HEAT TRACE OF NON-LOCAL OPERATORS

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ABSTRACT. This paper extends results of M. van den Berg on two-term asymptics for the trace of Schödinger operators when the Laplacian is replaced by non-local (integral) operators corresponding to rotationally symmetric stable processes and other closely related Lévy processes.

CONTENTS

1. INTRODUCTION

There is an extensive literature of trace formulæ for heat kernels of the Laplacian and its Schrödinger perturbations in spectral and scattering theory. For instance, van den Berg [7] and later Bañuelos and Sá Barreto [6], explicitly compute several coefficients in the asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ as $t \downarrow 0$ for potentials $V \in \mathcal{S}(\mathbb{R}^d)$, the class of rapidly decaying functions at infinity. In particular, in [6] a general formula is obtained for these coefficients by using elementary Fourier transform methods. For applications of these results to problems in scattering theory, see [6] and the references therein. The results in [6] were extended by Donnelly [19] to certain compact Riemannian manifolds. Heat asymptotic results have also been widely used in statistical mechanics, for details we refer the reader to the article [25] of E. H. Lieb on the second virial coefficient of a hard-sphere gas at low temperatures, and to the article [27] of M. D. Penrose, O. Penrose and G. Stell on the sticky spheres in quantum mechanics. We also refer the reader to Datchev and Hezari's overview article [23] for various other related spectral asymptotic results and applications. For a historical account, see also Arndt, Nittka, Peter and Steiner's article "Weyl's Law: Spectral properties of the Laplacian in mathematics and physics," [1, pp 1-71].

By a non-local operator in this paper we shall mean an operator which arises as the generator of a pure jump Lévy process whose action on a function is obtained as a principle value singular integral in the form of (2.8) and (5.1). For such operators many questions concerning spectral asymptotics are wide open at this point. These include, for example, the Weyl two-term asymptotics for the spectral counting function of the fractional Laplacian with Dirichlet boundary conditions and the McKean-Singer [26] result involving the Euler characteristic of the domain in the third-term asymptotics of the trace of the heat semigroup. On the other hand, a two-term trace asymptotics involving the volume of the domain

and the surface area of the boundary is proved in Bañuelos and Kulczycki [4] (for $C^{1,1}$ domains) and in Bañuelos, Kulczycki and Siudeja [5] (for Lipschitz domains) for the fractional Laplacian associated with symmetric stable processes. These results are in parallel to the results of van den Berg [8] and Brown [9] for the Laplacian. For further recent work on the Dirichlet case in domains of Euclidean space, see Frank and Geisinger, [21] and [22]. For other results related to the spectral theory of these operators, see [2], [3] and references therein. The purpose of this paper is to obtain analogues of the van den Berg result in [7] for the fractional Laplacian $\Delta^{\alpha/2}$ associated with symmetric α-stable processes, $0 < \alpha < 2$, in $\mathbb{R}^d$ and for other closely related non-local operators corresponding to sums of stable processes and the relativistic Brownian motion. These are all Lévy processes which are obtained by subordination of Brownian motion.

2. STABLE PROCESSES, STATEMENT OF RESULTS

Before we state our results precisely, we recall the basic definitions and elementary notions related to stable processes. Let X_t be the d-dimensional symmetric α–stable process of order $\alpha \in (0,2]$ in $\mathbb{R}^d$. The process X_t has stationary independent increments and its transition density $p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y)$, $t > 0$, $x, y \in \mathbb{R}^d$, is determined by its Fourier transform (characteristic function)

$$(2.1) \qquad e^{-t|\xi|^\alpha} = E(e^{i\xi \cdot X_t}) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t^{(\alpha)}(y) dy, \qquad t > 0, \quad \xi \in \mathbb{R}^d.$$

Denoting by P^x and E^x the probability and expectation, respectively, of this process starting at x we have that for any Borel set $B \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, $t > 0$,

$$P^x(X_t \in B) = \int_B p_t^{(\alpha)}(x-y) dy,$$

where

$$(2.2) \qquad p_t^{(\alpha)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi.$$

Here we can also write

$$(2.3) \qquad p_t^{(\alpha)}(x) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{\alpha/2}(s) \, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. While explicit formulae for the transition density of symmetric α-stable processes are only available for $\alpha = 1$ (the Cauchy process) and $\alpha = 2$ (the Brownian motion), these processes share many of the basic properties of Brownian motion. It also follows trivially from (2.3) that $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x. Thus exactly as in the Brownian motion case we have

$$(2.4) \qquad p_t^{(\alpha)}(x) \le t^{-d/\alpha} p_1^{(\alpha)}(0).$$

In fact, if we denote by ω_d the surface area of the unit sphere in $\mathbb{R}^d$ we can compute $p_1^{(\alpha)}(0)$ more explicitly.

$$p_1^{(\alpha)}(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi = \frac{\omega_d}{(2\pi)^d \alpha} \int_0^\infty e^{-s} s^{(\frac{d}{\alpha}-1)} ds$$

$$(2.5) \qquad\qquad\qquad\qquad\qquad = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}.$$

Throughout this paper we will deal with radial transition functions and use the notation

$$p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y).$$

In particular, $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0)$. Of importance for us in this paper is the scaling property of these processes. More precisely, by a simple change of variables in (2.2), we see that these processes are self-similar with scaling

$$(2.6) \qquad p_t^{(\alpha)}(x,y) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x, t^{-1/\alpha}y).$$

The transition densities $p_t^{(\alpha)}(x)$ satisfy the following well-known (see for example [30]) two sided inequality valid for all $x \in \mathbb{R}^d$ and $t > 0$,

$$(2.7) \qquad C_{\alpha,d}^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right) \leq p_t^{(\alpha)}(x) \leq C_{\alpha,d}\left(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\right),$$

where the constant $C_{\alpha,d}$ only depends on α and d. Here and throughout the paper we use the notation $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for any $a, b \in \mathbb{R}$.

For rapidly decaying functions $f \in \mathcal{S}(\mathbb{R}^d)$, we have the semigroup of the stable processes defined via the the Fourier inversion formula by

$$
\begin{aligned}
T_t f(x) &= E^x[f(X_t)] = E^0[f(X_t + x)] \\
&= \int_{\mathbb{R}^d} f(x+y)p_t(dy) = p_t * f(x) \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} \widehat{f}(\xi) d\xi.
\end{aligned}
$$

By differentiating this at $t = 0$ we see that its infinitesimal generator is $\Delta^{\alpha/2}$ in the sense that $\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi)$. This is a non-local operator such that for suitable test functions, including all functions in $f \in C_0^\infty(\mathbb{R}^d)$, we can define it as the principle value integral

$$(2.8) \qquad \Delta^{\alpha/2} f(x) = \mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y|>\epsilon\}} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy,$$

where

$$\mathcal{A}_{d,-\alpha} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2} \left|\Gamma\left(\frac{-\alpha}{2}\right)\right|}.$$

We will denote by H_0 the fractional Laplacian operator $\Delta^{\alpha/2}, \alpha \in (0,2]$ and by H its Schrödinger perturbation $H = \Delta^{\alpha/2} + V$, where $V \in L^\infty(\mathbb{R}^d)$. We let e^{-tH} and e^{-tH_0} be the associated heat semigroups and let p_t^H and $p_t^{(\alpha)}$ denote their corresponding transition densities (heat kernels). In fact, $p_t^{(\alpha)}$ is just as in (2.1) and the Feynman-Kac formula gives

$$(2.9) \qquad p_t^H(x,y) = p_t^{(\alpha)}(x,y) E_{x,y}^t\left(e^{-\int_0^t V(X_s)ds}\right),$$

where $E_{x,y}^t$ is the expectation with respect to the stable process (bridge) starting at x conditioned to be at y at time t. The main object of study in this paper is the trace difference

$$
\begin{aligned}
Tr(e^{-tH} - e^{-tH_0}) &= \int_{\mathbb{R}^d} (p_t^H(x,x) - p_t^{(\alpha)}(x,x))dx \\
&= p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx \\
&= t^{-d/\alpha} p_1^{(\alpha)}(0) \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx,
\end{aligned}
$$

(2.10)

where $p_1^{(\alpha)}(0)$ is the dimensional constant given by the right hand side of (2.5).

Before we go further let us observe that this quantity is well defined for all $t > 0$, provided $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Indeed, the elementary inequality $|e^z - 1| \leq |z|e^{|z|}$ immediately gives that

$$
\left| \int_{\mathbb{R}^d} E_{x,x}^t \left(e^{-\int_0^t V(X_s)ds} - 1 \right) dx \right| \leq e^{t\|V\|_\infty} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t |V(X_s)|ds \right) dx.
$$

However,

$$
\begin{aligned}
(2.11) \qquad E_{x,x}^t \left(\int_0^t |V(X_s)|ds \right) &= \int_0^t E_{x,x}^t |V(X_s)|ds \\
&= \int_0^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(x,y) p_{t-s}^{(\alpha)}(y,x)}{p_t^{(\alpha)}(x,x)} |V(y)|dyds.
\end{aligned}
$$

The Chapman–Kolmogorov equations and the fact that $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0,0)$ give that

$$
\int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(x,y) p_{t-s}^{(\alpha)}(y,x)}{p_t^{(\alpha)}(x,x)} dx = 1
$$

and hence

$$
(2.12) \qquad \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t |V(X_s)|ds \right) dx = t\|V\|_1.
$$

It follows then that

$$
(2.13) \qquad \left| Tr(e^{-tH} - e^{-tH_0}) \right| \leq t^{-d/\alpha+1} p_1^{(\alpha)}(0)\|V\|_1 e^{t\|V\|_\infty},
$$

valid for all $t > 0$ and all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

The previous argument also shows that for all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,

$$
(2.14) \qquad Tr\left(e^{-tH} - e^{-tH_0} \right) = p_t^{(\alpha)}(0) \sum_{k=1}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds \right)^k dx,
$$

where the sum is absolutely convergent for all $t > 0$.

We have the following two Theorems which parallel the results in van den Berg [7] for the Laplacian. Note however, that here (ii) in Theorem 2.1 (as well as Theorem 2.2) is more general as we do not assume that the potential is nonnegative.

Theorem 2.1. *(i) Let* $V : \mathbb{R}^d \to (-\infty, 0]$, $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *Then for all* $t > 0$

$$
(2.15) \qquad p_t^\alpha(0)t\|V\|_1 \leq Tr(e^{-tH} - e^{-tH_0}) \leq p_t^{(\alpha)}(0) \left(t\|V\|_1 + \frac{1}{2}t^2\|V\|_1\|V\|_\infty e^{t\|V\|_\infty} \right).
$$

In particular

$$Tr(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left(t\|V\|_1 + \mathcal{O}(t^2) \right)$$

(2.16)
$$= t^{-d/\alpha} p_1^{(\alpha)}(0) \left(t\|V\|_1 + \mathcal{O}(t^2) \right),$$

as $t \downarrow 0$.

(ii) If we only assume that $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then for all $t > 0$,

(2.17) $$\left| Tr(e^{-tH} - e^{-tH_0}) + p_t^{(\alpha)}(0)t \int_{\mathbb{R}^d} V(x)dx \right| \le p_t^{(\alpha)}(0)Ct^2\|V\|_1\|V\|_\infty e^{t\|V\|_\infty},$$

for some universal constant C. From this we conclude that

(2.18) $$Tr(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \mathcal{O}(t^2) \right),$$

as $t \downarrow 0$.

Theorem 2.2. *Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order γ (there exists a constant $M \in (0,\infty)$ such that $|V(x) - V(y)| \le M|x-y|^\gamma$, for all $x, y \in \mathbb{R}^d$) with $0 < \gamma < \alpha \wedge 1$, whenever $0 < \alpha \le 1$, and with $0 < \gamma \le 1$, whenever $1 < \alpha < 2$. Then for all $t > 0$,*

$$\left| \left(Tr(e^{-tH} - e^{-tH_0}) \right) + p_t^{(\alpha)}(0)t \int_{\mathbb{R}^d} V(x)dx - p_t^{(\alpha)}(0)\frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$

(2.19) $$\le C_{\alpha,\gamma,d}\|V\|_1 p_t^{(\alpha)}(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right),$$

where the constant $C_{\alpha,\gamma,d}$ depends only on α, γ and d. In particular,

(2.20) $$Tr(e^{-tH} - e^{-tH_0}) = p_t^{(\alpha)}(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right),$$

as $t \downarrow 0$.

Theorems 2.1 and 2.2 are proved in §3. In §5, we provide extensions to other non-local operators.

3. PROOF OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. Setting

$$a = \int_0^t V(X_s)ds, \quad \text{and} \quad b = t\|V\|_\infty,$$

we observe that $-b \le a \le 0$. We then use the following elementary inequality from [7]

(3.1) $$-a \le e^{-a} - 1 \le -a \left(1 + \frac{1}{2}be^b \right)$$

to conclude that

$$-\int_0^t V(X_s)ds \le \left(e^{-\int_0^t V(X_s)ds} - 1 \right) \le \left[-\int_0^t V(X_s)ds \right] \left(1 + \frac{1}{2}t\|V\|_\infty e^{t\|V\|_\infty} \right).$$

Taking expectations of both sides of this inequality with respect to $E_{x,x}^t$ and then integrating on $\mathbb{R}^d$ with respect to x, it follows exactly as in the derivation of the general bound (2.13), that (2.15) holds. This concludes the proof of (i) in Theorem 2.1.

For (ii), we observe that by (2.14) we have (again using (2.12))

$$(3.2) \qquad \left| Tr(e^{-tH} - e^{-tH_0}) + p_t^{(\alpha)}(0)t \int_{\mathbb{R}^d} V(x)dx \right|$$

$$\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left| \int_0^t V(X_s)ds \right|^k dx$$

$$\leq p_t^{(\alpha)}(0) \sum_{k=2}^{\infty} \frac{t^{k-1}\|V\|_\infty^{k-1}}{k!} \int_{\mathbb{R}^d} E_{x,x} \left(\int_0^t |V(X_s)|ds \right) dx$$

$$= p_t^{(\alpha)}(0)t\|V\|_1 \sum_{k=2}^{\infty} \frac{t^{k-1}\|V\|_\infty^{k-1}}{k!} \leq C p_t^{(\alpha)}(0)t^2\|V\|_1\|V\|_\infty e^{t\|V\|_\infty},$$

for some absolute constant C. This proves (ii) and concludes the proof of the theorem. $\qquad \square$

Remark 3.1. *We observe that the above holds for any operator which arises from subordination of a Brownian motion. For such operators the transition probabilities p_t are symmetric radial and decreasing, hence $p_t(x,x) = p_t(0)$. In section 5 below we give examples of other non-local operators for which Theorem 2.2 also holds.*

Proof of Theorem 2.2. We begin by observing that (exactly as in the proof of inequality (3.2)) we have

$$\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds \right]^2 \right|$$

$$(3.3) \qquad \leq \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_0^t |V(X_s)|ds,$$

for some constant C. By taking expectation of both sides of (3.3) with respect to $E_{x,x}^t$ and then integrating with respect to x, we obtain

$$\int_{\mathbb{R}^d} E_{x,x}^t \left(\left| e^{-\int_0^t V(X_s)ds} - 1 + \int_0^t V(X_s)ds - \frac{1}{2}\left[\int_0^t V(X_s)ds \right]^2 \right| \right) dx$$

$$\leq \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t |V(X_s)|ds \right) dx$$

$$= \quad C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} t\|V\|_1,$$

where we use again (2.12).

Returning to the definition of the trace differences in (2.10) we see that this leads to

$$\left| \frac{1}{p_t^{(\alpha)}(0)}(Tr(e^{-tH} - e^{-tH_0})) + t \int_{\mathbb{R}^d} V(x)dx - \frac{1}{2} \int_{\mathbb{R}^d} E_{x,x}^t \left(\left[\int_0^t V(X_s)ds \right]^2 \right) dx \right|$$

$$(3.4) \qquad \leq C(t\|V\|_\infty)^2 e^{t\|V\|_\infty} t\|V\|_1.$$

It remains to estimate the term $E_{x,x}^t([\cdot]^2)$. Since V is uniformly Hölder with exponent γ and constant M, we have

$$(3.5) \qquad |V(X_s + x) - V(x)| \leq M|X_s|^\gamma.$$

Hence,

$$
\begin{aligned}
\left| E_{x,x}^t \left[\int_0^t V(X_s)ds \right]^2 - t^2 V^2(x) \right| &= \left| E_{x,x}^t \left[\int_0^t V(X_s)ds \right]^2 - \left[\int_0^t V(x)ds \right]^2 \right| \\
&= \left| E_{x,x}^t \left(\left[\int_0^t V(X_s)ds \right]^2 - \left[\int_0^t V(x)ds \right]^2 \right) \right| \\
&= E_{0,0}^t \left(\left[\int_0^t (V(X_s + x) - V(x))ds \right] \cdot \right. \\
&\qquad \left. \left[\int_0^t V(X_s + x) + V(x))ds \right] \right).
\end{aligned}
$$

(3.6)

By employing (3.5), we obtain

$$
\left| E_{x,x}^t \left(\left[\int_0^t V(X_s)ds \right]^2 \right) - t^2 V^2(x) \right| \le M E_{0,0}^t \left(\left[\int_0^t |X_s|^\gamma ds \right] \left[\int_0^t (|V(X_s + x)| + |V(x)|)\, ds \right] \right).
$$

Integrating both sides of this inequality with respect to x and using Fubini's theorem, we have that the value of the second integral becomes $2t\|V\|_1$. Thus we arrive at

(3.7)
$$
\left| \int_{\mathbb{R}^d} E_{x,x}^t \left(\left[\int_0^t V(X_s)ds \right]^2 \right) dx - t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|
$$
$$
\le 2tM\|V\|_1 E_{0,0}^t \left(\int_0^t |X_s|^\gamma ds \right).
$$

Now, it remains to estimate the expectation on the right side of (3.7). As in (2.11) we have

(3.8)
$$
\begin{aligned}
E_{0,0}^t \left(\int_0^t |X_s|^\gamma ds \right) &= \int_0^t E_{0,0}^t(|X_s|^\gamma)ds \\
&= \int_0^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&\quad + \int_{t/2}^t \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds \\
&= 2 \int_0^{t/2} \int_{\mathbb{R}^d} \frac{p_s^{(\alpha)}(0,y)p_{t-s}^{(\alpha)}(y,0)}{p_t^{(\alpha)}(0,0)} |y|^\gamma dyds.
\end{aligned}
$$

To estimate the right hand side we observe that for all $0 < s < t/2$ and all $y \in \mathbb{R}^d$,

(3.9)
$$
p_{t-s}^{(\alpha)}(y,0) \le p_{t-s}^{(\alpha)}(0,0) \le p_{t/2}^{(\alpha)}(0,0).
$$

By scaling (Equation (2.6))

$$
\frac{p_{t/2}^{(\alpha)}(0,0)}{p_t^{(\alpha)}(0,0)} = 2^{d/\alpha}
$$

and therefore the right hand side of (3.8) is bounded above by

$$
\begin{aligned}
E_{0,0}^t \left(\int_0^t |X_s|^\gamma ds \right) &\leq 2^{d/\alpha+1} \int_0^{t/2} \int_{\mathbb{R}^d} p_s^{(\alpha)}(0,y)|y|^\gamma dy ds \\
&= 2^{d/\alpha+1} \int_0^{t/2} E^0(|X_s|^\gamma) ds \\
&= 2^{d/\alpha+1} \int_0^{t/2} s^{\gamma/\alpha} E^0(|X_1|^\gamma) ds \\
&= \frac{2^{d/\alpha+1}}{2^{\gamma/\alpha+1}} E^0(|X_1|^\gamma) \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1},
\end{aligned}
$$

where we used the scaling property of the stable process coming from (2.6). We now recall that $E^0(|X_1|^\gamma)$ is finite under our assumption that $\gamma < \alpha$. (This fact follows trivially from (2.7).) Thus we see that

$$
(3.10) \qquad E_{0,0}^t \left(\int_0^t |X_s|^\gamma ds \right) \leq C_{\alpha,\gamma,d} t^{\gamma/\alpha+1},
$$

where the constant $C_{\alpha,\gamma,d}$ depends only on α, γ and d. We conclude that

$$
\left| \int_{\mathbb{R}^d} E_{x,x}^t \left(\left[\int_0^t V(X_s) ds \right]^2 \right) dx - t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right| \leq 2tM\|V\|_1 E_{0,0}^t \left(\int_0^t |X_s|^\gamma ds \right)
$$

$$
(3.11) \hspace{9cm} \leq M\|V\|_1 C_{\alpha,\gamma,d} t^{\gamma/\alpha+2}.
$$

Returning to (3.4) we find that

$$
(3.12) \quad
\begin{aligned}
&\left| \frac{1}{p_t^\alpha(0)} \left(Tr(e^{-tH} - e^{-tH_0}) \right) + t \int_{\mathbb{R}^d} V(x) dx - \frac{1}{2} t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right| \\
&\leq C t^3 \|V\|_\infty^2 e^{t\|V\|_\infty} \|V\|_1 + M\|V\|_1 C_{\alpha,\gamma,d} t^{\gamma/\alpha+2} \\
&\leq C_{\alpha,\gamma,d} \|V\|_1 \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right).
\end{aligned}
$$

Rewriting this in the form stated in Theorem 2.2 we arrive at the announced bound

$$
(3.13) \quad
\begin{aligned}
&\left| (Tr(e^{-tH} - e^{-tH_0})) + p_t^\alpha(0) t \int_{\mathbb{R}^d} V(x) dx - p_t^{(\alpha)}(0) t^2 \frac{1}{2} \int_{\mathbb{R}^d} |V(x)|^2 dx \right| \\
&\leq C_{\alpha,\gamma,d} \|V\|_1 p_t^{(\alpha)}(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right),
\end{aligned}
$$

valid for all $t > 0$. □

Remark 3.2. *We remark that the estimate in (2.6) also works in the case of Brownian motion, $\alpha = 2$. However, in this case one may use the fact (as done in [7]) that*

$$
(3.14) \qquad \frac{p_s^{(2)}(0,y) p_{t-s}^{(2)}(y,0)}{p_t^{(2)}(0,0)} = p_{\frac{s}{t}(t-s)}^{(2)}(0,y)
$$

to explicitly compute the quantity in (3.8). It is also interesting to observe her that in the case of Brownian motion one can compute the quantity on the right hand side of (2.6) without the explicit estimate that comes from (3.14). Indeed, by observing that in law the Brownian bridge started at 0 and

conditioned to return to 0 in time t is the same as $B_s - \frac{s}{t}B_t$, where B_s is standard Brownian motion, a simple computation gives

$$
\begin{aligned}
E_{0,0}^t \left(\int_0^t |B_s|^\gamma ds \right) &= \int_0^t E^0 |B_s - \frac{s}{t}B_t|^\gamma ds \\
&\leq C_\gamma \int_0^t E^0 \left(|B_s|^\gamma + \left(\frac{s}{t}\right)^\gamma |B_t|^\gamma \right) ds \\
&= C_\gamma E^0 |B_1|^\gamma \int_0^t \left(s^{\gamma/2} + \left(\frac{s}{t}\right)^\gamma t^{\gamma/2} \right) ds \\
&= C_{\gamma,d}\, t^{\gamma/2+1},
\end{aligned}
\tag{3.15}
$$

where $C_{\gamma,d}$ depends only on γ and d.

Unfortunately, the simple path construction $X_s - \frac{s}{t}X_t$ only leads to the stable bridge when $\alpha = 2$ and we are not able to repeat (3.15) in the case $0 < \alpha < 2$, hence our use of (3.9). However, it may be that the construction by Chaumont in [11] (or perhaps some of the estimates of Fitzsimmons and Getoor in [20]) can be used to compute the expression in (2.6) explicitly.

4. ALTERNATIVE PROOFS OF THEOREMS 2.1 AND 2.2

In this section we present an alternative method for computing the first two coefficients in the expansion of the trace difference based on the expansion of powers of the Feynman-Kac functional in terms of multiple symmetric integrals. While this method does not give any new results, we believe the idea may lead the computation of more terms in the asymptotics. In addition, the argument is short and simple. We begin by observing that from formula (2.14) we have

$$
Tr\left(e^{-tH} - e^{-tH_0}\right) = p_t^{(\alpha)}(0) \sum_{k=1}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds \right)^k dx,
\tag{4.1}
$$

where as mentioned there the sum is absolutely convergent for all potentials $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and all $t > 0$. We note also that in fact, for every $N \geq 1$,

$$
Tr\left(e^{-tH} - e^{-tH_0}\right) = p_t^{(\alpha)}(0) \left(\sum_{k=1}^N \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds \right)^k dx + \mathcal{O}(t^{N+1}) \right),
\tag{4.2}
$$

as $t \downarrow 0$. This follows from the fact that

$$
\left| \sum_{k=N+1}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} E_{x,x}^t \left(\int_0^t V(X_s)ds \right)^k dx \right| \leq \sum_{k=N+1}^\infty \frac{1}{k!} t^k \|V\|_\infty^{k-1} \|V\|_1
$$
$$
\leq \frac{t^{N+1}\|V\|_\infty^N \|V\|_1}{(N+1)!} e^{t\|V\|_\infty}
\tag{4.3}
$$

Next observe that

$$
\left(\int_0^t V(X_s)ds \right)^k = k! J_k
$$

where

$$
J_k = \int_{0 \leq s_1 \leq s_2 \leq \cdots \leq s_k \leq t} V(X_{s_1})ds_1 V(X_{s_2})ds_2 \cdots V(X_{s_k})ds_k.
$$

Recalling that the finite dimensional distributions of the stable bridge are given by

$$P_{x,x}^t\{X_{s_1} \in dy_1, X_{s_2} \in dy_2, \dots, X_{s_k} \in dy_k\} = \frac{1}{p_t^{(\alpha)}(x,x)} \prod_{j=0}^k p_{s_{j+1}-s_j}^{(\alpha)}(y_{j+1} - y_j),$$

where $y_0 = y_{k+1} = x$, $s_0 = 0$, $s_{k+1} = t$, we can write

$$(4.4) \qquad Tr\left(e^{-tH} - e^{-tH_0}\right) = \sum_{k=1}^\infty (-1)^k \int_{0 \le s_1 \le s_2 \le \dots \le s_k \le t} \tilde{I}_k(s_1, s_2, \dots, s_k) ds_1 ds_2 \cdots ds_k,$$

where

$$\tilde{I}_k(s_1, s_2, \dots, s_k) = \int_{(R^d)^{k+1}} \prod_{j=0}^k p_{s_{j+1}-s_j}^{(\alpha)}(y_{j+1} - y_j) \prod_{j=1}^k V(y_j) dy_1 dy_2 \cdots dy_k dx.$$

Here (as several times before) we used the fact that $p_t^{(\alpha)}(x,x) = p_t^{(\alpha)}(0)$. By Fubini's theorem we may integrate with respect to x first and use the fact that

$$\int_{\mathbb{R}^d} p_{s_1}^{(\alpha)}(y_1 - x) p_{t-s_k}^{(\alpha)}(y_k - x) dx = p_{t-(s_k-s_1)}^{(\alpha)}(y_k - y_1)$$

to arrive at

$$(4.5) \qquad Tr\left(e^{-tH} - e^{-tH_0}\right) = \sum_{k=1}^\infty (-1)^k \int_{0 \le s_1 \le s_2 \le \dots \le s_k \le t} I_k(s_1, s_2, \dots, s_k) ds_1 ds_2 \cdots ds_k,$$

where

$$I_k(s_1, s_2, \dots, s_k) = \int_{(R^d)^k} \left(p_{t-(s_k-s_1)}^{(\alpha)}(y_k - y_1) V(y_1) \right) \cdot$$
$$\left(\prod_{j=1}^{k-1} p_{s_{j+1}-s_j}^{(\alpha)}(y_{j+1} - y_j) V(y_{j+1}) \right) dy_1 dy_2 \cdots dy_k.$$

We can then also re-write the right hand side of (4.2) as

$$(4.6) \qquad p_t^{(\alpha)}(0) \left(\sum_{k=1}^N \frac{(-1)^k}{p_t^{(\alpha)}(0)} \int_{0 \le s_1 \le s_2 \le \dots \le s_k \le t} I_k(s_1, s_2, \dots, s_k) ds_1 ds_2 \cdots ds_k + \mathcal{O}(t^{N+1}) \right).$$

From these formulas we can also compute coefficients in the asymptotic expansion of the trace as $t \downarrow 0$ fairly directly and in particular give alternative proofs of Theorems 2.1 and 2.2.

If $k = 1$, we see that

$$p_{t-(s_k-s_1)}^{(\alpha)}(y_k - y_1) = p_t^{(\alpha)}(0)$$

and

$$I_1(s_1) = p_t^{(\alpha)}(0) \int_{\mathbb{R}^d} V(y) dy,$$

which immediately leads to (2.18) in Theorem 2.1.

If $k = 2$, we write

$$(4.7) \qquad I_2(s_1, s_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) V(y_1) V(y_2) dy_1 dy_2$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) \left(V(y_1) - V(y_2) + V(y_2) \right) V(y_2) dy_1 dy_2$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) V^2(y_2) dy_1 dy_2$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) \left(V(y_1) - V(y_2) \right) V(y_2) dy_1 dy_2.$$

Integrating first with respect to y_1 we conclude that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) V^2(y_2) dy_1 dy_2 = p^{(\alpha)}_t(0) \int_{\mathbb{R}^d} |V(y)|^2 dy.$$

On the other hand, $|V(y_1) - V(y_2)| \leq M|y_1 - y_2|^\gamma$ an hence

$$(4.8) \qquad \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) \left(V(y_1) - V(y_2) \right) V(y_2) dy_1 dy_2 \right|$$

$$\leq M \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) |y_1 - y_2|^\gamma |V(y_2)| dy_1 dy_2$$

$$= M \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y_2 - y_1) p^{(\alpha)}_{(s_2-s_1)}(y_2 - y_1) |y_2 - y_1|^\gamma dy_1 \right) |V(y_2)| dy_2$$

$$= M \left(\int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y) p^{(\alpha)}_{(s_2-s_1)}(y) |y|^\gamma dy \right) \left(\int_{\mathbb{R}^d} |V(y)| dy \right).$$

Thus

$$\int_{0 \leq s_1 \leq s_2 \leq t} I_2(s_1, s_2) ds_1 ds_2 = p^{(\alpha)}_t(0) t^2 \int_{\mathbb{R}^d} |V(y)|^2 dy + R_t$$

where

$$R_t \leq M\|V\|_1 \int_{0 \leq s_1 \leq s_2 \leq t} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y) p^{(\alpha)}_{(s_2-s_1)}(y) |y|^\gamma dy ds_1 ds_2$$

$$= M\|V\|_1 \int_0^t \int_0^{s_2} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-(s_2-s_1)}(y) p^{(\alpha)}_{(s_2-s_1)}(y) |y|^\gamma dy ds_1 ds_2$$

$$= M\|V\|_1 \int_0^t \int_0^{s_2} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-s}(y) p^{(\alpha)}_s(y) |y|^\gamma dy ds ds_2$$

$$\leq M\|V\|_1 \int_0^t \int_0^t \int_{\mathbb{R}^d} p^{(\alpha)}_{t-s}(y) p^{(\alpha)}_s(y) |y|^\gamma dy ds ds_2$$

$$= M\|V\|_1 t \int_0^t \int_{\mathbb{R}^d} p^{(\alpha)}_{t-s}(y) p^{(\alpha)}_s(y) |y|^\gamma dy ds$$

$$= 2M\|V\|_1 t \int_0^{t/2} \int_{\mathbb{R}^d} p^{(\alpha)}_{t-s}(y) p^{(\alpha)}_s(y) |y|^\gamma dy ds$$

$$\leq C_{\gamma,\alpha,d} E^0(|X_1|^\gamma) p^{(\alpha)}_t(0) t^{\gamma/\alpha+2},$$

using the argument of (3.9). As in (3.12), this gives another proof of (2.20) in Theorem 2.2.

5. Extension to other non-local operators

Taking $0 < \beta < \alpha < 2$ and $a \geq 0$, we consider the process $Z_t^a = X_t + aY_t$, where X_t and Y_t are independent α-stable and β-stable processes, respectively. This process is called the independent sum of the symmetric α-stable process X and the symmetric β-stable process Y with weight a. The infinitesimal generator of Z_t^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. This again is a non-local operator. Acting on functions $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$(5.1) \qquad \left(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} \right) f(x) =$$

$$\mathcal{A}_{d,-\alpha} \lim_{\epsilon \to 0^+} \int_{\{|y| > \epsilon\}} \left(\frac{1}{|y|^{d+\alpha}} + \frac{a^\beta}{|y|^{d+\beta}} \right) [f(x+y) - f(x)]dy,$$

where $\mathcal{A}_{d,-\alpha}$ is as in (2.8).

Properties of the heat kernel (transition probabilities) for this operator have been studied by several people in recent years. See for example Chen and Kumagai [18], Chen, Kim and Song [16] or Jakubowski and Szczypkowski [24], and references given there. If we denote the heat kernel of this operator by $p_t^a(x)$, we have that

$$(5.2) \qquad p_t^a(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t(|\xi|^\alpha + a^\beta |\xi|^\beta)} d\xi$$

$$(5.3) \qquad = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^a(s) \, ds,$$

where $\eta_t^a(s)$ is be the density function of the sum of the $\alpha/2$-stable subordinator and a^2-times the $\beta/2$-stable subordinator. Again, this density is radial, symmetric, and decreasing in $|x|$. It is proved in [18] and [16] that there is a constant $C_{\alpha,\beta,d}$ depending only on the parameters indicated such that for all $x \in \mathbb{R}^d$ and $t > 0$,

$$(5.4) \qquad C_{\alpha,\beta,d}^{-1} f_t^a(x) \leq p_t^a(x) \leq C_{\alpha,\beta,d} f_t^a(x)$$

where

$$f_t^a(x) = \left((a^\beta t)^{-d/\beta} \wedge t^{-d/\alpha} \right) \wedge \left(\frac{t}{|x|^{d+\alpha}} + \frac{a^\beta t}{|x|^{d+\beta}} \right).$$

We note that for $a = 0$ this is just the estimate in (2.7). For any $\gamma > 0$ with $0 < \gamma < \beta < \alpha$ we have for any $t > 0$,

$$(5.5) \qquad
\begin{aligned}
\int_0^t E^0(|Z_s^a|^\gamma)ds & \leq C_\gamma \left(\int_0^t E^0(|X_s|^\gamma)ds + a^\gamma \int_0^t E^0(|Y_s|^\gamma)ds \right) \\
& = C_\gamma \left(E^0(|X_1|^\gamma) \frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1} + a^\gamma E^0(|Y_1|^\gamma) \frac{t^{\gamma/\beta+1}}{\gamma/\beta+1} \right) \\
& = C_{a,\alpha,\beta,d} \left(\frac{t^{\gamma/\alpha+1}}{\gamma/\alpha+1} + \frac{t^{\gamma/\beta+1}}{\gamma/\beta+1} \right).
\end{aligned}$$

With (5.5) both proofs above for Theorem 2.2 can be repeated to obtain the same result for the operator $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$, replacing $\Delta^{\alpha/2}$. More precisely we have

Theorem 5.1. *Let $a \geq 0$, $0 < \beta < \alpha < 2$ and let $H_0^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and that it is also uniformly Hölder continuous of order γ, with $0 < \gamma < \beta \wedge 1$, whenever*

$0 < \beta \leq 1$, and with $0 < \gamma \leq 1$, whenever $1 < \beta < 2$. Let $H^a = \Delta^{\alpha/2} + a^\beta \Delta^{\beta/2} + V$. Then for all $t > 0$,

$$\left| (Tr(e^{-tH^a} - e^{-tH_0^a})) + p_t^a(0)t \int_{\mathbb{R}^d} V(x)dx - p_t^a(0)\frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$

$$(5.6) \qquad \leq C_{a,\alpha,\beta,\gamma,d}\|V\|_1 p_t^a(0) \left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} + t^{\gamma/\beta+2} \right),$$

where the constant $C_{a,\alpha,\beta,\gamma,d}$ depends only on a, α, β, γ and d. In particular,

$$(5.7) \qquad Tr(e^{-tH^a} - e^{-tH_0^a}) = p_t^a(0) \left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right),$$

as $t \downarrow 0$.

We next consider another stochastic process associated with a non-local operator that has been of considerable interest to many people in recent years, the so called relativistic Brownian motion and its more general versions, the α-stable relativistic process. This is again a Lévy process denote by X_t^m with characteristic function

$$(5.8) \qquad e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)} = E(e^{i\xi \cdot X_t^m}) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t^{(m,\alpha)}(y)dy,$$

for any $m \geq 0$ and $0 < \alpha < 2$. The case $\alpha = 1$ is the relativistic Brownian motion and $m = 0$ yields back the stable processes of order α. Henceforth we assume that $m > 0$. As in the case of stable processes, X_t^m is a subordination of Brownian motion and in fact

$$(5.9) \qquad \begin{aligned} p_t^{(m,\alpha)}(x) = \quad &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)} d\xi \\ &= \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} \eta_t^{m,\alpha}(s)\, ds, \end{aligned}$$

where $\eta_t^{m,\alpha}(s)$ is the density of the subordinator with Bernstein function $\Phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$. As before we see that $p_t^{(m,\alpha)}(x)$ is radial, symmetric, and decreasing in $|x|$. By a simple change of variables it is easy to see that the transition densities have the scaling property

$$(5.10) \qquad p_t^{(m,\alpha)}(x) = m^{d/\alpha} p_{mt}^{(1,\alpha)}(m^{1/\alpha}x).$$

These properties can all be found in Ryznar [29] where it is also proved that

$$(5.11) \qquad p_t^{(m,\alpha)}(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x|^2}{4s}} e^{-m^{2/\alpha}s} \eta_t^{\alpha/2}(s)\, ds,$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator. By scaling

$$\eta_t^{\alpha/2}(s) = t^{-2/\alpha} \eta_1^{\alpha/2}(st^{-2/\alpha}).$$

Hence changing variables in (5.11) leads to

$$(5.12) \qquad \lim_{t\downarrow 0} e^{-mt} t^{d/\alpha} p_t^{(m,\alpha)}(0) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \eta_1^{\alpha/2}(s)\, ds = p_1^\alpha(0) = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha},$$

where the last equality follows from (2.5).

The infinitesimal generator of X_t^m is given by $m - \left(m^{2/\alpha} - \Delta\right)^{\alpha/2}$. The case $\alpha = 1$ gives the generator $m - \sqrt{-\Delta + m^2}$ which is the free relativistic Hamiltonian. For more on these notions we refer the reader to Carmona, Masters and Simon [10].

Estimates for the global transition probabilities $p_t^{(m,\alpha)}(x)$ and their Dirichlet counterparts for various domains have been studied by many authors. We refer the reader to [12], [13], [14], [15], [17], [18], [28], [29]. In particular, it follows from Chen, Kim and Song [14, Theorem 2.1] that there exists a constant $C_{\alpha,m,d}$ depending on α, m and d such that for all $x \in \mathbb{R}^d$ and all $t \in (0, 1]$,

$$(5.13) \qquad C_{\alpha,m,d}^{-1}\, t^{-d/\alpha} \wedge \frac{t\,\Psi(m^{\frac{1}{\alpha}}|x|)}{|x|^{d+\alpha}} \leq p_t^{(m,\alpha)}(x) \leq C_{\alpha,m,d}\, t^{-d/\alpha} \wedge \frac{t\,\Psi(m^{\frac{1}{\alpha}}|x|)}{|x|^{d+\alpha}},$$

where

$$\Psi(r) = 2^{-(d+\alpha)}\Gamma\left(\frac{d+\alpha}{2}\right)^{-1}\int_0^\infty s^{\frac{d+\alpha}{2}-1}e^{-s/4}e^{-r^2/s}ds$$

which is a decreasing function of r^2 with $\Psi(0) = 1$, $\Psi(r) \leq 1$ and with

$$c_1^{-1}e^{-r}r^{(d+\alpha-1)/2} \leq \Psi(r) \leq c_1 e^{-r}r^{(d+\alpha-1)/2},$$

for all $r \geq 1$. (See also Chen and Song,[17, p. 277].) Thus the relativistic transition probabilities behave similarly to those of stable processes. We note in particular that $p_t^{(m,\alpha)}(x,y) = p_t^{(m,\alpha)}(x-y)$ and that $p_t^{(m,\alpha)}(x,0)$ is radial (a function of $|x|$), decreasing with its maximum at 0 and that $p_t^{(m,\alpha)}(0,0)$ is decreasing as a function of t.

Next, we need to compute bounds for the γ moments of X_s^m. More precisely we want to understand the behavior of $\int_0^t E^0|X_r^m|^\gamma dr$, as $t \downarrow 0$. Let us denote the standard Brownian motion in $\mathbb{R}^d$ by B_s and recall that $E^0|B_{2s}|^\gamma = 2^{\gamma/2}s^{\gamma/2}E^0|B_1|^\gamma$ and $2^{\gamma/2}E^0|B_1|^\gamma = C_{\gamma,d}$, where the latter is an explicit and computable constant whose exact value is not relevant for our calculations here. Using this, (5.10) and (5.11) we have

$$
\begin{aligned}
(5.14) \qquad \int_0^t E^0|X_r^m|^\gamma ds &= m^{\frac{d\gamma}{\alpha}}\int_0^t E^0|X_{mr}^1|^\gamma dr = m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt} E^0|X_r^1|^\gamma dr \\
&= m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt}\int_{\mathbb{R}^d}|x|^\gamma p_r^{(1,\alpha)}(x)dx\,dr \\
&= m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt}e^r\left(\int_0^\infty E^0|B_{2s}|^\gamma e^{-s}\eta_r^{\alpha/2}(s)ds\right)dr \\
&= C_{\gamma,d}\,m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt}e^r\left(\int_0^\infty s^{\gamma/2}e^{-s}\eta_r^{\alpha/2}(s)ds\right)dr, \\
&= C_{\gamma,d}\,m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt}e^r\left(\int_0^\infty s^{\gamma/2}e^{-s}r^{-2/\alpha}\eta_1^{\alpha/2}(r^{-2/\alpha}s)ds\right)dr \\
&= C_{\gamma,d}\,m^{\left(\frac{d\gamma}{\alpha}-1\right)}\int_0^{mt}r^{\gamma/\alpha}e^r\left(\int_0^\infty s^{\gamma/2}e^{-sr^{2/\alpha}}\eta_1^{\alpha/2}(s)ds\right)dr,
\end{aligned}
$$

where we used the scaling property of the stable subordinator for the second to the last line and a change variables for the last line. But

$$\int_0^\infty s^{\gamma/2}e^{-sr^{2/\alpha}}\eta_1^{\alpha/2}(s)ds \leq \int_0^\infty s^{\gamma/2}\eta_1^{\alpha/2}(s)ds = C_\gamma E^0|X_1|^\gamma < \infty,$$

whenever $\gamma < \alpha$. (Here, X_1 is the stable process of order α at time 1.) Thus,

$$\int_0^t E^0 |X_r^m|^\gamma dr \leq C_{\alpha,\gamma,m,d}\, t^{\gamma/\alpha+1} e^{mt}$$

and this is valid for all $t > 0$.

With this inequality we follow the reasoning in (3.9) to conduced that

$$
(5.15) \qquad
\begin{aligned}
E_{0,0}^t \left(\int_0^t |X_s^m|^\gamma ds \right)
&= C_{\alpha,m,d} \int_0^{t/2} \int_{\mathbb{R}^d} p_s^{(m,\alpha)}(0,y)|y|^\gamma dy ds \\
&= C_{\alpha,m,d} \int_0^{t/2} E^0(|X_s^m|^\gamma) ds \\
&\leq C_{\alpha,\gamma,m,d}\, t^{\gamma/\alpha+1},
\end{aligned}
$$

for all $t > 0$.

With (5.15), the proof of Theorem 2.2 can be repeated to obtain a similar result for the operator associated with the relativistic stable processes. More precisely we have

Theorem 5.2. *Let* $H_0^m = m - \left(m^{2/\alpha} - \Delta \right)^{\alpha/2}$. *Suppose* $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ *and that it is also uniformly Hölder continuous of order* γ, *with* $0 < \gamma < \alpha \wedge 1$, *whenever* $0 < \alpha \leq 1$, *and with* $0 < \gamma \leq 1$, *whenever* $1 < \alpha < 2$. *Let* $H^m = m - \left(m^{2/\alpha} - \Delta \right)^{\alpha/2} + V$. *Then for all* $t > 0$,

$$\left| (Tr(e^{-tH^m} - e^{-tH_0^m})) + p_t^\alpha(0)t \int_{\mathbb{R}^d} V(x)dx - p_t^\alpha(0)\frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx \right|$$

$$(5.16) \qquad \leq C_{\alpha,\gamma,m,d}\|V\|_1 p_t^{(m,\alpha)}(0)\left(\|V\|_\infty^2 e^{t\|V\|_\infty} t^3 + t^{\gamma/\alpha+2} \right),$$

where the constant $C_{\alpha,\gamma,m,d}$ *depends only on* α, γ, m *and* d.

In particular,

$$(5.17) \quad Tr(e^{-tH^m} - e^{-tH_0^m}) = p_t^{(m,\alpha)}(0)\left(-t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2}t^2 \int_{\mathbb{R}^d} |V(x)|^2 dx + \mathcal{O}(t^{\gamma/\alpha+2}) \right),$$

as $t \downarrow 0$.

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