result is proved for $1 < p \leq 2$. The results are derived from martingale inequalities which are of independent interest.

1. INTRODUCTION

The Beurling-Ahlfors transform is the singular integral operator acting on $L^p(\mathbb{C})$, defined by the formula

$$Bf(z) = -\frac{1}{\pi}\text{p.v.} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} \mathrm{d}w,$$

where p.v. means the principal value and the integration is with respect to the Lebesgue measure on the plane. This operator plays a fundamental role in the study of quasiconformal mappings and partial differential equations. For an overview of some of these applications, see [1] and references therein. An important and interesting open problem is that of finding the precise value of the L^p norms of this operator. This question has gained considerable interest in the literature and the long standing conjecture of T. Iwaniec [12] states that for $1 < p < \infty$,

$$||B||_{L^p(\mathbb{C}) \to L^p(\mathbb{C})} = p^* - 1,$$

where $p^* = \max\{p, p/(p-1)\}$. While the lower bound of $p^* - 1$ was obtained by Lehto [15], the question about the upper bound remains open. So far, the best result in this direction is the inequality $||B||_{L^p(\mathbb{C}) \to L^p(\mathbb{C})} \leq 1.575(p^* - 1)$, established in [2] using the probabilistic techniques of Burkholder [7], [8]. Asymptotically a better estimate has been obtained in [6] where it is proved that $||B||_{L^p(\mathbb{C}) \to L^p(\mathbb{C})} \leq 1.3992\, p$, as $p \to \infty$, which is an improvement over the previous well known asymptotic estimate $||B||_{L^p(\mathbb{C}) \to L^p(\mathbb{C})} \leq \sqrt{2}\, p$, as $p \to \infty$

As a Calderón–Zygmund singular integral, the Beurling-Ahlfors operator is also of weak-type $(1, 1)$, i.e., it maps $L^1(\mathbb{C})$ into weak–$L^1(\mathbb{C})$. A problem of interest also is to determine the best constant in the weak $(1, 1)$ inequality; see [18]. In [3], the action of the operator on the space of real-valued radial functions on $\mathbb{C}$ is studied and the weak-type constant on such functions is identified. We say that $F : \mathbb{C} \to \mathbb{R}$ is radial if it satisfies $F(z) = F(|z|)$ for all $z \in \mathbb{C}$. Here is one of the results proved in [3].

Theorem 1.1. *Suppose that F is a real-valued radial function. Then for any $\lambda > 0$ we have*

$$\lambda|\{z \in \mathbb{C} : |BF(z)| \geq \lambda\}| \leq \frac{1}{\log 2}||F||_{L^1(\mathbb{C})}$$

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and the inequality is sharp.

The paper [3] contains also some related L^p inequalities for the Hardy–type operators that arise when studying the action of the Beurling-Ahlfors operator on radial functions; see §4 and §5 below. Here we also refer the reader to Gill's paper [14] which identifies the optimal weak-type $(1,1)$ constant for the adjoint of the operator used in [3] in the proof of Theorem 1.1. We would also like to mention here the paper [19] by Volberg, which is devoted to the study of related objects.

The objective of this paper is to strengthen the above results in several directions. First of all, we shall work in the vector-valued setting. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. With no loss of generality of the results, this space will be assumed to be $\ell^2_{\mathbb{R}}$. For any p-integrable function $f=(f_1,f_2,\ldots):\mathbb{C}\to\mathcal{H}$ we define Bf coordinatewise: that is, we set

$$Bf=(Bf_1,Bf_2,\ldots)\in\ell^2_{\mathbb{C}}.$$

We shall determine the best constant in the corresponding weak type (p,p) estimate for the range $1\leq p\leq 2$. It will follow from the considerations of Section 2 that for $p\in[1,2]$, the equation

$$e^x\int_x^\infty e^{-s}|1-s|^p\mathrm{d}s=|1-x|^p$$

has a unique nonnegative solution $x=x_p$. Define

$$C_p=|1-x_p|^{-p}. \tag{1.1}$$

It is simple to verify that $C_1=\frac{1}{\log 2}$ and that $C_2=1$. We shall establish the following statement.

Theorem 1.2. *Suppose that $F:\mathbb{C}\to\mathcal{H}$ is a radial function. Then for any $\lambda>0$ and $1\leq p\leq 2$ we have*

$$\lambda^p|\{z\in\mathbb{C}:|BF(z)|\geq\lambda\}|\leq C_p||F||^p_{L^p(\mathbb{C})}. \tag{1.2}$$

The constant is the best possible.

In contrast with [3] which reduces to a careful analysis of a Hardy operator, our proof here will depend heavily on probabilistic techniques. We shall relate (1.2) to a certain sharp martingale inequality, which is interesting on its own. Furthermore, we shall combine this approach with classical estimates of Burkholder and obtain refined versions of certain L^p-bounds studied in [3].

Before we proceed, we remark that since $||B||_{L^p(\mathbb{C})\to L^p(\mathbb{C})}\leq 1.575(p^*-1)$, as mentioned above, Chebyshev's inequality immediately implies that

$$\lambda^p|\{z\in\mathbb{C}:|Bf(z)|\geq\lambda\}|\leq(1.575(p^*-1))^p\,||f||^p_{L^p(\mathbb{C})} \tag{1.3}$$

for all $1<p<\infty$ and all $f:\mathbb{C}\to\mathbb{C}$. A better upper bound than (1.3) is obtained in [16]. That paper also gives a lower bound estimate on the best constant in this weak-type inequality and for other weak-type inequalities for operators obtained from projections of martingale transforms of stochastic integrals. For a precise statement of these results, we refer the reader to [16]. (See also Remark (4.1) to see how these vector valued type inequalities for $\ell^2_{\mathbb{R}}$ imply the same results for $\ell^2_{\mathbb{C}}$.)

We have organized this paper as follows. The next section contains some auxiliary technical material, which is needed in our further considerations. Section 3 is devoted to the probabilistic version of Theorem 1.2. In Section 4, we deduce the estimate (1.2) and construct a family of examples which prove the optimality of the constant C_p. Finally, in Section 5 we study sharp L^p estimates for the Beurling-Ahlfors operator acting on radial functions. While these type of results have been obtained in [3] (see also [5]), our probabilistic proof here may be of independent interest.

2. Some technical lemmas

Consider the family $(\varphi_a)_{a\in(0,\infty)}$ of functions on $\mathbb{R}$, given by

$$\varphi_a(t) = \begin{cases} |t|^2 & \text{if } |t| \leq a, \\ 2a|t| - a^2 & \text{if } |t| > a \end{cases}$$

and for any $a > 0$, define $g_a : [0,\infty) \to \mathbb{R}$ by the formula

$$g_a(x) = e^x \int_x^\infty e^{-s}\varphi_a(s-1)\mathrm{d}s = \int_0^\infty e^{-s}\varphi_a(s+x-1)\mathrm{d}s. \tag{2.1}$$

A direct computation leads to the following explicit expressions for these functions:

$$g_a(x) = \begin{cases} 2e^{x+a-1}(1-e^{-2a}) - 2ax - a^2 & \text{if } x < (1-a)_+, \\ x^2 + 1 - 2e^{x-a-1} & \text{if } (1-a)_+ \leq x < 1+a, \\ 2ax - a^2 & \text{if } x \geq a+1. \end{cases}$$

Here, as usual, $x_+ = \max\{x, 0\}$ denotes the positive part of the real number x. In what follows, we shall require some additional properties of the function g_a, which are proved in the Lemmas below.

Lemma 2.1. *For any $a > 0$ we have*

$$g_a(0) < \varphi_a(1). \tag{2.2}$$

Proof. When $a < 1$, the estimate reads $2e^{a-1}(1-e^{-2a}) - 2a < 0$ and it suffices to note that the function $a \mapsto 2e^{a-1} - 2e^{-a-1} - 2a$ is strictly convex on $[0,1]$ and nonpositive at the endpoints of this interval. For $a \geq 1$, (2.2) can be rewritten as $1 - 2e^{-a-1} < 1$, which is evident. □

We turn to the main technical fact. We use the notation $x' = x/|x|$ for $x \neq 0$ and $x' = 0$ otherwise.

Lemma 2.2. *Let $a > 0$ and suppose that x, $d \in \mathcal{H}$ satisfy $|d| \geq 1$. Then*

$$\varphi_a(|x+d|) - g_a(|x|) + \big(-g_a(|x|) + \varphi_a(1-|x|)\big)\langle x', d\rangle \geq 0. \tag{2.3}$$

Proof. We consider several cases separately.

1° The case $|x| \leq 1-a$. Then $|x+d| \geq |d| - |x| \geq a$ and the estimate is equivalent to

$$a(|x+d| + |x| + \langle x', d\rangle) - e^{|x|+a-1}(1-e^{-2a})(1+\langle x', d\rangle) \geq 0. \tag{2.4}$$

We carry out the following optimization procedure. Note that if we add to d a vector v which is orthogonal to x, then all the summands above are unchanged, except for $|x+d|$. If v is chosen properly, this norm can be decreased (and thus the bound strengthens) - unless $|d| = 1$ or x and d are linearly dependent. Consequently, it suffices to prove the estimate under the assumption that one of these two possibilities occurs.

(a) If $|d| = 1$, fix $|x|$, let $s = \langle x', d\rangle \in [-1,1]$, and denote the left hand side of (2.4) by $L(s)$. Then L is convex, $L(-1) = 0$ and

$$\lim_{s\downarrow -1} L'(s) = ae^{|x|-1}\left(\frac{e^{-|x|+1}}{1-|x|} - \frac{2\sinh a}{a}\right).$$

Since $a \mapsto \sinh a/a$ is increasing, the expression in the parentheses is not smaller than

$$\frac{e^{-|x|+1}}{1-|x|} - \frac{2\sinh(1-|x|)}{1-|x|} = \frac{e^{|x|-1}}{1-|x|} > 0.$$

Therefore, $L \geq 0$ on $[-1,1]$.

(b) Now we turn to the second possibility, in which x and d are linearly dependent. If $\langle x', d\rangle < 0$, then $|x+d| + |x| + \langle x', d\rangle \geq 0$ and $1 + \langle x', d\rangle \leq 0$, so (2.4) holds. If $\langle x', d\rangle \geq 0$, the inequality becomes

$$2a(|x|+|d|) - e^{|x|+a-1}(1-e^{-2a})(1+|d|) \geq 0.$$

The left-hand side, as a function of $|d|$, is increasing: we have $2a > 1 - e^{-2a}$ and $e^{|x|+a-1} \leq 1$. Thus it suffices to prove the bound for $|d| = 1$ and we have already done this in (a).

2° The case $1 - a < |x| < 1 + a$*,* $|x+d| \leq a$. The inequality (2.3) is equivalent to

$$|d|^2 - 1 + 2e^{|x|-1-a}(1 + \langle x', d\rangle) \geq 0.$$

However, $|d|^2 - 1 = (|d|+1)(|d|-1) \geq 2(|d|-1)$ and $2e^{|x|-1-a}(1+\langle x', d\rangle) \geq 2(1-|d|)$, so the bound is valid.

3° The case $1 - a < |x| < 1 + a$*,* $|x+d| > a$. The inequality (2.3) becomes

$$2a|x+d| - a^2 - |x|^2 - 2xd - 1 + 2e^{|x|-a-1}(1 + \langle x', d\rangle) \geq 0. \tag{2.5}$$

Arguing as in the first case, we see that it suffices to prove this for $|d| = 1$ or when x and d are linearly dependent. If $|d| = 1$, the inequality can be transformed into

$$2e^{|x|-1}(1 + \langle x', d\rangle) \geq (|x+d| - a)^2 e^a. \tag{2.6}$$

The derivative of the right-hand side, with respect to a, is

$$e^a(|x+d|-a)(|x+d|-a-2) \leq e^a(|x+d|-a)(|x|-a-1) \leq 0.$$

In addition, we have $a > |1 - |x||$ by the assumptions of 3°. Thus we will be done if we show (2.6) for $a = 1 - |x|$ or $a = |x| - 1$, depending on whether $|x| \leq 1$ or $|x| \geq 1$. The first case has been already considered in 1° above, the second one will be studied in 4° and 5° below. Next, suppose that x, d are linearly dependent. If $\langle x, d\rangle \geq 0$, then (2.5) reads

$$2a|x| + 2a|d| - a^2 - |x|^2 - 2|x||d| - 1 + 2e^{|x|-a-1}(1+|d|) \geq 0.$$

The left hand side is a nondecreasing function of $|d|$, so it suffices to check the bound for $|d| = 1$, which has been already done above. If $\langle x, d\rangle < 0$, then $|x+d| = |d| - |x|$ (otherwise the assumptions of the Case would be violated: $a < |x+d| = |x| - |d| < 1 + a - |d|$) and (2.5) becomes

$$2a|d| - 2a|x| - a^2 - |x|^2 + 2|x||d| - 1 + 2e^{|x|-a-1}(1-|d|) \geq 0.$$

The left hand side is a nondecreasing function of $|d|$, because $a + |x| > 1 \geq e^{|x|-a-1}$. Thus we deduce the bound from the case $|d| = 1$ which has been already considered.

4° The case $|x| \geq 1 + a$*,* $|x+d| < a$. This time (2.3) takes the form

$$|x+d|^2 - 2a|x| + a^2 - 2a\langle x', d\rangle \geq 0.$$

However, since $|x| + \langle x', d\rangle \leq |x+d|$, the left hand side is not smaller than $|x+d|^2 - 2a|x+d| + a^2 = (|x+d|-a)^2 \geq 0$.

5° The case $|x| \geq 1 + a$*,* $|x+d| \geq a$. Then (2.3) is equivalent to the estimate

$$2a(|x+d| - |x| - \langle x', d\rangle) \geq 0,$$

which is obvious.

The proof is complete. □

Now we introduce another family of functions. For $1 \leq p \leq 2$, let $h_p : [0, \infty) \to \mathbb{R}$ be given by

$$h_p(x) = e^x \int_x^\infty e^{-s}|1-s|^p \mathrm{d}s = \int_0^\infty e^{-s}|1-s-x|^p \mathrm{d}s.$$

It is clear from the second integral that the function h_p is convex (even strictly convex, provided $p > 1$) and $\lim_{x\to\infty} h_p(x) = \infty$. Furthermore, for $1 < p < 2$ there is a very useful representation of h_p in terms of the previous family $\{g_a\}_{a\in(0,\infty)}$. Namely, we have the identity

$$|t|^p = \frac{p(p-1)(2-p)}{2} \int_0^\infty a^{p-3}\varphi_a(t)\mathrm{d}s, \qquad t \in \mathbb{R}, \tag{2.7}$$

and hence it follows from Fubini's theorem that

$$h_p(x) = \frac{p(p-1)(2-p)}{2} \int_0^\infty a^{p-3} g_a(x)\mathrm{d}x. \tag{2.8}$$

We shall need the following further properties of h_p.

Lemma 2.3. *Let $1 \leq p \leq 2$ be fixed.*

(i) If $p \neq 2$, then h_p is decreasing in a certain neighborhood of 0. The function h_2 is increasing on $[0, \infty)$.

(ii) For all x, $d \in \mathcal{H}$ with $|d| \geq 1$ we have

$$|x+d|^p - h_p(|x|) + (-h_p(|x|) + |1-|x||^p)d \geq 0. \tag{2.9}$$

Proof. (i) First we consider the case in which p lies strictly between 1 and 2. We derive that

$$\lim_{x\downarrow 0} h_p'(x) = h_p(0) - 1 = \frac{p(p-1)(2-p)}{2} \int_0^\infty a^{p-3}\big[g_a(0) - \varphi_a(1)\big]\mathrm{d}a,$$

which is negative in view of (2.2). If $p = 1$, then we compute that

$$h_1(x) = \begin{cases} -x + 2e^{x-1} & \text{if } x \leq 1, \\ x & \text{if } x > 1 \end{cases}$$

and the monotonicity is obvious. Finally, when $p = 2$, we have $h_2(x) = 1 + x^2$ so the claim is trivial.

(ii) If $1 < p < 2$, the bound is an immediate consequence of (2.3), (2.7), (2.8) and Fubini's theorem. The case $p \in \{1, 2\}$ follows by continuity. □

Combining the above facts, we see that if $1 \leq p \leq 2$, then the function h_p attains its minimum at a certain point x_p. Furthermore, $x_p \leq 1$, since $h_p'(1) = h_p(1) > 0$, and $x_p > 0$ if and only if $p < 2$. The final observation is that $h_p'(x) = h_p(x) - |1-x|^p$, so the constant C_p given by (1.1) satisfies

$$C_p = \big[h_p(x_p)\big]^{-1}.$$

3. A MARTINGALE INEQUALITY

As we have already announced in the Introduction, the heart of the matter lies in proving an appropriate martingale inequality. Let us start with introducing the necessary background. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$ be an adapted continuous-time martingale, taking values in the Hilbert space $\mathcal{H}$. As usual, we assume that the trajectories of X are right-continuous and have limits from the left. We shall also use the notation $\Delta X_t = X_t - X_{t-}$ for the jump of X at time $t > 0$. The symbol $[X, X]$ will stand for the quadratic covariance process of X: that is, we put $[X, X]_t = \sum_{j=1}^{\infty}[X^j, X^j]_t$, where X^j denotes the j-th coordinate of X. Here for real valued martingale M, $[M, M]$ is the usual square bracket of M. That is, $[M, M]_t = |M_0|^2 + [M^c, M^c]_t + \sum_{0<s\leq t}|\Delta M_s|^2$, where $[M^c, M^c]_t$ is the quadratic variation of the pathwise continuous part of the martingale M and in fact $[M^c, M^c]_t = [M, M]_t^c$, the continuous part of $[M, M]_t$. For the details of this construction, see Dellacherie and Meyer [10]. Furthermore, $X^* = \sup_{t\geq 0}|X_t|$ and $||X||_p = \sup_{t\geq 0}||X_t||_p$ will denote the maximal function and the p-th moment of the process X, respectively.

The main result of this section can be stated as follows. Recall C_p given by (1.1).

Theorem 3.1. *For any $1 \leq p \leq 2$ and any martingale X we have*

$$\mathbb{P}((\Delta X)^* \geq 1) \leq C_p||X||_p^p. \tag{3.1}$$

The constant is the best possible.

The case $p = 1$ of this result is closely related to the inequality proved by Cox and Kemperman in [9] for discrete martingales. However, our proof here is entirely different from that in [9]. We shall make use of a novel approach which can be regarded as a modification of Burkholder's method: namely, we shall exploit the properties of certain special functions. (For the detailed description of Burkholder's method, see the survey [8] concerning the discrete-time case and consult Wang [20] for the necessary changes which must be implemented in the continuous-time setting).

For a fixed $1 \leq p \leq 2$, introduce U_p, $V_p : \mathcal{H} \times \{0, 1\} \to \mathbb{R}$ by the following formulas:

$$U_p(x, y) = \begin{cases} C_p^{-1} - |x|^p & \text{if } y = 1, \\ 0 & \text{if } |x| \leq x_p,\ y = 0, \\ C_p^{-1} - h_p(|x|) & \text{if } |x| > x_p,\ y = 0 \end{cases}$$

and $V_p(x, y) = C_p^{-1}1_{\{y=1\}} - |x|^p$. Here 1_A denotes the indicator of a set A.

Lemma 3.1. *Let $1 \leq p \leq 2$. The functions U_p and V_p enjoy the following properties.*

(i) We have

$$U_p(x, 0) \leq 0 \qquad \text{for } x \in \mathcal{H}. \tag{3.2}$$

(ii) For any x, $d \in \mathcal{H}$ and $y \in \{0, 1\}$ we have

$$U_p(x + d, 1 - (1 - y)1_{\{|d|<1\}}) \leq U_p(x, y) + \langle U_{px}(x, y), d\rangle. \tag{3.3}$$

(iii) We have the majorization

$$U_p(x, y) \geq V_p(x, y) \qquad \text{for } x \in \mathcal{H} \text{ and } y \in \{0, 1\}. \tag{3.4}$$

Proof. The condition (i) is obvious with our definition of C_p. To show (ii), suppose first that $y = 1$: then (3.3) follows immediately from the concavity of the function $U_p(\cdot, 1)$. If $y = 0$ and $|d| < 1$, then the estimate can be rewritten in the form

$$U_p(x + d, 0) \le U_p(x, 0) + \langle U_{px}(x, 0), d\rangle$$

and follows from concavity of $U_p(\cdot, 0)$ (which is evident in view of the results from the previous section). Finally, suppose that $y = 0$ and $|d| \ge 1$; then (3.3) reads

$$U_p(x + d, 1) \le U_p(x, 0) + \langle U_{px}(x, 0), d\rangle.$$

If $|x| \ge x_p$, this is equivalent to (2.9); if $|x| < x_p$, the bound becomes $C_p^{-1} - |x+d|^p \le 0$. It suffices to observe that by the triangle inequality,

$$C_p^{-1} - |x + d|^p \le C_p^{-1} - |1 - x_p|^p = 0.$$

(iii) We may restrict ourselves to $\mathcal{H} = \mathbb{R}$ and it suffices to prove the majorization for $x \ge 0$ and $y = 0$. The bound is evident on $[0, x_p]$. We shall prove that the difference $D_p(x) = U_p(x, 0) - V_p(x, 0)$ is increasing on $[x_p, \infty)$; this will immediately give the claim. We have $D_p'(x) = e^x G_p(x)$, where

$$G_p(x) = -\int_x^\infty e^{-s}|1 - s|^p \mathrm{d}s + e^{-x}(|1 - x|^p + px^{p-1}), \qquad x \ge 0.$$

Next, $G_p'(x) = -pe^{-x}H_p(x)$, with

$$H_p(x) = x^{p-1} - |x - 1|^{p-1}\mathrm{sgn}\,(x - 1) - (p - 1)x^{p-2}, \qquad x > 0,$$

and, finally, we have

$$H_p'(x) = (p - 1)[x^{p-2} - (1 - x)^{p-2} - (p - 2)x^{p-3}].$$

The expression in the square brackets is decreasing on $(0, 1)$ and the limits at the endpoints 0, 1 are equal to $+\infty$ and $-\infty$, respectively. Thus, H increases on $[0, a_p]$ and decreases on $[a_p, 1]$ for some $a_p \in (0, 1)$. Furthermore, $\lim_{x\downarrow 0} H_p(x) = -\infty$ and $H_p \ge 0$ on $[1, \infty)$, due to the mean value property. These facts imply that G_p increases on $[0, b_p]$ and decreases on $[b_p, \infty)$ for some $b_p \in [0, 1]$. However, we easily check that $G_p(x_p) = pe^{-x_p}x_p^{p-1} > 0$ and $\lim_{x\to\infty} G_p(x) = 0$. Consequently, G_p is positive on the half-line $[x_p, \infty)$ and we are done. □

Proof of (3.1). By standard approximation, we may assume that $\mathcal{H}$ is finite-dimensional: $\mathcal{H} = \mathbb{R}^d$ for some $d \ge 1$. Fix a positive number δ and let $\varphi : \mathbb{R}^d \to [0, \infty)$ be a C^∞ function, supported on the unit ball of $\mathbb{R}^d$ and satisfying $\int_{\mathbb{R}^d} \varphi = 1$. Let U^δ, V^δ : $\mathbb{R}^d \times \{0, 1\} \to \mathbb{R}$ be given by the convolutions

$$U_p^\delta(x, y) = \int_{\mathbb{R}^d} U_p(x + \delta r, y)\varphi(r)\mathrm{d}r,$$

$$V_p^\delta(x, y) = \int_{\mathbb{R}^d} V_p(x + \delta r, y)\varphi(r)\mathrm{d}r.$$

These two functions inherit the properties listed in Lemma 3.1. We have

$$U_p^\delta(x, 0) \le 0 \qquad \text{for } x \in \mathbb{R}^d, \tag{3.5}$$

$$U_p^\delta(x + d, 1 - (1 - y)1_{\{|d|<1\}}) \le U_p^\delta(x, y) + \langle U_{px}^\delta(x, y), d\rangle, \tag{3.6}$$

for x, $d \in \mathbb{R}^d$, $y \in \{0, 1\}$, and

$$U_p^\delta(x, y) \ge V_p^\delta(x, y) \ge C_p^{-1}1_{\{y=1\}} - ||x| - \delta|^p \tag{3.7}$$

for $x \in \mathbb{R}^d$ and $y \in \{0,1\}$. Indeed, (3.5) and (3.7) are obvious and (3.6) follows from the fact that U_p is of class C^1. Finally, observe that $U_p^\delta(\cdot,0)$ is concave, since so is U_p.

Now fix a martingale X as in the statement. Obviously, we may and do assume that X is bounded in L^p, since otherwise there is nothing to prove. Introduce the stopping time $\tau = \inf\{s \geq 0 : |\Delta X_s| \geq 1\}$. We see that $U_p^\delta(\cdot,0)$ is of class C^∞ and the process $(X_t, 1_{\{|\Delta X_t|\geq 1\}})$ takes values in $\mathbb{R}^d \times \{0\}$ on the time interval $[0,\tau)$. Therefore, an application of Itô's formula for processes with jumps (see Protter [17]) yields

$$U_p^\delta(X_{\tau\wedge t}, 1_{\{|\Delta X_{\tau\wedge t}|\geq 1\}}) = I_0 + I_1 + I_2/2 + I_3, \tag{3.8}$$

where

$$\begin{aligned}
I_0 &= U_p^\delta(X_0, 0),\\
I_1 &= \int_{0+}^{\tau\wedge t} U_{px}^\delta(X_{s-},0)\cdot \mathrm{d}X_s,\\
I_2 &= \sum_{k=1}^d\sum_{\ell=1}^d \int_{0+}^{\tau\wedge t} U_{px_kx_\ell}^\delta(X_{s-},0)\mathrm{d}[X^k,X^\ell]_s^c,\\
I_3 &= \sum_{0<s\leq\tau\wedge t}\Big[U_p^\delta(X_s, 1_{\{|\Delta X_s|\geq 1\}}) - U_p^\delta(X_{s-},0) - \langle U_{px}^\delta(X_{s-},0), \Delta X_s\rangle\Big].
\end{aligned}$$

Let us analyze the terms I_0 through I_3. The first expression is nonpositive, in virtue of (3.5). The second one has zero expectation, by the properties of stochastic integrals. Next, we have $I_2 \leq 0$, because the function $U_p^\delta(\cdot,0)$ is concave, as we have observed above. Finally, each summand in I_3 is nonpositive, which is guaranteed by (3.6). Consequently, integrating both sides of (3.8) gives $\mathbb{E}U_p^\delta(X_{\tau\wedge t}, 1_{\{|\Delta X_{\tau\wedge t}|\geq 1\}}) \leq 0$, and combining this with (3.7) yields $\mathbb{P}(|\Delta X_{\tau\wedge t}| \geq 1) \leq C_p\mathbb{E}||X_{\tau\wedge t}| - \delta|^p$. Letting $\delta \to 0$ and using Lebesgue's dominated convergence theorem, we obtain

$$\mathbb{P}(|\Delta X_{\tau\wedge t}| \geq 1) \leq C_p\mathbb{E}|X_{\tau\wedge t}|^p \leq C_p||X||_p^p.$$

Next, for each $\varepsilon > 0$ we have

$$\{(\Delta X)^* \geq 1\} \subset \bigcup_{t\geq 0}\{|\Delta X_{\tau\wedge t}| \geq 1-\varepsilon\}$$

and the events on the right are increasing. Consequently, applying the preceding inequality to the martingale $X/(1-\varepsilon)$ and letting $t\to\infty$ gives

$$\mathbb{P}((\Delta X)^* \geq 1) \leq C_p(1-\varepsilon)^{-p}||X||_p^p.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Sharpness. This will be clear by the examples considered in the next section. □

4. The weak-type bound for the Beurling-Ahlfors operator

We start from describing the action of the Beurling-Ahlfors operator on the class of radial functions. For any square-integrable $f : [0,\infty) \to \mathbb{R}$, let F be the associated radial function, given by $F(z) = f(|z|^2)$, $z \in \mathbb{C}$. Then, as proved in [3], we have

$$BF(z) = \frac{\bar{z}^2}{|z|^2}\Lambda f(|z|^2),$$

where the operator Λ is defined by

$$\Lambda f(u) = \frac{1}{u}\int_0^u f(v)\mathrm{d}v - f(u). \tag{4.1}$$

This can be easily extended to the vector valued case. For any $f = (f_1, f_2, \ldots) : [0,\infty) \to \mathcal{H}$, we see that

$$BF(z) = \left(\frac{\bar{z}^2}{|z|^2}\Lambda f_1(|z|^2), \frac{\bar{z}^2}{|z|^2}\Lambda f_2(|z|^2), \ldots\right) \in \ell^2_{\mathbb{C}}. \tag{4.2}$$

We can also define the action of Λ on $\mathcal{H}$-valued functions, simply by noting that (4.1) makes sense in the vector setting.

Remark 4.1. We would like to stress here that although we have assumed that $\mathcal{H}$ is a real Hilbert space, our results are also valid for functions taking values in complex Hilbert spaces $\mathcal{K}$. To see this, assume that $\mathcal{K} = \ell^2_{\mathbb{C}}$, take $f = (f_1 + ig_1, f + ig_2, \ldots) : [0,\infty) \to \mathcal{K}$ and let $F = (F_1 + iG_1, F_2 + iG_2, \ldots)$ be the corresponding radial function on $\mathbb{C}$. Obviously, we have

$$|F(z)|_{\ell^2_{\mathbb{C}}} = |(F_1, G_1, F_2, G_2, \ldots)(z)|_{\ell^2_{\mathbb{R}}}, \qquad z \in \mathbb{C}.$$

Furthermore, for any k and $z \in \mathbb{C}$,

$$\begin{aligned}|B(F_k + iG_k)(z)| &= \left|\frac{\bar{z}^2}{|z|^2}\left(\Lambda f_k(|z|^2) + i\Lambda g_k(|z|^2)\right)\right| \\ &= \left(|\Lambda f_k(|z|^2)|^2 + |\Lambda g_k(|z|^2)|^2\right)^{1/2} = |(BF_k(z), BG_k(z))|,\end{aligned}$$

which gives $|BF(z)|_{\ell^2_{\mathbb{C}}} = |B(F_1, G_1, F_2, G_2, \ldots)|_{\ell^2_{\mathbb{R}}}$. Thus any bound for radial $\ell^2_{\mathbb{R}}$-valued functions yields the corresponding estimate for radial functions with values in $\ell^2_{\mathbb{C}}$.

Now we turn to Theorem 1.2.

Proof of (1.2). Let $f : [0,\infty) \to \mathcal{H}$ be a bounded function, supported on a certain finite interval $[0, M]$. Consider the probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) = ([0, M], \mathcal{B}([0, M]), m(\cdot)/M),$$

where m denotes the Lebesgue's measure. For any $t \in [0, M]$, let $\mathcal{F}_t$ be the smallest complete σ-field which contains the interval $[0, M-t]$ and all Borel subsets of $[M-t, M]$; for $t > M$, put $\mathcal{F}_t = \mathcal{F}$. Obviously, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration, f can be regarded as an integrable random variable and thus the process

$$X = (X_t)_{t\geq 0} = (\mathbb{E}(f|\mathcal{F}_t))_{t\geq 0}$$

is a martingale. It is easy to see that for almost all $\omega \in \Omega$,

$$X_t(\omega) = \begin{cases} f(\omega) & \text{if } t \geq M - \omega, \\ \frac{1}{M-t}\int_0^{M-t} f(s)\mathrm{d}s & \text{if } t < M - \omega. \end{cases}$$

Of course, we have

$$(\Delta X)^*(\omega) \geq |\Delta X_{M-\omega}| = \left|\frac{1}{\omega}\int_0^\omega f(s)\mathrm{d}s - f(\omega)\right| = |\Lambda f(\omega)|$$

and hence

$$\begin{aligned} m(\{x \in (0, M) : |\Lambda f(x)| \geq 1\}) &\leq M\mathbb{P}(|\Delta X|^* \geq 1) \\ &\leq C_p M||X||_p^p = C_p\int_0^\infty |f(x)|^p \mathrm{d}x. \end{aligned}$$

Letting $M \to \infty$ gives

$$m(\{x \in (0,\infty) : |\Lambda f(x)| \geq 1\}) \leq C_p ||f||^p_{L^p([0,\infty))}. \tag{4.3}$$

However, by (4.2),

$$\begin{aligned} |\{z \in \mathbb{C} : |BF(z)| \geq 1\}| &= |\{z \in \mathbb{C} : |\Lambda f(|z|^2)| \geq 1\}| \\ &= \int_0^\infty 1_{\{|\Lambda f(r^2)| \geq 1\}}(s) \cdot 2\pi s \,\mathrm{d}s \\ &= \pi \int_0^\infty 1_{\{|\Lambda f(r)| \geq 1\}}(s) \,\mathrm{d}s \\ &= \pi m(\{r \in [0,\infty) : |\Lambda f(r)| \geq 1\}) \end{aligned} \tag{4.4}$$

and, by a similar calculation, $||F||^p_{L^p(\mathbb{C})} = \pi ||f||^p_{L^p(0,\infty)}$. Plugging these two identities into (4.3) yields the weak-type estimate for compactly supported functions. The general case follows immediately by a standard approximation. □

Sharpness. Fix $1 \leq p \leq 2$ and consider the function $f : [0,\infty) \to \mathbb{R}$ given by $f(s) = x_p - 1 - \log s$ for $s \in [0,1]$ and $f(s) = 0$ for remaining s. We easily derive that for $u \leq 1$,

$$\Lambda f(u) = \frac{1}{u}\int_0^u (x_p - 1 - \log v)\mathrm{d}v - (x_p - 1 - \log u) = 1$$

and

$$||f||^p_{L^p([0,\infty))} = \int_0^1 |x_p - 1 - \log s|^p \mathrm{d}s = \int_0^\infty |x_p - 1 + t|^p e^{-t}\mathrm{d}t = h_p(x_p) = C_p^{-1},$$

so that the corresponding radial function F satisfies

$$\begin{aligned} &|\{z \in \mathbb{C} : |BF(z)| \geq 1\}| - C_p ||F||^p_{L^p(\mathbb{C})} \\ &\qquad = \pi \Big[m(\{u : |\Lambda f(u)| \geq 1\}) - C_p ||f||^p_{L^p([0,\infty))} \Big] = 0. \end{aligned}$$

This proves the optimality of the constant C_p. □

5. L^p-ESTIMATES

The above stochastic representation of Λ, combined with Burkholder's estimate for differentially subordinate martingales, yields the following statement.

Theorem 5.1. *Assume that $\mathcal{H}_1$, $\mathcal{H}_2$ are orthogonal subspaces of $\mathcal{H}$ and let f_1, f_2 be two functions on $[0,\infty)$, taking values in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then for any $1 < p < \infty$ we have*

$$||\Lambda f_1 + f_2||_{L^p([0,\infty))} \leq (p^* - 1)||f_1 + f_2||_{L^p([0,\infty))}. \tag{5.1}$$

If $1 < p \leq 2$, then the constant is the best possible. It is already the best possible if $f_2 \equiv 0$ and f is assumed to be real-valued.

This generalizes Theorem 4.1 and Theorem 4.2 of [3]. More precisely, in the first statement it is proved in [3] that $||\Lambda||_{L^p([0,\infty);\mathbb{R})} = p^* - 1$ for $1 < p \leq 2$, while in the second (5.1) is studied for real-valued f_1 and purely imaginary f_2, with a slightly worse constant.

Recall that a martingale Y is differentially subordinate to a martingale X if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. Here is a celebrated L^p-estimate of Burkholder [7].

Theorem 5.2. *If X, Y are $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X, then*

$$||Y||_p \leq (p^* - 1)||X||_p, \qquad 1 < p < \infty.$$

The constant is the best possible.

Proof of (5.1). We may and do assume that f_1 and f_2 are supported on a certain interval $[0, M]$, where M is a positive integer. Denote by X^1, X^2 the associated martingales on the probability space $([0, M], \mathcal{B}([0, M]), m(\cdot)/M)$: see the previous section. By the very definition, we have that X^1 takes values in $\mathcal{H}^1$ and X^2 takes values in $\mathcal{H}^2$. Fix a positive integer n and consider an $\mathcal{H} \oplus \ell^2(\mathcal{H})$-valued martingale Y^n, given as follows: $Y_0^n = (X_0^2, X_0^1, 0, 0, \ldots)$ and, if $t > 0$ and k is an integer satisfying $k2^{-n} < t \leq (k+1)2^{-n}$, then Y_t^n equals

$$(X_t^2, X_0^1, X_{2^{-n}}^1 - X_0^1, X_{2\cdot 2^{-n}}^1 - X_{2^{-n}}^1, \ldots, X_{k2^{-n}}^1 - X_{(k-1)2^{-n}}^1, X_t^1 - X_{k2^{-n}}^1, 0, 0, \ldots).$$

In other words, we insert X^2 on the first coordinate, the initial value of X^1 on the second coordinate and, for $k = 0, 1, 2, \ldots$, the increment $X_{(k+1)2^{-n}\wedge t}^1 - X_{k2^{-n}\wedge t}^1$ on $k + 3$rd coordinate of Y^n.

We have $[Y^n, Y^n]_t = [X^1, X^1]_t + [X^2, X^2]_t$ for all t and hence Y^n is differentially subordinate to the martingale (X^1, X^2). Therefore, using Burkholder's result and the orthogonality of the ranges of f_1 and f_2, we may write

$$||Y^n||_p \leq (p^* - 1)||\sqrt{|X^1|^2 + |X^2|^2}||_p = (p^* - 1)||f_1 + f_2||_{L^p([0,\infty))}/M. \tag{5.2}$$

On the other hand,

$$|Y_M^n|^2 = |X_M^2|^2 + |X_0^1|^2 + \sum_{k=0}^{M2^n - 1} |X_{(k+1)2^{-n}}^1 - X_{k2^{-n}}^1|^2 \to |X_M^2|^2 + [X^1, X^1]_M$$

in probability, as $n \to \infty$ (see e.g. Doléans [11]). Since $[X^1, X^1]_M \geq (\Delta X^1)^* \geq |\Lambda f_1|$ and the ranges of Λf_1 and f_2 are orthogonal, we obtain, by Fatou's lemma,

$$\liminf_{n\to\infty} ||Y^n||_p \geq ||\Lambda f_1 + f_2||_{L^p([0,\infty))}/M.$$

It suffices to use (5.2) to complete the proof of (5.1). □

Sharpness for $1 < p \leq 2$. To see that $p^* - 1 = (p-1)^{-1}$ cannot be decreased, as in [3] use the functions $f(x) = x^{\varepsilon - 1/p} 1_{[0,1]}(x)$ for $\varepsilon > 0$ sufficiently small. The calculations are similar to those from the previous section; we omit the details. □

Repeating the reasoning from (4.4), we obtain the following fact.

Theorem 5.3. *For any radial function $F : \mathbb{C} \to \mathcal{H}$ we have*

$$||BF||_{L^p(\mathbb{C})} \leq (p^* - 1)||F||_{L^p(\mathbb{C})}, \qquad 1 < p < \infty. \tag{5.3}$$

If $1 < p \leq 2$, then the constant is the best possible.

Remark 5.1. The bound (5.3) is also proved in Volberg [19] by entirely different methods. Unfortunately we have not succeeded in obtaining the best L^p-constant for the rage $p > 2$, but it is not difficult to find a constant which is of optimal order as $p \to \infty$. Indeed, applying Doob's maximal inequality we get $||(\Delta X)^*||_p \leq 2||X^*||_p \leq \frac{2p}{p-1}||X||_p$ for any $\mathcal{H}$-valued martingale X. Thus, using the above machinery, we obtain

$$||BF||_{L^p(\mathbb{C})} \leq \frac{2p}{p-1}||F||_{L^p(\mathbb{C})}, \qquad 2 < p \leq \infty,$$

for any radial function $F : \mathbb{C} \to \mathcal{H}$ where in case $p = \infty$ we restrict ourselves to radial functions in $L^{p_0}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ for some $2 \le p_0 < \infty$ in order to have BF well defined. This bound may look a bit unexpected, especially in view of the conjecture of Iwaniec, since the constant does not explode as p tends to infinity. Though the estimate does not seem to be sharp for finite p, we can show that the constant *is* optimal for $p = \infty$. This was observed in [3, p. 355]. Repeating the argument there, take $f = 1_{[0,1]} - 1_{[1,2]}$ and note that

$$\Lambda f(u) = \begin{cases} 0 & \text{if } u \le 1 \text{ or } u > 2, \\ 2/u & \text{if } 1 < u \le 2. \end{cases}$$

Therefore, $||f||_{L^\infty([0,\infty))} = 1$, $||\Lambda f||_{L^\infty([0,\infty))} = 2$ and hence $||BF||_{L^\infty(\mathbb{C})} = 2||F||_{L^\infty(\mathbb{C})}$ for the corresponding radial function.

For related inequalities we refer the reader to [5], specially to the results contained in §6. Also relevant for the calculations leading to the results in [3] is Iwaniec's paper [13] in which he finds that when restricted to functions in $L^2(\mathbb{C}) \cap L^\infty(\mathbb{C})$, the BMO norm of B is 3.

Finally, let us mention that [3] also studies higher order versions of the operator Λ. Indeed, for any integer $m \ge 0$ define

$$\begin{aligned} \Lambda_m f(u) &= \frac{2m+2}{m+2} \int_0^1 f(uv^{\frac{2}{m+2}}) dv - f(u) \\ &= \int_0^1 f(uv)(m+1) v^{\frac{m}{2}} dv - f(u). \end{aligned} \tag{5.4}$$

Then Λ_0 is just the operator Λ as defined in (4.1). It is observed in [3, p. 355] that

$$||\Lambda_m||_{L^\infty} = \frac{2(m+1)}{(m+2)} + 1,$$

for all m and it is also conjectured there that

$$||\Lambda_m||_{L^p} = \frac{mp+2}{p(m+2)-2}, \quad 1 < p \le 2, \tag{5.5}$$

for all m.

In [14], the best constant in the weak-type $(1,1)$ inequality for the adjoint Λ^* of the operator Λ is investigated and it is shown that in fact its value is also $\frac{1}{\log 2}$. In addition, letting $\|\Lambda_m\|_w$ denote the best constant in the weak-type $(1,1)$ inequality for the operator Λ_m, it is conjectured in [14] that for any $m \ge 0$,

$$\|\Lambda_m\|_w = \frac{m 2^{\frac{2}{2+m}}}{(2+m)(2 - 2^{\frac{2}{2+m}})}, \tag{5.6}$$

where the case $m = 0$ is interpreted as the limit as $m \to 0$ of this expression, which is in fact $\frac{1}{\log 2}$. It would be of interest to have a probabilistic understanding of the operators Λ^* and Λ_m for all $m \ge 1$, and to gain further insights into their properties and particularly to investigate the validity of (5.5) and (5.6).

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