Lifetime of Conditioned Brownian Motion and Related Estimates for Heat Kernels Eigenfunctions and Eigenvalues in Euclidean Domains
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Lecture I Lifetime of Brownian Motion, Conditioned Brownian Motion in Euclidean Domains: Some Results, **Some Problems**.

Lecture II Intrinsic Ultracontractivity: Heat kernels, Eigenfunctions, Eigenvalues, Connections to I. Some Results, **Some Problems**.

Basic Question How does the Euclidean geometry of the domain (volume, diameter, inradius) and hyperbolic-quasihyperbolic geometry (growth of hyperbolic and quasihyperbolic distance) affect the above Probabilistic–Analytical Objects?

Aim To obtain **sharp Estimates**, often of isoperimetric–type.

I. Classical Result (≈ 50 years): $D \subset \mathbb{R}^n$, $\mathrm{vol}(D) < \infty$. B_t Brownian motion in $\mathbb{R}^n$, τ_D = exit time of B_t from D.

Then

$$\sup_{x \in D} E_x(\tau_D) \leq E_0(\tau_{D^*}),$$

$$D^* = \text{ball of same volume as } D.$$

That is

$D^* = B(0,r)$, $C_n r^n = \mathrm{vol}(D)$, c_n = volume of unit ball. Itô applied to $f(x) = |x|^2$ gives

$$E_0(\tau_{D^*}) = \frac{1}{n} r^2 = \frac{1}{n} \left(\frac{\mathrm{vol}(D)}{c_n} \right)^{2/n}.$$

More (≈ 20 years): For all $x \in D$, $t > 0$,

$$P_x\{\tau_D > t\} \leq P_0\{\tau_{D^*} > t\}.$$

More general results in C. Bandle [11].

Proof (Works just as well for symmetric stable!):

Set $P_t(x-y) = \frac{1}{(2\pi t)^{n/2}} \, e^{-\frac{|x-y|^2}{2t}}$.

$$P_x\{\tau_D > t\} = \lim_{m\to\infty} P_x \left\{ B_{\frac{jt}{m}} \in D, \; j = 1, 2, \ldots, m \right\}$$

$$= \lim_{m\to\infty} \int_D \cdots \int_D P_{t/m}(x_1 - x) \prod_{j=2}^{m} P_{t/m}(x_j - x_{j-1}) dx_1 \ldots dx_m$$

$$\leq \lim_{m\to\infty} \int_{D^*} \cdots \int_{D^*} P_{t/m}(x_1) \prod_{j=2}^{m} P_{t/m}(x_j - x_{j-1}) dx_1 \ldots dx_m$$

$$= \lim_{m\to\infty} P_0 \left\{ B_{\frac{jt}{m}} \in D^*, \; j = 1, 2, \ldots, m \right\} = P_0\{\tau_{D^*} > t\}.$$

(Brascamp-Lieb-Luttinger 1974)

M. Kac: $\qquad -\lambda_1^D = \lim_{t\to\infty} \frac{1}{t} \, \log P_x\{\tau_D > t\}.$

$$\lambda_1^{D^*} \leq \lambda_1^D \qquad \text{(Faber–Krahn)}$$

$$d_D(x) = \text{dist}(x, D^c) = \text{dist}(x, \partial D).$$

$$R_D = \sup_{x \in D} d_D(x) = \text{"radius of largest ball in } D\text{"}.$$

$$R_D = \text{inradius of } D.$$

Theorem: Suppose there is a constant C_1 such that

$$(*) \qquad \text{Cap}(B(x, 2R_D) \cap D^c) \geq C_1 R_D^{n-2}, \quad \text{for all } x \in D,$$

$n \geq 3$ with a similar assumption for $n = 2$. **(Capacity boundary condition.)** Then,

$$\sup_{x \in D} E_x(\tau_D) \leq C_2 R_D^2$$

Proof: $(*) \Leftrightarrow \sup_{x \in D} P_x\{B_t \text{ exits } B(x, 2R) \text{ before } D\} = P_D < 1.$

$$E_x(\tau_D) \leq \frac{1}{n} (2R_D)^2 \sum_{j=0}^{\infty} P_D^j = \frac{4R_D^2}{n} \frac{1}{1 - P_D}.$$

For $D \subset \mathbb{C}$ simply connected (S.C), no other assumptions .

$$P_D \le \frac{4}{\pi} \arctan\left(\frac{1}{\sqrt{2}}\right),$$

(Beurling's Thesis). Thus for some universal constant C_0

$$\frac{1}{2} R_D^2 \le \sup_{x \in D} E_x(\tau_D) \le C_0 R_D^2$$

for all S.C. planar domains with $R_D < \infty$.

Problems 1: Find the best C_0 and extremal D. Or, amongst all S.C. planar D with $R_D = 1$ find the one (ones) with largest

$$\sup_{x \in D} E_x(\tau_D).$$

Problem 2: Let $P(2) = \sup\{P_D \colon D, \text{ S.C. }, R_D = 1\}$. Find $P(2)$ and the extremal $D's$.

It would be of interest if $\quad P(2) < \frac{1}{2}$.

Why are Problems 1 and 2 of interest?

$$\varphi_1 = \text{first eigenfunction of } \frac{1}{2}\Delta \text{ in } D.$$

$$\varphi_1(B_{t\wedge\tau_D}) - \varphi_1(x) = \int_0^{t\wedge\tau_D} \nabla\varphi(B_s)\cdot dB_s - \lambda\int_0^{t\wedge\tau_D}\varphi_1(B_s)ds$$

$$\varphi_1(x) = \lambda_1 E_x\left(\int_0^{\tau_D}\varphi_1(B_s)ds\right)$$

$$\frac{1}{\sup\limits_{x\in D} E_x(\tau_D)} \le \lambda_1.$$

For S.C. planar D, there is a universal constant e_0 such that

$$\frac{e_0}{R_D^2} \le \lambda_1 \le \frac{j_0^2}{2R_D^2},\ j_0^2 = 5.781\ldots$$

Problem 3: Find the best e_0 and extremal(s) $D's$. Or, amongst all "drums" of inradius 1, find the one(s) that produce the lowest fundamental tone. (Open since before 1951; Polya–Szegö book.) References to Recent Progress: [17], [19], [24], [26]. In particular, Bañuelos- Carroll [17] disproves a conjecture of R. Osserman and gives a conjecture for a possible extremal "drum." Problems 2 and 3 are closely related (as explained in [17]) to famous open problem (open for about 60 years) concerning the schlicht Bloch–Landau constant.

II. Conditioned Brownian motion: (Doob 1953, [49]) If h is positive and harmonic in D, then the Brownian motion conditioned by h (the Doob h–process) is determined by the transition probabilities

$$P_t^h(w, z) = \frac{1}{h(w)}\ P_t^D(w, z)h(z)$$

$P_t^D(w, z) =$ transitions for killed B.M. in D. The processes has generator

$$L_h = \frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla.$$

Basic processes: (1) $h(z) = K_0^D(z, \xi) = K_\xi^D(z), z \in D, \xi \in \partial D$, the Martin (Poisson kernel), (2) $h(z) = G_D(z, w)$, Green function in D. The first gives the Brownian motion conditioned to exit the domain at the point ξ and the second gives Brownian motion conditioned to go from z to w without leaving the domain. The general h–processes are mixtures of these.

Doob used $D = \mathbb{R}_+^{n+1}$ to study boundary behavior of harmonic function. Other beautiful uses in analysis were given in the 70's by Burkholder–Gundy–Silverstein.

Theorem (M. Cranston, T. McConnell [41] 1983)).

(i) For any $D \subset \mathbb{R}^2$, $\sup_{\substack{x\in D \\ h\in H^+(D)}} E_x^h(\tau_D) \le C_0 \text{ area}(D)$

(ii) $\exists$ bounded $D \subset \mathbb{R}^3$ and $h \in H^+(D)$ such that

$$E_z^h(\tau_D) = \infty, \ \forall z \in D.$$

(Question posed by K. L. Chung)

Cranston–McConnell Basic Estimate: For any $D \subset \mathbb{R}^n$,

For any integer m, define (for any fixed $x_0 \in D$)

$$\begin{aligned} D_m &= \{x \in D \colon 2^{m-1}h(x_0) < h(x) < 2^{m+1}h(x_0)\} \\ C_m &= \{x \in D \colon h(x) = 2^m h(x_0)\}. \end{aligned}$$

Then

$$E_x^h(\tau_D) \le 8 \sum_{m=-\infty}^{\infty} \sup_{x\in C_m} E_x(\tau_{D_m})$$

An elegant (in Chung's words "perspicacious") proof of the basic estimate can be found in Chung [36]. An analytic proof, which I leaned from T. Wolff and which has been used recently by Aikawa [1]–[5] and others for various extensions of the basic estimate, can be found in [14].

The hyperbolic metric: Assume $D \subset \mathbb{R}^2$ is simply connected. The hyperbolic distance between $z_0, z \in D$ is

$$\rho_D(z_0, z) = \rho_{D_0}(f(z_0), f(z)),$$

where D_0 is the unit disk and $f : D_0 \to D$ conformally. If σ_{D_0} and σ_D denote the corresponding hyperbolic densities for D_0 and D, we have

$$\rho_{D_0}(z_0, z) = \inf_{\gamma} \int_0^1 \sigma_{D_0}(\gamma(t))|\gamma'(t)|dt,$$

where the inf is over all curves γ contained in D_0 with $\gamma(0) = z_0$, $\gamma(1) = z$. with a similar expression for ρ_D. $\sigma_D(z) = \frac{1}{(1-|z|^2)}$ and by the Koebe–1/4 theorem,

$$\sigma_D(z) \approx \frac{1}{d_D(z)},$$

where as before $d_D(z)$ is the distance from z to the boundary of D. The minimizing curve is called the hyperbolic geodesic. Note that in the case of D_0 we have

$$\rho_{D_0}(0, z) = \log\left(\frac{1+|z|}{1-|z|}\right) \approx \log\left(\frac{1}{d_{D_0}(z)}\right).$$

In general this remains true for "nice enough" domain. That is if the domain D is "nice enough" (so called Hölder domains [13]) we have

$$\rho_D(z_0, z) \leq C_1 \log\left(\frac{1}{d_D(z)}\right) + C_2$$

We need the "hyperbolic metric" for arbitrary domains. What is that?

Quasi–hyperbolic distance: Let $\mathcal{F} = \{\mathcal{Q}_|\}_{|=\infty}^{\infty}$ be a Whitney decomposition of D: (1) $\cup_{j=1}^{\infty} Q_j = D$, (2) $Q_j^0 \cap Q_k^0 = \emptyset$, (3) $\mathrm{diam}(Q_k) \approx \mathrm{dist}(Q_k, D^c)$.

A Whitney chain from x to y in D is a subset $\{Q_1, Q_2, \ldots, Q_m\}$ of $\mathcal{F}$ with $x \in Q_1,\ y \in Q_m$ and $\partial Q_k \cap \partial Q_{k+1} \neq \emptyset$.

$$
\begin{aligned}
d_W(x,y) &= \#\text{ cubes in minimal chain.} \\
&= \inf_{\gamma} \int_{\gamma} \frac{ds}{d_D(s)} \;=\; \text{quasihyperbolic metric.}
\end{aligned}
$$

Note: If $D \subset \mathbb{R}^2$ is simply connected, this is "equivalent" to the hyperbolic metric as defined above. Now, a simple application.

Assume $D \subset \mathbb{R}^n, n \geq 2$ has **capacity boundary** and $d_W(x_0, x) \leq \dfrac{C_1}{d_D(x)^{\alpha}} + C_2$, $x_0 \in D$ fixed. Harnack $\Rightarrow h(x) \geq e^{-c d_w(x,x_0)} h(x_0)$.

If $D =$ unit disk $= \{z \in \mathbb{C}\colon |z| < 1\} = D_0$ ([50], [24]),

$$\sup_{\substack{x\in D\\ h\in H^+(D)}} E_x^h(\tau_{D_0}) = E_{-1}^1(\tau_{D_0}) = 4\log 2 - 2 \approx .7728$$

and for the rectangle $R_n = [-n, n] \times [-\pi/n, \pi/n]$, $x_1 = -n, x_2 =$

n,

$E_{x_1}^{x_2}(\tau_{R_n}) \leq 1$

$$\lim_{n\to\infty} E_{x_1}^{x_2}(\tau_{R_n}) = 1.$$

Thus, a long thin rectangles beats the disk.

Problem 4: Amongst all domains $D \subset \mathbb{R}^2$ of area 1, find the one(s) that maximizes $\sup_{\substack{z\in D\\ h\in H^+(D)}} E_x^h(\tau_D)$. That is, find the best constant C in the Cranston-McConnell inequality.

We will see shortly that $\exists\ D \subset \mathbb{R}^2$ with area$(D) = \infty$ and

$$\sup_{\substack{x\in D\\ h\in H^+(D)}} E_x^h(\tau_D) < \infty.$$

Thus volume is not the "determinant" factor.

Problem 5: Give a geometric characterization for those D's for which $\sup E_x^h(\tau_D) < \infty$.

What is known about problem 4?

Theorem: (P. Griffen, T. McConnell, G. Verchota, [50]): Let $D \subset \mathbb{R}^2$ be simply connected. Set

$$L_D(\alpha, \beta) = E_\alpha^\beta(\tau_D), \ \alpha, \ \beta \in \overline{D}.$$

$$\sup_{\alpha,\beta \in \overline{D}} L_D(\alpha, \beta) = \sup_{\alpha,\beta \in \partial D} L_D(\alpha, \beta) \tag{1}$$

$$\sup_{\alpha,\beta \in \partial D} L_D(\alpha, \beta) < \frac{1}{\pi} \text{ area}(D). \tag{2}$$

(3) If D is convex symmetric P = perimeter, d = diameter, R_D = inradius. Then

$$\sup_{\alpha,\beta \in \partial D} L_D(\alpha, \beta) \geq \frac{\text{Area}(D)}{\pi} - \frac{4}{\pi} R_D(P - 2d).$$

In particular, if

$$R_n = \left[-\frac{n}{2}, \frac{n}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

Then

$$\frac{1}{|R_n|} E_{-\frac{n}{2}}^{\frac{n}{2}}(\tau_R) \geq \frac{1}{\pi} - \frac{4}{\pi n}.$$

Remark: Xu [68] had proved before that there is a universal constant C such that for all convex domains $D \subset \mathbb{R}^2$,

$$\sup_{\alpha,\beta \in \overline{D}} L_D(\alpha, \beta) \geq C \text{Area}(D).$$

Problem 6 (Conjecture) Prove that amongst all planar convex domains of area π, the unit disk minimizes

$$\sup_{\alpha,\beta\in\partial D} L_D(\alpha,\beta).$$

That is, $D \subset \mathbb{R}^2$ convex, area $(D) = \pi \Rightarrow$

$$\sup_{\alpha,\beta\in\partial D} L_D(\alpha,\beta) \geq E^1_{-1}(\tau_{D_0}).$$

What is known about Problem 5? Necessary and sufficient conditions for finiteness of $L(\alpha,\beta)$ for domains above the graph of a function in the plane.

$$D_f = \{(x,y)\colon 0 < x < 1,\ f(x) < y < 1\},$$

f non-positive uppersemicontinuous. For $z \in D$, let

$A_\varepsilon(z) =$ Area of Whitney cubes which intersect the geodesic from $\left(\frac{1}{2},\frac{1}{2}\right)$ to z and which have length less than ε.

$$A(\varepsilon) = \sup_{z\in D} A_\varepsilon(z)$$

Theorem (R. Bañuelos, B. Davis [23]): $\sup_{\alpha,\beta\in\overline{D}} L_D(\alpha,\beta) < \infty$ if and only if $A(\varepsilon) \leq C$, for all $0 < \varepsilon \leq 1$.

This theorem easily provides examples of domains of infinite area for which the Cranston-McConnell estimate holds. (The first examples of domains of infinite area with bounded conditional lifetime were given in Xu [68].) More general examples will be given below.

The simply connected domain D has the Γ–condition if there is a fixed point $0 \in D$ such that for any $x \in D$ there is a curve $\Gamma \subset D$ from 0 to x with the property that

$$\text{dist}\,(z, D^c) \geq c\ \text{dist}(x, D^c),\ \forall z \in \Gamma.$$

Notice that D_f has the Γ–condition.

Theorem(R. Smits [63]: Best known necessary and sufficient condition.) Suppose D has the Γ–condition. Then $\sup_{\alpha,\beta \in \overline{D}} L(\alpha, \beta) < \infty$ if and only if $A(\varepsilon) \leq C$ ("big O") , for all $0 < \varepsilon \leq 1$.

Clearly domains above the graph of a function satisfy the Γ condition. The following are two examples of domains satisfying the conditions of Smits' theorem which are not above the graph of a function. The examples are from [63]., (draw a picture).

Example 1. Let $D_k = \{(x, y) : \cos(\frac{\pi}{3^k}) < x < \infty, > y| < f_k(x) = C_k/(x+1)^2\}$ where $C_k = (\cos(\pi/3^k + 1))^2/2^{k+1}$. Let $z_k^j, j = 1, \ldots, 2^{k-1}$ be the centers of the intervals removed to create the Cantor set. That is, $z_1^1 = \frac{1}{2}(\frac{1}{3}+\frac{2}{3}) = \frac{1}{2}, z_2^1 = \frac{1}{2}(\frac{1}{9}+\frac{2}{9}) = \frac{1}{6}, \ldots$. Set $W_k = \cup_j e^{2\pi i z_k^j} D_k$ where $e^{i\theta} D = \{e^{i\theta} z : z \in D\}$. Finally, set $D = D_0\,(\cup_{k=1}^{\infty} W_k)$ where D_0 is the unit disk. This domains, (need to check) satisfies the conditions of the theorem and

$$\text{area}(D) \geq \sum_{k=1}^{\infty} 2^{k-1} \int_1^{\infty} f_k(x)dx = \infty.$$

Example 2. Let $\tilde{D} = \{(x, y)$ such that $\frac{\sqrt{2}}{2} < x < \infty$ and $|y| < \frac{1}{x^2}\}$. Let $e^{i\theta}\tilde{D}$ be as in the previous example for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. From the complex plane we delete the x and y axes except for the portion inside the unit ball. For l = 2,3, ... delete the $4l^2$ equally spaced line segments between $x = l-1$ and $l, y = l-1$ and $l, x = -l$ and $-l+1$ and $y = -l$ and $-l+1$ which are parallel to the axes and don't intersect $\cup_j e^{\frac{ij\pi}{4}}\tilde{D} \quad j = 1, 3, 5$ and 7. The resulting domain satisfies the conditions of the theorem and is not above the graph of a function. The domain is almost the whole plane in the sense that its complement has Lebesgue measure zero

Remark: Unfortunately, there are examples of domains of infinite area which do not satisfy the γ condition but which have bounded conditional expected lifetime, [18]. Thus, the geometric necessary and sufficient condition for planar simply connected domains to have the Crason–McConnell estimate may be very difficult to find.

Key to above results: Conditioned Brownian paths "follow" hyperbolic geodesics.

Theorem (Bañuelos–Carroll [18]) Let $Q \subset D$ be a geodesic ball of radius 1. Let $z \in D$, $\xi \in \partial D$ and γ the geodesic from z to ξants C_1 and C_2 such that

$$\begin{aligned} C_2\, e^{-2\ \rho_D(\gamma,Q)} &\leq P_x^{\xi}\{B_t \in Q \text{ for some } t < \tau_D\} \\ &\leq C_1\, e^{-2\ \rho_D(\gamma,Q)} \end{aligned}$$

$$\rho_D(\gamma, Q) = \text{ hyperbolic distance from } \gamma \text{ to } Q.$$

Also, for any $\alpha, \beta \in \overline{D}$,

$$\frac{1}{4}\int_D e^{-\rho_D(z,\gamma)} dz \leq E_\alpha^\beta\,(\tau_D) \leq 4\int_D e^{-\rho_D(z,\gamma)} dz,$$

where $\rho_D(z,\gamma)$ is the hyperbolic distance from the point $z \in \gamma$ and γ is the hyperbolic geodesic through α and β.

See also A. Ancona [7], [8].

Problem 7: Is there a sharp estimate similar to $(*)$ for Cartan–Hadamard manifolds of curvature between two negative constants?

Lecture II

Intrinsic Ultracontractivity:

Heat kernels, Eigenfunctions, Eigenvalues, Connections to I.

Some Results, Some Problems

Basic Question: 'How does the Euclidean geometry of the domain (volume, diameter, inradius) and hyperbolic-quasihyperbolic geometry (growth of hyperbolic and quasihyperbolic distance) affect these Objects?

Aim: As in Lecture I, our goal is to describe **sharp Estimates**

Recall: $T_t = e^{-tH}$ a symmetric Markovian semigroup on $L^2(X, dx)$,

(self–adjoint, positive preserving, and a contraction on L^p, $1 \leq p \leq \infty$) is said to be *ultracontractive* if

$$T_t\colon L^1 \to L^\infty, \qquad \forall\, t > 0.$$

Equivalent to:

$$T_f f(x) = \int_X P_t(x,y) f(y) dy$$

where $P_t(x,y)$ is a symmetric kernel satisfying

$$0 \leq P_t(x,y) \leq C_t \quad \forall\, x, y \in D,\ t > 0,$$

C_t independent of x and y.

Example 1. The semigroup of Brownian motion in $\mathbb{R}^n$.

$$P_t(x,y) = \frac{1}{(2\pi t)^{n/2}}\, e^{-\frac{|x-y|^2}{2t}} \le \frac{1}{(2\pi t)^{n/2}}.$$

Example 2. The Cauchy semigroup in $\mathbb{R}^n$.

$$P_t(x,y) = \frac{c_n t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} \le \frac{c_n}{t^n}$$

(and other symmetric stable processes with $0 < \alpha \le 2$).

Example 3. Semigroup T_t^D of killed Brownian motion in $D \subset \mathbb{R}^n$, D any domain.

$$\begin{aligned} P_t^D(x,y) &= \text{ Dirichlet heat kernel for } \frac{1}{2}\Delta \text{ in } D \\ &\le \frac{1}{(2\pi t)^{n/2}}\, e^{\frac{-|x-y|^2}{2t}} \le \frac{1}{(2\pi t)^{n/2}}. \end{aligned}$$

The Intrinsic Semigroup: Let φ_1 be the first eigenfunction for D normalized by $\int\limits_D \varphi_1^2 dx = 1$, corresponding to λ_1. Set

$$\begin{aligned} \widetilde{P}_t^D(x,y) &= \frac{e^{\lambda_1 t} P_t^D(x,y)}{\varphi_1(x)\varphi_1(y)}, \\ \widetilde{T}_t^D f(x) &= \int_D \widetilde{P}_t^D(x,y) f(y)\, d\mu(y),\ d\mu = \varphi_1^2 dx_1,\ f \in L^2(D, d\mu). \end{aligned}$$

Remark: $\widetilde{T}_t^D$ is the semigroup of Brownian motion conditioned to remain forever in D.

(i) $\widetilde{P}_t^D(x,y) = \widetilde{P}_t^D(y,x)$ (symmetry)

(ii) $\displaystyle\int_D \widetilde{P}_t(x,y)d\mu(y) = \frac{e^{\lambda_1 t}}{\varphi_1(x)} \int_D P_t(x,y)\varphi_1(y)dy = 1$

(iii) $\displaystyle\widetilde{T}_t^D f(x) = \frac{e^{\lambda_1 t}}{\varphi_1(x)} \int_D P_t^D(x,y)f(y)\varphi_1(y)dy = \frac{e^{\lambda_1 t}}{\varphi_1(x)}\, T_t^D(f\varphi_1)(x)$

(i)—(iii) $\Rightarrow \widetilde{T}_t^D$ is a symmetric Markovian semigroup on $L^2(D,d\mu)$.

Definition (E.B. Davies–B. Simon [44]): $\{T_t^D\}$ is said to be *intrinsically ultracontractive* **(IU)** if the intrinsic semigroup $\{\widetilde{T}_t^D\}$ is *ultracontractive.* That is,

$$\widetilde{T}_t^D\colon\ L^1(D,d\mu) \to L^\infty(D,d\mu),\ \forall t>0.$$

Or,

$$P_t^D(x,y) \le a_t e^{-\lambda_1 t}\varphi_1(x)\varphi_1(y),\ \forall t>0,$$

a_t independent of x, and y. In fact, $a_t = \|\widetilde{T}_t^D\|_{1,\infty}$ and $a_t \le c_t$ for some non–increasing (in t) c_t.

We will say that D is intrinsically ultracontractive **(IU)** in this case.

Consequences of IU

Theorem (Well known and easy; see for example Smits [64]):

Suppose $D \subset \mathbb{R}^n$ is IU. $\exists\, C > 0$ such that $\forall\, t > 1$,

$$(*) \qquad e^{-(\lambda_2-\lambda_1)t} \le \sup_{x,y\in D} |\widetilde{P}_t^D(x,y) - 1| \le Ce^{-(\lambda_2-\lambda_1)t}$$

$\lambda_2 =$ second Dirichlet eigenvalue. $(\lambda_2 - \lambda_1) =$ spectral gap.

(**Remark:** Ture for any intrinsic ultracontractive semigroup.)

Corollary: If $D \subset \mathbb{R}^n$ is IU then

(i) $\displaystyle \sup_{\substack{x\in D\\ h\in H^+(D)}} \left(\frac{\varphi_1(x)}{h(x)} \int_D \varphi_1(y)h(y)dy \right) < \infty$

(ii) $\displaystyle \sup_{\substack{x\in D\\ h\in H^+(D)}} E_x^h(\tau_D) < \infty$

(iii) $\displaystyle \lim_{t\to\infty} e^{\lambda_1 t} P_x^h\{\tau_D > t\} = \frac{\varphi_1(x)}{h(x)} \int_D \varphi_1(y)h(y)dy$

In particular,

$$-\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \, \log P_x^h\{\tau_D > t\}.$$

All follows from $(*)$ and

$$P_x^h\{\tau_D > t\} = \frac{1}{h(x)} \int_D P_t^D(x,y)h(y)dy.$$

Indeed, let $t_0 > 1$ be such that for all $t \geq t_0$, and all $x, y \in D$,

$$\frac{1}{2} \leq \frac{e^{\lambda_1 t} P_t^D(x,y)}{\varphi_1(x)\varphi_1(y)} \leq \frac{3}{2}$$

$$\frac{1}{2}\, e^{-\lambda_1 t_0}\, \frac{\varphi_1(x)}{h(x)} \int_D h(y)\varphi_1(y)dy \leq \frac{1}{h(x)} \int_D P_{t_0}^D(x,y)h(y)dy \leq 1.$$

For $t \geq t_0$

$$\begin{aligned} P_x^h\{\tau_D > t\} &= \frac{1}{h(x)} \int_D P_t^D(x,y)h(y)dy \\ &\leq \frac{3}{2}\, e^{-\lambda_1 t} \sup_{x \in D} \left(\frac{\varphi_1(x)}{h(x)} \int_D \varphi_1(y)h_1(y)dy \right) \end{aligned}$$

Remark. There are many other interesting consequences of IU.

Here are just two more examples:

1. $(\varphi_n, \lambda_n) = n$–eigenfunction $, n$–eigenvalue. $IU \Rightarrow$

$$|e^{-\lambda_n t}\varphi_n(x)| = \left| \int_D P_t^D(x,y)\varphi_n(y)dy \right| \leq a_t e^{-\lambda_1 t}\varphi(x) \int_D |\varphi_n(y)|\varphi_1(y)dy.$$

Thus for all n, $|\varphi_n(x)| \leq a_t e^{-\lambda_1 t} e^{\lambda_n t}\varphi_1(x),\ \forall n,\ \forall t > 0.$

2. IU $\Rightarrow$

$$P_x\{\tau_D > t\} = \int_D P_t(x,y)dy \leq a_t\varphi(x)e^{-\lambda_1 t}.$$

Problem 8 (The van den Berg 1983 Conjecture on Dirichlet spectral gaps [33]): Let $D \subset \mathbb{R}^n$ be a convex domain of diameter d. Then

$$\frac{3\pi^2}{2d^2} < \lambda_2 - \lambda_1$$

with the lower bound approached for thin rectangles.

First result: SWYY [62] 1985: $\dfrac{\pi^2}{8d^2} < \lambda_2 - \lambda_1$.

Time to equilibrium: For $\varepsilon > 0$, define

$$T_\varepsilon^D = \inf\left\{t > 0\colon \sup_{x,y\in D} |\tilde{P}_t^D(x,y) - 1| \le \varepsilon\right\}.$$

Brownian ____________________

Theorem (R. Smits [64]): For $D \subset \mathbb{R}^n$ convex diameter d,

$$T_\varepsilon \le C_D^1 + \frac{2d^2}{\pi^2}\log\frac{1}{\varepsilon}.$$

Problem $8'$: Amongst all convex domains of fixed diameter the rate to equilibrium is smallest (takes longer time to reach it) for thin rectangles.

Remark: For fixed area, there are arbitrarily large rates to equilibrium. For more on this and the intuition comparing time to equilibrium for Brownian motion conditioned to remain forever in D to that of reflected Brownian motion in D, see R. Smits[64] "Spectral gaps and rates to equilibrium on convex domains."

L. Saloff-Coste[60]: "Precise estimates on the rate at which certain diffusions tend to equilibrium" (preprint).

Problem 9: (The "hot spots" conjecture of J. Rauch (1974)). Let $D \subset \mathbb{R}^n$. Let ψ_2 be any eigenfunction for the Neumann Laplacian in D corresponding to μ_2 (the first nontrivial eigenvalue). Then ψ_2 has its maximum (and minimum) on the boundary and only on the boundary of D.

Progress:

- Not true for arbitrary domains: (Werner, ... Details in a coming paper of Burdzy and Werner).

- Bañuelos–Burdzy (1997 preprint): True for several types of planar convex domains.

- Jerison–Naridashvili: Claim to have proved it for all planar convex domains. (No paper yet!)

Problem 10 (Conjecture) Prove that the Brownian motion conditioned to remain forever in D, D convex in $\mathbb{R}^n$, also has the "hot spots" property. Namely, that φ_2/φ_1 also has its maximum (and minimum) on the boundary and only on the boundary of D.

Problem 11: Give a geometric characterization of IU.

Sufficient Conditions: Let $f\colon [0,1] \to (-\infty, 0]$.

$$D_f = \{(x,y)\colon 0 < x < 1,\ f(x) < y < 1\}.$$

- First Group:

1. "1." B. Davies–B. Simon (1984) [44]: f Lipschitz $\Rightarrow D_f$ IU.
2. "2." Fabes–Garofalo–Salsa (1986) : f Lipschitz $\Rightarrow D_f$ IU.
3. "3." D. DeBlassie (1987) [48]: f Lipschitz with small constant $\Rightarrow$ IU.
4. "4." C. Kenig–J. Pipher (1988) [52]: f Lipschitz $\Rightarrow$ IU
5. "1." Burgess Davis (1990) [47]: f bounded $\Rightarrow D_f$ IU.
6. "2." R. Bass—K. Burdzy (1990) [31]: $f \in L^p,\ p > 1 \Rightarrow D_f$ IU.
7. "3." R. Bañuelos (1990) [13]: $D \subset \mathbb{R}^n$ with **capacity boundary condition** and $d_w(x_0, x) \leq \dfrac{C_1}{d_D(x)^\alpha} + C_2,\ 0 < \alpha < 2$.
   (1) Then D IU.
   (2) $\forall\ \alpha \geq 2,\ \exists\ D$ s.t. $d_W(x_0, x) \sim \dfrac{1}{d_D(x)^\alpha}$ and D not IU.

Theorem (Bañuelos–Davis [23]). D_f is IU if and only if $\Leftrightarrow$ $\lim\limits_{\varepsilon \to 0} A(\varepsilon) = 0$.

Problem 12 (Conjecture): Let $D \subset \mathbb{R}^2$ be s.c. with the Γ–condition. D is IU $\Leftrightarrow \lim\limits_{\varepsilon \to 0} A(\varepsilon) = 0$.

Probabilistic characterization of IU (Bass–Burdzy [31]):

IU $\Leftrightarrow$ For each $t > 0$ $\exists$ compact $K_t \subset D$ such that for all $x \in D$,

$$P_x\{B_t \in K_t | \tau_D > t\} > a_t$$

where a_t is independent of x. (This was used in Bass–Burdzy [31] and Bañuelos–Davis [23].)

Analytic: (Log–Sobolev, E. B. Davies [43]) $T_t = e^{-tA}$.

Suppose $\forall\ \varepsilon > 0\ \exists \beta(\varepsilon)$ such that for all $u \geq 0,\ u \in \mathrm{Dom}(A)$,

$$\int_X u^2 \log u dx \leq \varepsilon Q(u,u) + \beta(\varepsilon)\|u\|_2^2 + \|u\|_2^2 \log \|u\|_2^2,$$

$Q(u,u) = \langle -Au, u\rangle$. Then

$$\begin{aligned} \|T_t u\|_{L^\infty} &\leq e^{CM(t)}\|u\|_1 \\ M(t) &= \frac{1}{t}\int_0^t \beta(\varepsilon) d\varepsilon < \infty. \end{aligned}$$

Converse true with $\beta(\varepsilon) = M\left(\frac{\varepsilon}{4}\right) + 2$.

Since $P_t^D(x,y) \leq \dfrac{1}{(2\pi t)^{n/2}}$ we have $\forall\ u \geq 0,\ u \in C_0^\infty(D)$,

For $\tilde{T}_t^D = e^{-tA},\ A = \dfrac{1}{2}\ \Delta + \dfrac{\nabla\varphi}{\varphi} \cdot \nabla$ and

$$Q(u,u) = \langle -Au, u\rangle = \frac{1}{2}\int_D |\nabla u|^2 \varphi_1^2(x)dx.$$

Suppose we can prove: $\forall\ 0 \leq u \in C_0^\infty(D)$

$$(2) \qquad \int_D |u|^2 \log\ \frac{1}{\varphi_1}\ dx \leq \varepsilon \int_D |\nabla u|^2 dx + \beta(\varepsilon)\|u\|^2_{L^2(dx)}.$$

Adding (1) and (2) gives for all $u \in C_0^\infty(D)$,

$$\begin{aligned}\int |u|^2 \log\left(\frac{u}{\varphi}\right) dx \ &\leq\ 2\varepsilon \int_D |\nabla u|^2 dx + (cn \log\ \frac{1}{\varepsilon}\ + \beta(\varepsilon) + 2)\|u\|_{L^2(dx)} \\ &+\ \ \|u\|^2_{L^2(dx)} \log \|u\|^2_{L^2(dx)}.\end{aligned}$$

Need to have good estimates on $\log \dfrac{1}{\varphi_1(x)}$. Often Harnack is enough:

$$\varphi_1(x) > \varphi(x_0) e^{-C d_w(x, x_0)}$$

or

$$\log\left(\frac{1}{\varphi_1(x)}\right) \leq C_1 \; d_w(x, x_0) + C_2.$$

Suppose: $\quad d_W(x, x_0) \leq \dfrac{C_3}{d_D(x)^\alpha} + C_4, \quad 0 < \alpha < 2.$

$$\log \frac{1}{\varphi_1(x)} \leq \frac{C_1}{d_D(x)^\alpha} + C_2 \leq C_1 \; \frac{\varepsilon}{d(x)^2} + C_2 \; \varepsilon^{-\frac{\alpha}{2-\alpha}}.$$

$$\left(ab \leq \frac{a^p}{p} + \frac{b^q}{q}\right)$$

$$\begin{aligned}\int_D u^2 \log \frac{1}{\varphi_1(x)} \, dx &\leq C_1 \; \varepsilon \int_D \frac{|u|^2}{d(x)^2} \, dx + C_2 \; \beta(\varepsilon) \int_D (u)^2 dx \\ &\leq C_1 \; \varepsilon \int_D |\nabla u|^2 dx + C_2 \; \beta(\varepsilon) \|u\|_2.\end{aligned}$$

(By Ancona [6], provided D has capacity boundary condition.)

Remark: The log-Sobolev approach to (IU) leads to many interesting and difficult questions concerning the sharp decay at the boundary of Dirichlet eigenfunctions. There is now a large body of research concerning this topic. Here we mention only two of the earlier results and give references to many of the more recent ones. First, a non-sharp result.

Theorem (B. Davies–B. Simon [44]): Let $f : C^2[0,1] \to \mathbb{R}$, $f \downarrow 0$, $f \in L'(\mathbb{R}^+)$, $|f'| \le M$, $f'(x)/f(x) \to 0$ as $x \to \infty$. Let $\varphi_1(x,y)$ be the first eigenfunction for $D_f = \{(x,y) : -f(x) < y < f(x)\}$. Then

$$C_1 \; e^{-\frac{C_2 x}{f(x)}} \le \varphi_1(x,0) \le C_3 \; e^{-\frac{C_4 x}{f(x)}}.$$

Here are the two examples of sharp results:

Theorem (R. Bañuelos–B. Davis [16]): Suppose $|f'(x)| \le M$, $f \downarrow 0$, $f \in L^1$.

$$C_1 \; e^{-\frac{\pi}{2}\int\limits_1^x \frac{dt}{f(t)} - \frac{\pi}{24}\int\limits_1^x \frac{f'(t)^2}{f(t)}dt} \le \varphi_1(x,0) \le C_2 \; e^{-\frac{\pi}{2}\int\limits_1^x \frac{dt}{f(t)}}.$$

Theorem: (Bañuelos [20]) $D \subset \mathbb{R}^2$, area$(D) < \infty$, s.c. Fix $z_0 \in D$.

$$C_2 \; e^{-2\rho_D(z_0,z)} \le \varphi_1(z) \le C_1 \; e^{-2\rho_D(z_0,z)}.$$

Remark: The last theorem was motivated by the fact that for these domains

$$\rho_D((1,0),(x,0)) \approx \frac{\pi}{4}\int_1^x \frac{ds}{f(s)}.$$

Many other sharp eigenfunction estimates in: Bañuelos [20], Lapidus–Pang [55], Bañuelos–van den Berg [21], Lindeman–Pang–Zhao [56], Cranston–Li [40], van den Berg–Bolthausen [32]. Gradient estimates: Cranston-Zhao [19], Bañuelos-Pang [27], Athanasopoulos-Caffarelli-Salsa [10],. . . .

Estimates for eigenfunctions, Green functions, Poisson kernels and Intrinsic Ultracontractivity for symmetric stable processes can be found in the recent work of Zhen–Qing Chen and Renming Song, [33], [34].

Neumann: Julian Edwards (Florida International University) "Eigenfunction decay for the Neumann Laplacian on Horn-like domains," (preprint) and references there.

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Note: Not included in this list are many references to applications of conditioned Brownian motion to boundary behavior of harmonic and caloric functions (Martin boundary, boundary Harnack principle, etc.), nor to conditional gage theorems and their applications to Harnack inequalities and potential theory for Schrödinger operators. Some of these topics will be discussed in the lectures of Professor Richard Bass.

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