Level sets of Neumann eigenfunctions

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Abstract

In this paper we prove that the level sets of the first non-constant eigenfunction of
the Neumann Laplacian on a convex planar domain have only finitely many connected
components. This problem is motivated, in part, by the “hot spots” conjecture of J.
Rauch.

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1 Introduction

Let $\Omega$ be a simply connected bounded domain in the complex plane $\mathbb{C}$ for which the spectrum of the Laplacian in $\Omega$ with Neumann boundary conditions is discrete. Let $\varphi$ be the eigenfunction corresponding to the lowest nonzero eigenvalue $\mu$. It is well known (and follows easily from the comparison of the Neumann eigenvalue to the Dirichlet eigenvalue) that the nodal line $\{z \in \Omega : \varphi(z) = 0\}$ cannot enclose a subdomain of $\Omega$. However, very little, if anything, seems to be known about the geometry of the level sets for values other than zero. The purpose of this paper is to prove that under the assumption that the domain $\Omega$ is strictly convex with $C^\infty$ boundary, the level sets have finitely many connected components.

The desire to understand the geometry of the level sets of the first Neumann eigenfunction is motivated, in part, by the “hot spots” conjecture. This conjecture has received a lot of attention since it was proposed by J. Rauch in 1974. It asserts that the maximum and the minimum of any eigenfunction corresponding to $\mu$ are attained on the boundary and only on the boundary of the domain. The conjecture, as it turns out, is false for general planar domains, but it remains open even for arbitrary convex planar domains. We refer the reader to [1], [2] and [3] for some history, recent results, and additional references on the conjecture. Our main result in this paper is

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}$ be a bounded convex domain with $C^\infty$ boundary. Suppose that $\partial \Omega$ has positive curvature at each of its points. Let $\varphi$ be an eigenfunction corresponding to the smallest positive eigenvalue $\mu$ of the Neumann Laplacian $-\Delta_N$ on $\Omega$. For $c > 0$, let

$$N(c) = \{z \in \overline{\Omega} : \varphi(z) = c\}$$

be a level set of $\varphi$. Then $N(c)$ has finitely many connected components.

One can see by a compactness argument that Theorem 1.1 is equivalent to

**Theorem 1.2.** If $P(c)$ is a connected component of $N(c)$, then

(1.1) $\text{dist}(P(c), N(c) \setminus P(c)) > 0$.

It is this equivalent form of Theorem 1.1 that we shall prove. Theorem 1.2 will be proved by contradiction. We shall assume that (1.1) is false. If

(1.2) $\text{dist}(P(c), N(c) \setminus P(c)) = 0$,

then, since $P(c)$ is compact, there exist a point $\alpha \in P(c)$ and a sequence of points $\{b_n\}_{n=1}^\infty$ in $N(c) \setminus P(c)$ such that

(1.3) $\lim_{n \to \infty} b_n = \alpha$.

This will lead to a contradiction as we shall show that there is no $\alpha \in \overline{\Omega}$ for which this can happen.

**Remark 1.1.** It should be mentioned here that if the domain has analytic boundary (even if it is not uniformly convex) then the result follows easily. Indeed, by Theorem 5.7.1' on p. 169 of [12], if the coefficients of a second order elliptic equation are real analytic on a
bounded analytic domain $D$ up to the boundary, then for every point $z$ on $\partial D$ there exists a ball $B$ centered at $z$ such that solutions of the elliptic equation can be extended to be real analytic functions on $B$. The result follows easily from this fact (see the remark following Proposition 2.1 below). Even more general results are available from the theory of analytic and subanalytic sets in the case of analytic domains. A careful examination of our proof shows that our result holds even if the boundary of the domain is only assumed to be of class $C^n$, for a sufficiently large $n$. Of course, we still need to assume that $\partial \Omega$ has positive curvature at every point.

The paper is organized as follows. In §2 we will state several known results and prove a proposition for conformal mappings which will give us various regularity properties of the eigenfunction. Since, by the remark following Proposition 2.1 below, it is easy to see that $\alpha \notin \Omega$, in §3 and §4 we will show, by mapping to the half space and using some of the results of §2, that there is no point $\alpha \in \partial \Omega$ for which (1.3) can hold. This will complete the proof.

2 Preliminary Results

In this section we will derive some results which will play a key role in our proofs in the subsequent sections. We begin by recalling the following result from the theory of real analytic functions. It can be found in [11], Theorem 6.5.12.

**Proposition 2.1.** Let $D$ be a simply connected domain in the plane and let $u$ be a real analytic function on $D$. Consider a level set $\Gamma = \{z \in D : u(z) = c\}$ of $u$. Then any compact subset of $D$ intersects at most finitely many connected components of $\Gamma$.

**Remark 2.1.** We recall that solutions of an elliptic equation with real analytic coefficients are real analytic and hence any eigenfunction in $\Omega$ is real analytic inside the domain $\Omega$, see [4], p. 136. This result together with Proposition 2.1 imply that if $\alpha$ is as in Equation (1.3), then $\alpha \notin \Omega$. For if we have $\alpha \in \Omega$, then for a sufficiently small closed disc $K \subseteq \Omega$ center at $\alpha$, only a finite number of connected components of $N(c)$ can intersect $K$. Since the distance from $\alpha$ to each of these connected components is positive, the distance from $\alpha$ to $N(c) \setminus P(c)$ is positive.

**Lemma 2.1.** Let $D$ be a bounded $C^\infty$ planar domain and let $\psi$ be an eigenfunction of the Neumann Laplacian $-\Delta_N$ on $D$. Then $\psi \in C^\infty(D)$.

**Remark 2.2.** This lemma follows from [10], Theorem 2.4.2.7 and 2.5.1.1 and [8], p. 262, using a standard iteration argument.

In the proposition below, we will denote by $B(z, \delta)$ a disc centered at a point $z \in \mathbb{R}^2$ with radius $\delta$.

**Proposition 2.2.** Let $\Omega \subset \mathbb{C}$ be a bounded convex domain with a positively curved $C^\infty$ boundary. That is, $\Omega$ satisfies the hypothesis of Theorem 1. Let $\alpha_0 \in \partial \Omega$. Then there exist a conformal map $h : \Omega \to \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane

$$\mathbb{H} : \{z = x + iy \in \mathbb{C} : y > 0\},$$

where ...
and \( \delta > 0 \) such that \( h \mid_{B(\alpha_0, \delta) \cap \Omega} \) can be extended to a homeomorphism on \( \overline{B(\alpha_0, \delta) \cap \Omega} \) and \( h' \mid_{B(\alpha_0, \delta) \cap \Omega} \) can be extended to a continuous functions on \( \overline{B(\alpha_0, \delta) \cap \Omega} \) and such that

\[
\partial_h \left( \left( \left( h^{-1} \right)' \right)^2 \right)(h(\alpha_0)) \neq 0
\]

and

\[
\partial_y \left( \left( \left( h^{-1} \right)' \right)^2 \right)(h(\alpha_0)) < 0.
\]

**Remark 2.3.** In the proof below and for the rest of the paper, we will sometimes use the following convention: if \( F \) is a function defined on a planar domain \( D \) and if we write \( z = x + iy \) to denote a point in \( D \), we will often write \( F(x, y) \) for \( F(z) \).

**Proof.** By the Kellog-Warschawski Theorem (see [13], p.49), it is easy to check that there exists a conformal map \( g : \mathbb{H} \to \Omega \) with the following properties:

(i) there exists \( \epsilon > 0 \) such that \( g \) can be extended to a continuous function on \( \{(x, 0) \in \mathbb{R}^2 : -\epsilon < x < \epsilon\} \),

(ii) \( g(0, 0) = \alpha_0 \),

(iii) there exists \( \delta > 0 \) such that \( g^{-1} \mid_{B(\alpha_0, \delta) \cap \Omega} \) can be extended to a homeomorphism on \( \overline{B(\alpha_0, \delta) \cap \Omega} \) and that \( (g^{-1})' \mid_{B(\alpha_0, \delta) \cap \Omega} \) can be extended to a continuous function \( (g^{-1})' \mid_{B(\alpha_0, \delta) \cap \Omega} : \overline{B(\alpha_0, \delta) \cap \Omega} \to \mathbb{C} \),

(iv) \( (g^{-1})'(\alpha_0) \neq 0 \), Writing \( g(z) = u(z) + iv(z) \) where \( u \) and \( v \) are real harmonic functions, since \( \partial \Omega \) has positive curvature, we have

\[
(2.3) \quad -\frac{\partial^2 u}{\partial x^2} \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} \frac{\partial u}{\partial x} > 0.
\]

Using the Cauchy-Riemann equations, we see that (2.3) implies that

(v) \( \frac{\partial}{\partial y}(\left| g' \right|^2)(0, 0) < 0. \)

If

\[
\frac{\partial}{\partial y}(\left| g' \right|^2)(0, 0) \neq 0,
\]

then we can put \( h = g^{-1} \) and the proposition is proved. So we assume that

\[
(2.4) \quad \frac{\partial}{\partial x}(\left| g' \right|^2)(0, 0) = 0.
\]

Let \( G : \mathbb{H} \to \Omega \) be the conformal map

\[
(2.5) \quad G(z) = g(\hat{x}^{-1} - z^{-1}) = g \circ U(z), \quad z \in \mathbb{H}
\]
where $\hat{x} \in \mathbb{R}\setminus\{0\}$ and
\begin{equation}
U(z) = (\hat{x}^{-1} - z^{-1}), \quad z \in \mathbb{H}.
\end{equation}
Then
\begin{equation}
G(\hat{x}, 0) = \alpha_0,
\end{equation}
and, by the assumption (2.4),
\begin{equation}
\frac{\partial}{\partial x}(|G(\hat{x}, 0)|^2) \neq 0.
\end{equation}
Writing $z = x + iy$, we have
\begin{align*}
U(z) &= \frac{1}{x} - \frac{1}{x + iy} \\
&= \frac{[(x - \hat{x}) + iy][x - iy]}{\hat{x}(x^2 + y^2)} \\
&= \frac{(x - \hat{x})x + y^2 + i\{xy - (x - \hat{x})y\}}{\hat{x}(x^2 + y^2)}.
\end{align*}
Let
\begin{align*}
X = \Re(U(z)) &= \frac{(x - \hat{x})x + y^2}{\hat{x}(x^2 + y^2)} \\
Y &= \Im(U(z)) = \frac{xy - (x - \hat{x})y}{\hat{x}(x^2 + y^2)} = \frac{y}{(x^2 + y^2)}.
\end{align*}
Then
\begin{align*}
\frac{\partial X}{\partial y} &= \frac{2y\hat{x}(x^2 + y^2) - \{(x - \hat{x})x + y^2\}2\hat{y}y}{x^2(x^2 + y^2)^2} = \frac{2x\hat{x}y}{x^2(x^2 + y^2)^2}.
\end{align*}
and
\[
\frac{\partial Y}{\partial y} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.
\]

Therefore
\[
\frac{\partial}{\partial y}(|G'|^2)(\hat{x},0) = \frac{\partial}{\partial y}(|g' \circ U|^2|U'|^2)(\hat{x},0)
\]
\[
= \left\{ \frac{\partial}{\partial y} (|g' \circ U|^2)|U'|^2 + |g' \circ U|^2 \frac{\partial}{\partial y}(|U'|^2) \right\} (\hat{x},0)
\]
\[
= \left\{ \left[ \frac{\partial}{\partial X} (|g'|^2)(U(\cdot)) \frac{\partial X}{\partial y} + \frac{\partial}{\partial Y} (|g'|^2)(U(\cdot)) \frac{\partial Y}{\partial y} \right]|U'|^2 \right\} (\hat{x},0)
\]
\[
+ |g' \circ U|^2 \frac{\partial}{\partial y}(|U'|^2)(\hat{x},0)
\]
\[(2.10)\]
\[
= \frac{\partial}{\partial Y}(|g'|^2)(0,0)\hat{x}^{-6}.
\]

Since
\[
\frac{\partial}{\partial y}(|g'|^2)(0,0) < 0,
\]
by property (v), we have
\[(2.11)\]
\[
\frac{\partial}{\partial y}(|G'|^2)(\hat{x},0) < 0.
\]

By (2.5), (2.6) and property (iii) on \(g\) at the beginning of the proof, we see that there exists \(\delta_1 > 0\) such that \(G^{-1} \mid_{B(\alpha_0, \delta_1) \cap \Omega}\) can be extended to a homeomorphism on \(\overline{B(\alpha_0, \delta_1) \cap \Omega}\) and \((G^{-1})' \mid_{B(\alpha_0, \delta_1) \cap \Omega}\) can be extended to continuous functions on \(\overline{B(\alpha_0, \delta_1) \cap \Omega}\) and
\[
(G^{-1})'(\alpha_0) = (G'(\hat{x},0))^{-1} \neq 0.
\]

Hence the proposition follows, from (2.9) and (2.11), by putting \(h = G^{-1}\).

\[\square\]

\section{The case \(\nabla \psi(0,0) = (0,0)\) and \(\frac{\partial^2 \psi}{\partial y^2} \neq 0\)}

Let us set some notations first. Let \(\alpha\) be as in (1.3) and let \(h : \Omega \to \mathbb{H}\) be the conformal map constructed in Proposition 2.2 with the point \(\alpha_0 = \alpha\). We may assume that
\[
h(\alpha) = 0
\]
and
\[
h(\hat{\alpha}) = \infty
\]
for some \(\hat{\alpha} \in \partial \Omega\). Let
\[(3.1)\]
\[
\psi = \varphi \circ h^{-1}.
\]
where $\varphi$ is any Neumann eigenfunction of the lowest positive eigenvalue $\mu$. By a simple change of variables, we see that $\psi$ satisfies the Schrödinger equation

$$\tag{3.2} -\Delta \psi - \mu |(h^{-1})'|^2 \psi = 0$$

on $\mathbb{H}$. By [13], Theorem 3.5, $\psi$ satisfies the boundary condition

$$\tag{3.3} \frac{\partial \psi}{\partial y}(x, 0) = 0, \quad x \in \mathbb{R}.$$ 

By Lemma 2.1, $\varphi \in C^\infty(\overline{\Omega})$. So, by Proposition 2.2 and some standard results on conformal maps (see [13], Corollary 2.8, Theorems 3.5 and 3.6), there exists $\delta > 0$ such that

$$\tag{3.4} \psi \in C^\infty(\mathbb{H} \cap B(0, \delta)).$$

We now reflect $\psi$ along the real axis

$$\{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$$

by defining

$$\tag{3.5} \psi(x, y) = \psi(x, -y), \quad y < 0.$$ 

Then, by (3.3) and (3.4),

$$\tag{3.6} \psi \in C^2(B(0, \delta))$$

and satisfies

$$\tag{3.7} -\Delta \psi + V \psi = 0, \quad \text{on } B(0, \delta),$$

where

$$\tag{3.8} V(x, y) = \begin{cases} -\mu |(h^{-1})'|^2(x, y), & ((x, y) \in B(0, \delta) \text{ with } y \geq 0) \\ -\mu |(h^{-1})'|^2(x, -y), & ((x, y) \in B(0, \delta) \text{ with } y < 0). \end{cases}$$

If $\nabla \psi(0, 0) \neq (0, 0)$, then, by the implicit function theorem, the level set $\{\psi = \psi(0, 0)\}$ is a smooth curve near $(0, 0)$. So $h(\alpha) \neq (0, 0)$ in this case, which is a contradiction, and there is nothing to prove. Therefore we shall assume that

$$\nabla \psi(0, 0) = (0, 0).$$

We have two possibilities: either $\frac{\partial^2 \psi}{\partial y^2}(0, 0) \neq 0$ or $\frac{\partial^2 \psi}{\partial y^2}(0, 0) = 0$. In this section we shall deal with the first possibility $\frac{\partial^2 \psi}{\partial y^2}(0, 0) \neq 0$. The second possibility $\frac{\partial^2 \psi}{\partial y^2}(0, 0) = 0$ will be dealt with in Section 4.
Assuming that \( \partial^2\psi/\partial y^2(0,0) \neq 0 \), we shall show in Cases 3.1 and 3.2 below that \( \psi \) cannot oscillate near \((0,0)\) along the real axis in the following sense: There exists \( \hat{N} \in \mathbb{N} \) such that for all \( n \geq \hat{N} \), there exist \( x_{n,-},x_{n,0},x_{n,+} \in (\neg n^{-1},0) \) such that
\[
\begin{cases}
\frac{\partial \psi}{\partial x}(x_{n,-},0) < 0, \\
\frac{\partial \psi}{\partial x}(x_{n,0},0) = 0, \\
\frac{\partial \psi}{\partial x}(x_{n,+},0) > 0
\end{cases}
\]
(3.9) and that
\[
\cdots < x_{n-1,+} < x_{n,-} < x_{n,0} < x_{n,+} < x_{n+1,-} < \ldots.
\]
(3.10)

We shall show that (3.9) and (3.10) imply a contradiction. Then in Case 3.3 we shall deal with the case when \( \partial^2\psi/\partial y^2(0,0) \neq 0 \) and \( \psi \) does not oscillate as in (3.9) and (3.10).

So we first assume that (3.9) and (3.10) hold for \( n \geq \hat{N} \). Since, by Lemma 2.1 and [13], p. 49, \( \psi \in C^\infty(\mathbb{H}) \), it follows that \( s_{n,-,\ell},s_{n,-,r},s_{n,+,\ell},s_{n,+r} \in \mathbb{R} \) such that
\[
\frac{\partial \psi}{\partial x}(x,0) < 0, \quad s_{n,-,\ell} < x < s_{n,-,r}
\]
(3.11) and
\[
\frac{\partial \psi}{\partial x}(x,0) > 0, \quad s_{n,+\ell} < x < s_{n,+r}
\]
(3.12)

where
\[
s_{n,-,\ell} = \inf \left\{ x \in \mathbb{R} : x < x_{n,-} \text{ and } \frac{\partial \psi}{\partial x}(t,0) < 0, \text{ for all } x < t \leq x_{n,-} \right\}
\]
(3.13) and
\[
s_{n,+\ell} = \inf \left\{ x \in \mathbb{R} : x < x_{n,+} \text{ and } \frac{\partial \psi}{\partial x}(t,0) > 0, \text{ for all } x < t \leq x_{n,+} \right\},
\]
(3.14)

with similar definitions for \( s_{n,r} \) with “sup” replacing “inf”. Let \( G_{n,\ell} \) and \( G_{n,r} \) be the connected component of
\[
\left\{ z \in \overline{\mathbb{H} \cap B(0,\delta)} : \frac{\partial \psi}{\partial x}(z) \neq 0 \right\}
\]
that contains the segments
\[
\sigma_{n,\ell} = \left\{ x \in \mathbb{R} : s_{n,-,\ell} < x < s_{n,-,r} \right\}
\]
(3.15) and
\[
\sigma_{n,r} = \left\{ x \in \mathbb{R} : s_{n,+\ell} < x < s_{n,+r} \right\},
\]
(3.16)

respectively. Let \( \Gamma_{n,\ell} \) and \( \Gamma_{n,r} \) be connected compact subsets of \( \partial G_{n,\ell} \) and \( \partial G_{n,r} \), respectively. Since \( \frac{\partial \psi}{\partial x} > 0 \) on \( \sigma_{n-1,r} \cup \sigma_{n,r} \) and \( \frac{\partial \psi}{\partial x} < 0 \) on \( \sigma_{n,\ell} \), we can assume that the following two conditions on \( \Gamma_{n,\ell} \) and \( \Gamma_{n,r} \) are satisfied:
(i) $\Gamma_{n,\ell}$ separates $\sigma_{n,\ell}$ from $\sigma_{n-1,r} \cup \sigma_{n,r}$ and

(ii) $\Gamma_{n,r}$ separates $\sigma_{n,r}$ from $\sigma_{n,\ell} \cup \sigma_{n+1,\ell}$.

Let

$$
\Gamma = \left\{ z \in \overline{H} \cap B(0, \delta) : \text{there exists a sequence of points} \ \{\gamma_n\} \ \text{with} \ \gamma_2 \in \Gamma_{2,\ell}, \gamma_3 \in \Gamma_{2,r}, \gamma_4 \in \Gamma_{3,\ell}, \ldots, \ \text{that has a subsequence} \ \gamma_{n_i} \ \text{with} \ \lim_{i \to \infty} \gamma_{n_i} = z, \ \text{and that there exists} \ M \in \mathbb{N} \ \text{such that} \ z \notin \bigcup_{n \geq M} (\Gamma_{n,\ell} \cup \Gamma_{n,r}) \right\}.
$$

By Proposition 2.1 with $D = \overline{H} \cap B(0, \delta)$ and $u = \frac{\partial \psi}{\partial x}$, we have

$$
(3.18) \ \Gamma \subseteq \partial(\overline{H} \cap B(0, \delta)).
$$

We shall now consider the following cases.

**Case 3.1:** Suppose $\Gamma = \{(0,0)\}$. We first claim that for all sufficiently small $\delta_1 \in (0, \delta)$ there exists $M \in \mathbb{N}$ such that for all $n \geq M$ we have

$$
(3.19) \ \Gamma_{n,i} \subseteq \overline{H} \cap B(0, \delta_1), \ \ i = \ell, r.
$$

To see this suppose that (3.19) is not true. Then there exists an increasing sequence of positive integers $\{n_k\}$ such that

$$
\Gamma_{n_k,j} \not\subseteq \overline{H} \cap B(0, \delta_1), \ \ j = \ell \ \text{or} \ r.
$$

Let $p_{n_k}$ be the point where $\Gamma_{n_k,j}$ intersects $\partial(\overline{H} \cap B(0, \delta_1))$ and let $p$ be a limit point of the sequence $\{p_{n_k}\}$. If $p \notin \mathbb{R}$, then we have a contradiction since $\frac{\partial \psi}{\partial x}$ is analytic on $\overline{H} \cap B(0, \delta)$ and so cannot oscillate along an analytic arc in $\overline{H} \cap B(0, \delta)$. If $p \in \mathbb{R}$, then there exists a subsequence of $\{\Gamma_{n_k,j}\}$ which “approaches” a subinterval $I$ of $(-1,1)$ on $\mathbb{R}$ of positive length. (To see this, we note that for every $\epsilon > 0$ only a finite number of the connected sets $\Gamma_{n,r}$ and $\Gamma_{n,\ell}$ can intersect the horizontal line $y = \epsilon$. For otherwise $\bigcup_n (\Gamma_{n,\ell} \cup \Gamma_{n,r})$ would have a limit point on the horizontal line $y = \epsilon$. That is, a limit point inside $\overline{H}$, and this contradicts the analyticity of $\psi$ since an analytic function cannot oscillate on a straight line segment in this manner. Thus the connected sets $\Gamma_{n,i}$, $i = r, l$ will get closer and closer to the real axis as $n \to \infty$.) Thus $\frac{\partial \psi}{\partial x} = 0$ on $I$. But we have

$$
-\Delta \left( \frac{\partial \psi}{\partial x} \right) + V \frac{\partial \psi}{\partial x} = -\frac{\partial V}{\partial x} \psi,
$$

and this contradicts (2.1) on $I$.

Since $\Gamma_{n,\ell}$ (respectively $\Gamma_{n,r}$) separates $\sigma_{n,\ell}$ (respectively $\sigma_{n,r}$) from $\sigma_{n-1,r} \cup \sigma_{n,r}$ (respectively $\sigma_{n,\ell} \cup \sigma_{n+1,\ell}$), writing

$$
R(\Gamma_{n,i}) = \{ z = x - iy : y > 0 \ \text{and} \ x + iy \in \Gamma_{n,i} \}, \ \ i = \ell, r
$$

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for the reflection of $\Gamma_{n,i}$ along the real axis, for $n \geq M$

(3.20) \[ \hat{\Gamma}_{n,i} = [\Gamma_{n,i} \cup R(\Gamma_{n,i})] \setminus \{x + i0 : x \in \mathbb{R}\}, \]

is a closed curve in $B(0, \delta_1)$. Since $\psi \in C^2(B(0, \delta))$ by (3.6), we can consider $\frac{\partial \psi}{\partial x}|_{B(0, \delta_1)}$.

Let $\hat{Q}_{n,i}$ be the domain enclosed by $\hat{\Gamma}_{n,i}$. Then by (3.7) we have

(3.21) \[ -\Delta \left( \frac{\partial \psi}{\partial x} \right) + V \left( \frac{\partial \psi}{\partial x} \right) = -\left( \frac{\partial V}{\partial x} \right) \psi \text{ in } \hat{Q}_{n,i}, \]

\[ \frac{\partial \psi}{\partial x} = 0 \text{ on } \partial \hat{Q}_{n,i} \subseteq \hat{\Gamma}_{n,i}. \]

By Proposition 2.2, if $\delta_1 \in (0, \delta)$ is sufficiently small, then

(3.22) \[ -\frac{\partial V}{\partial x}(z)\psi(z) \neq 0 \quad (z \in B(0, \delta_1)). \]

Also if $\delta_1 \in (0, \delta)$ is sufficiently small, then the Schrödinger operator

(3.23) \[ L_{n,i} = -\Delta + V \]

defined on $\hat{Q}_{n,i}$, with Dirichlet boundary conditions, has a bounded inverse with Green’s function $G_{n,i}(z, \omega) > 0$. Thus, by (3.21) and (3.22), we have

(3.24) \[ \frac{\partial \psi}{\partial x}|_{\hat{Q}_{n,i}} = L_{n,i}^{-1} \left( -\left( \frac{\partial V}{\partial x} \right) \psi|_{\hat{Q}_{n,i}} \right) \]

\[ = \int_{\hat{Q}_{n,i}} -G_{n,i}(z, \omega) \left( \frac{\partial V}{\partial x} \right)(\omega)\psi(\omega)d\omega. \]

By (3.22) and (3.24) either

(3.25) \[ \begin{cases} 0 < \frac{\partial \psi}{\partial x}|_{\hat{Q}_{n,i}}(z) \\ 0 > \frac{\partial \psi}{\partial x}|_{\hat{Q}_{n,i}}(z) \end{cases} \]

for all $z \in \hat{Q}_{n,i}$, $i = \ell, r$, if $n$ is sufficiently large. This contradicts (3.11) and (3.12). Hence this case cannot occur.

**Case 3.2:** $\Gamma$ contains points in $\partial((\mathbb{H} \cap B(0, \delta))$ other than $(0, 0)$. In this case $\Gamma$ must contain an interval $[0, \gamma]$ for some $\gamma > 0$. (As in the proof of Case 3.1, to see this, we note that for every $\epsilon > 0$ only a finite number of the connected sets $\Gamma_{n,\ell}$ and $\Gamma_{n,r}$ can intersect the horizontal line $y = \epsilon$. For otherwise $\bigcup (\Gamma_{n,\ell} \cup \Gamma_{n,r})$ would have a limit point on the horizontal line $y = \epsilon$. That is, a limit point inside $\mathbb{H}$, and this contradicts the analyticity of $\psi$. Thus the connected sets $\Gamma_{n,i}$, $i = r, l$ will get closer and closer to the real axis as $n \to \infty$.) Therefore either infinitely many $\Gamma_{n,\ell}$ or infinitely many $\Gamma_{n,r}$ would intersect the imaginary axis

(3.26) \[ \mathcal{I} = \{0 + iy : y \in \mathbb{R}\}. \]
So suppose that $\Gamma_{n,i}$ intersects $I$ at $y_n$ for infinitely many $n \in \mathbb{N}$, where $i = \ell$ or $r$. Then, by Proposition 2.1, we can assume, by taking a subsequence if necessary, that

\begin{equation}
(3.27) \quad y_n \downarrow 0, \text{ as } n \to \infty.
\end{equation}

By the assumptions (3.9)–(3.14) we have

\begin{equation}
(3.28) \quad \frac{\partial^k \psi}{\partial x^k}(0,0) = 0, \quad k \in \mathbb{N}.
\end{equation}

By (3.6) and (3.7) we have

\begin{equation}
(3.29) \quad -\Delta \left( \frac{\partial \psi}{\partial x} \right) + V \left( \frac{\partial \psi}{\partial x} \right) = -\left( \frac{\partial V}{\partial x} \right) \psi \quad \text{on } B(0, \delta).
\end{equation}

Let

\begin{equation}
(3.30) \quad I_n = \{0 + iy : 0 < y < y_n\}.
\end{equation}

Consider $\frac{\partial \psi}{\partial x} \big|_{I_n}$ and the one-dimensional Dirichlet Laplacian $-\frac{d^2}{dy^2}$ defined on $I_n$. By (3.29)

\begin{equation}
(3.31) \quad \begin{cases}
-\frac{d}{dy} \left( \frac{\partial \psi}{\partial y} \right) = \left( \frac{\partial^3 \psi}{\partial x^3} \right) - V \left( \frac{\partial \psi}{\partial x} \right) - \left( \frac{\partial V}{\partial x} \right) \psi \\
\frac{\partial \psi}{\partial x}(0,0) = \frac{\partial \psi}{\partial x}(0,y_n) = 0.
\end{cases}
\end{equation}

By (2.1), there exists $\delta_1 > 0$ such that, for all sufficiently large $n \in \mathbb{N}$, we have

\begin{equation}
(3.32) \quad \left| \frac{\partial V}{\partial x}(0,y)\psi(0,y) \right| > \delta_1, \quad 0 < y < y_n.
\end{equation}

So, by (3.28) and (3.32), there exists $\delta_2 > 0$ such that, for all sufficiently large $n \in \mathbb{N}$, we have

\begin{equation}
(3.33) \quad \left| \frac{\partial^3 \psi}{\partial x^3} - V \left( \frac{\partial \psi}{\partial x} \right) - \left( \frac{\partial V}{\partial x} \right) \psi \right| \geq \delta_2 \quad \text{on } I_n.
\end{equation}

Let $G_n(y, y') > 0$ be the Green’s function of $-\frac{d^2}{dy^2}$ on $I_n$. Then, by (3.31),

\begin{equation}
(3.34) \quad \frac{\partial \psi}{\partial x}(0,y) = \int_0^{y_n} G_n(y,y') \left[ \frac{\partial^3 \psi}{\partial x^3} - V \frac{\partial \psi}{\partial x} - \psi \frac{\partial V}{\partial x} \right](0,y') dy'
\end{equation}

for all $y \in (0, y_n)$. Hence (3.33) and (3.34) imply that

\begin{equation}
(3.35) \quad \frac{\partial \psi}{\partial x}(0,y) \neq 0, \quad 0 < y < y_n.
\end{equation}

This gives a contradiction since $y_{n+1} < y_n$ and, as $y_k \in \Gamma_{k,i}$, $\frac{\partial \psi}{\partial x}(0,y_{n+1}) = 0$. Hence the Case 3.2 cannot occur. Since the Cases 3.1 and 3.2 exhaust all the possibilities under
the assumptions (3.9) and (3.10), we see that \( \psi \) cannot oscillate near \((0,0)\) along the real axis.

**Case 3.3:** We now assume that \( \frac{\partial^2 \psi}{\partial y^2}(0,0) \neq 0 \) and that \( \psi \) does not oscillate along the real axis in the sense of (3.9) and (3.10). By (3.6), there exists \( \delta_2 \in (0, \delta/\sqrt{2}) \) such that

\[
(3.36) \quad \left| \frac{\partial^2 \psi}{\partial y^2}(x,y) \right| > 0, \ 0 < |x| < \delta_2, \ 0 < y < \delta_2.
\]

Hence (3.3) and (3.36) imply that the function \( y \mapsto \psi(x,y) \) has a local maximum or local minimum at \( y = 0 \) for all \( x \) with \( 0 < |x| < \delta_2 \). Since we are assuming that \( \nabla \psi(0,0) = (0,0) \) and that \( \psi \) is not oscillating in the sense of (3.9) and (3.10), we can first assume that

\[
\frac{\partial \psi}{\partial x}(x,0) < 0, \ x \in (-\delta_2,0)
\]

and that

\[
\frac{\partial^2 \psi}{\partial y^2}(x,y) > 0, \ 0 < |x| < \delta_2, \ 0 < y < \delta_2.
\]

Then

\[ \psi(x,y) > c = \psi(0,0), \ -\delta_2 < x < 0, \ 0 < y < \delta_2. \]

So the level set \( \{ \psi = \psi(0,0) = c \} \) does not exist on the rectangle \( \{(x,y) \in \mathbb{R}^2 : \ -\delta_2 < x < 0 \text{ and } 0 < y < \delta_2 \} \). Next assume that

\[ \frac{\partial \psi}{\partial x}(x,0) > 0, \ x \in (-\delta_2,0) \]

and that

\[ \frac{\partial^2 \psi}{\partial y^2}(x,y) > 0, \ 0 < |x| < \delta_2, \ 0 < y < \delta_2, \]

then the level set \( \{ \psi = \psi(0,0) = c \} \) near \((0,0)\) either does not exist or is a continuous curve on the rectangle

\[ -\delta_2 < x < 0, \ 0 < y < \delta_2. \]

To see that if the level set \( \{ \psi = \psi(0,0) = c \} \) near \((0,0)\) exists in the rectangle, then it has to be a continuous curve, we first note that, since \( \frac{\partial^2 \psi}{\partial y^2}(x,y) > 0 \) and \( \frac{\partial \psi}{\partial x}(x,0) = 0 \), the function \( y \mapsto \psi(x,y) \) is increasing for all \( x \in (-\delta_2,\delta_2) \) and \( 0 \leq y \leq \delta_2 \). Since we are assuming that \( \frac{\partial \psi}{\partial x}(x,0) > 0 \) for \( x \in (-\delta_2,0) \), \( \psi(x,0) < c \) for all \( x \in (-\delta_2,0) \). Since \( \psi(0,\frac{1}{\delta_2} \delta_2) > c \), any curve \( \gamma \) in the rectangle \( \{(x,y) \in \mathbb{R}^2 : -\delta_2 < x < 0, 0 \leq y \leq \delta_2 \} \) joining \((0,\frac{1}{\delta_2} \delta_2)\) to \((x,0)\), where \(-\delta_2 < x < 0\), must contain a point at which \( \psi \) takes the value \( c \). Hence the level set \( \{ \psi = \psi(0,0) = c \} \) near \((0,0)\) must be a continuous curve. Indeed, since the function \( y \mapsto \psi(x,y) \) is increasing for \( x \in (-\delta_2,0) \), this continuous curve must be the graph of a continuous function. Similarly one can show that if
\[
\frac{\partial^2 \psi}{\partial y^2}(x, y) < 0, \quad 0 < |x| < \delta_2, \quad 0 < y < \delta_2.
\]
then the level set \(\{\psi = \psi(0, 0) = c\}\) near \((0, 0)\) must be the graph of a continuous function or does not exist. By similar arguments, we see that the level set \(\{\psi = \psi(0, 0)\}\) near \((0, 0)\) either does not exist or is a continuous curve on the rectangle
\[
0 < x < \delta_2, \quad 0 < y < \delta_2.
\]
Therefore the level set \(\{\psi = \psi(0, 0)\}\) near \((0, 0)\) consists either of a single point \(\{(0, 0)\}\) or is a continuous curve. Hence \(h(\alpha) \neq (0, 0)\), which is a contradiction. Thus this case cannot occur either.

4 The case \(\nabla \psi(0, 0) = 0\) and \(\frac{\partial^2 \psi}{\partial y^2}(0, 0) = 0\)

Suppose

\[\frac{\partial^2 \psi}{\partial y^2}(0, 0) = 0.\] (4.1)

In this case, by (3.7), we have

\[\frac{\partial^2 \psi}{\partial x^2}(0, 0) = (V \psi)(0, 0) < 0.\] (4.2)

Therefore we can assume that \(\delta_1 > 0\) is sufficiently small so that

\[\frac{\partial^2 \psi}{\partial x^2}(x, 0) \leq -\delta_1, \quad -\delta_1 < x < \delta_1.\] (4.3)

We shall consider only \(0 < x < \delta_1\). The arguments for \(-\delta_1 < x < 0\) are similar. Since we are still assuming that \(\nabla \psi(0, 0) = (0, 0)\), (4.2) implies we can assume that

\[\psi(x, 0) < c = \psi(0, 0), \quad 0 < x < \delta_1.\] (4.4)

By (3.7) we have

\[-\frac{\partial^3 \psi}{\partial y^3} - \frac{\partial^3 \psi}{\partial x^2 \partial y} - V \frac{\partial \psi}{\partial y} - \psi \frac{\partial V}{\partial y} \]
on \(\mathbb{H} \cap B(0, \delta)\). By (3.3) we have

\[\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} - V \left(\frac{\partial \psi}{\partial y}\right)\right)(0, 0) = 0.\] (4.6)

By (2.2), (3.8), (4.5) and (4.6) and the assumption that

\[\varphi(\alpha) = \psi(0, 0) = c > 0,\] (4.7)
there exists $\delta_1 \in (0, \delta/\sqrt{2})$ such that

$$\frac{\partial^3 \psi}{\partial y^3}(x, y) \geq \delta_1, \quad -\delta_1 < x < \delta_1, \quad 0 \leq y < \delta_1.$$  

(4.8)

Hence, for all $x \in (-\delta_1, \delta_1)$, the graph of $y \mapsto \frac{\partial \psi}{\partial y}(x, y)$ looks like either

Figure I: The graph of $y \mapsto \frac{\partial \psi}{\partial y}(x, y)$

or

Figure II: The graph of $y \mapsto \frac{\partial \psi}{\partial y}(x, y)$

Hence, for all $x \in [0, \delta_1)$, the graph of the function $y \mapsto \psi(x, y)$ looks like either
By (4.4), the two possible graphs of \( y \mapsto \psi(x, y) \) given in Figures III and IV, and arguments similar to those used in Case 3.3, we see that the level set \( \{ \psi = \psi(0, 0) = c \} \) near \( (0, 0) \) in the rectangle \( \{(x, y) \in \mathbb{R}^2 : 0 < x < \delta_1, 0 \leq y < \delta_1 \} \), is the graph of a continuous function. Applying similar arguments to the rectangle \( \{(x, y) \in \mathbb{R}^2 : -\delta_1 < x \leq 0, 0 \leq y < \delta_1 \} \), we see that the level set \( \{ \psi = \psi(0, 0) \} \) is a continuous curve near \( (0, 0) \). Hence \( h(\alpha) \neq (0, 0) \), which is a contradiction. Thus this case cannot occur.

We collect the above results of this section and §3 into the following proposition.

**Proposition 4.1.** Let \( \alpha \) be as in (1.3). Then \( \alpha \notin \partial \Omega \).

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**References**


