

TWO-TERM TRACE ESTIMATES FOR RELATIVISTIC STABLE PROCESSES

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ABSTRACT. We prove trace estimates for the relativistic α -stable process extending the result of Bañuelos and Kulczycki (2008) in the stable case.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

For $m > 0$, an \mathbb{R}^d -valued process with independent, stationary increments having the following characteristic function

$$\mathbb{E}e^{i\xi \cdot X_t^{\alpha,m}} = e^{-t\{(m^{2/\alpha} + |\xi|^2)^{\alpha/2} - m\}}, \quad \xi \in \mathbb{R}^d,$$

is called relativistic α -stable process with mass m . We assume that sample paths of $X_t^{\alpha,m}$ are right continuous and have left-hand limits a.s. If we put $m = 0$ we obtain the symmetric rotation invariant α -stable process with the characteristic function $e^{-t|\xi|^\alpha}$, $\xi \in \mathbb{R}^d$. We refer to this process as isotropic α -stable Lévy process. For the rest of the paper we keep α, m and $d \geq 2$ fixed and drop α, m in the notation, when it does not lead to confusion. Hence from now on the relativistic α -stable process is denoted by X_t and its counterpart isotropic α -stable Lévy process by \tilde{X}_t . We keep this notational convention consistently throughout the paper, e.g., if $p_t(x - y)$ is the transition density of X_t , then $\tilde{p}_t(x - y)$ is the transition density of \tilde{X}_t .

In Ryznar [13] Green function estimates of the Schrödinger operator with the free Hamiltonian of the form

$$(-\Delta + m^{2/\alpha})^{\alpha/2} - m,$$

were investigated, where $m > 0$ and Δ is the Laplace operator acting on $L^2(\mathbb{R}^d)$. Using the estimates in Lemma 2.6 below and proof in Bañuelos and Kulczycki (2008) we provide an extension of the asymptotics in [3] to the relativistic α stable processes for any $0 < \alpha < 2$.

Brownian motion has characteristic function

$$\mathbb{E}^0 e^{i\xi \cdot B_t} = e^{-t|\xi|^2}, \quad \xi \in \mathbb{R}^d.$$

Let $\beta = \alpha/2$. Ryznar showed that X_t can be represented as a time-changed Brownian motion. Let $T_\beta(t)$, $t > 0$, denote the strictly β -stable subordinator with the following

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Laplace transform

$$(1.1) \quad \mathbb{E}^0 e^{-\lambda T_\beta(t)} = e^{-t\lambda^\beta}, \quad \lambda > 0.$$

Let $\theta_\beta(t, u)$, $u > 0$, denote the density function of $T_\beta(t)$. Then the process $B_{T_\beta(t)}$ is the standard symmetric α -stable process.

Ryznar [13, Lemma 1] showed that we can obtain $X_t = B_{T_\beta(t, m)}$, where a subordinator $T_\beta(t, m)$ is a positive infinitely divisible process with stationary increments with probability density function

$$\theta_\beta(t, u, m) = e^{-m^{1/\beta}u + mt} \theta_\beta(t, u), \quad u > 0.$$

Transition density of $T_\beta(t, m)$ is given by $\theta_\beta(t, u - v, m)$. Hence the transition density of X_t is $p(t, x, y) = p(t, x - y)$ given by

$$(1.2) \quad p(t, x) = e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-\frac{|x|^2}{4u}} e^{-m^{1/\beta}u} \theta_\beta(t, u) du.$$

Then

$$p(t, x, x) = p(t, 0) = e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-m^{1/\beta}u} \theta_\beta(t, u) du.$$

The function $p(t, x)$ is a radially symmetric decreasing and that

$$(1.3) \quad p(t, x) \leq p(t, 0) \leq e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} \theta_\beta(t, u) du = e^{mt} t^{-d/\alpha} \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d\alpha}},$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in \mathbb{R}^d . For an open set D in \mathbb{R}^d we define the first exit time from D by $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$.

We set

$$(1.4) \quad r_D(t, x, y) = \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t],$$

and

$$(1.5) \quad p_D(t, x, y) = p(t, x, y) - r_D(t, x, y),$$

for any $x, y \in \mathbb{R}^d$, $t > 0$. For a nonnegative Borel function f and $t > 0$, let

$$P_t^D f(x) = \mathbb{E}^x[f(X_t) : t < \tau_D] = \int_D p_D(t, x, y) f(y) dy,$$

be the semigroup of the killed process acting on $L^2(D)$, see, Ryznar [13, Theorem 1].

Let D be a bounded domain (or of finite volume). Then the operator P_t^D maps $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. This follows from (1.3), (1.4), and the general theory of heat semigroups as described in [10]. It follows that there exists an orthonormal basis of eigenfunctions $\{\varphi_n : n = 1, 2, 3, \dots\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n : n = 1, 2, 3, \dots\}$ of the generator of the semigroup P_t^D satisfying

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. By definition, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x), \quad x \in D, \quad t > 0.$$

Under such assumptions we have

$$(1.6) \quad p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y).$$

In this paper we are interested in the behavior of the trace of this semigroup

$$(1.7) \quad Z_D(t) = \int_D p_D(t, x, x) dx.$$

Because of (1.6) we can write (1.7) as

$$(1.8) \quad Z_D(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_D \varphi_n^2(x) dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}.$$

We denote d -dimensional volume of D by $|D|$.

Our first result is the Weyl's asymptotic for the eigenvalues of the relativistic Laplacian

Proposition 1.1.

$$(1.9) \quad \lim_{t \rightarrow 0} t^{d/\alpha} e^{-mt} Z_D(t) = C_1 |D|,$$

where $C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d\alpha}}$.

Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which do not exceed λ . It follows from (1.9) and the classical Tauberian theorem (see for example [11], p.445 Theorem 2) where $L(t) = C_1 |D| e^{m/t}$ is our slowly varying function at infinity that

$$(1.10) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-d/\alpha} e^{-m/\lambda} N(\lambda) = \frac{C_1 |D|}{\Gamma(1 + d/\alpha)}.$$

This is the analogue for relativistic stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

Remark 1.2. The first author presented (1.10) at a conference in Vienna at the Schrödinger Institute in 2009 (see [1]) and at the 34th conference in stochastic processes and their applications in Osaka in 2010 (see [2]). Thus this result has been known to the authors, and perhaps to others, for number of years.

Our goal in this paper is to obtain the second term in the asymptotics of $Z_D(t)$ under some additional assumptions on the smoothness of D . Our result is inspired by result for trace estimates for stable processes by Bañuelos and Kulczycki [3].

To state our main result we need the following property of the domain D .

Definition 1.3. The boundary, ∂D , of an open set D in \mathbb{R}^d is said to be R -smooth if for each point $x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D)$ and $\partial B_1 \cap \partial B_2 = x_0$.

Theorem 1.4. Let $D \subset \mathbb{R}^d, d \geq 2$, be an open bounded set with R -smooth boundary. Let $|D|$ denote the volume (d -dimensional Lebesgue measure) of D and $|\partial D|$ denote its surface area ($(d-1)$ -dimensional Lebesgue measure) of its boundary. Suppose $\alpha \in (0, 2)$. Then

$$(1.11) \quad \left| Z_D(t) - \frac{C_1(t)e^{mt}|D|}{t^{d/\alpha}} + C_2(t)|\partial D| \right| \leq \frac{C_3 e^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}}, \quad t > 0,$$

where

$$C_1(t) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty z^{-d/2} e^{-(mt)^{1/\beta} z} \theta_\beta(1, z) dz \rightarrow C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^{d\alpha}}, \quad \text{as } t \rightarrow 0,$$

$$C_2(t) = \int_0^\infty r_H(t, (x_1, 0, \dots, 0), (x_1, 0, \dots, 0)) dx_1 \leq \frac{C_4 e^{2mt} t^{1/\alpha}}{t^{d/\alpha}}, \quad t > 0$$

$$C_4 = \int_0^\infty \tilde{r}_H(1, (x_1, 0, \dots, 0), (x_1, 0, \dots, 0)) dx_1,$$

$C_3 = C_3(d, \alpha), H = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ and r_H is given by (1.4).

Remark 1.5. When $m = 0, 0 < \alpha \leq 2$, $C_2(t) = C_4 t^{1/\alpha} / t^{d/\alpha}$. Then the result in Theorem 1.4 becomes, for bounded domains with R -smooth boundary,

$$(1.12) \quad \left| Z_D(t) - \frac{C_1|D|}{t^{d/\alpha}} + \frac{C_4|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \right| \leq \frac{C_7|D|t^{2/\alpha}}{R^2 t^{d/\alpha}},$$

where C_1, C_4 are as in Theorem 1.4. This was established by Bañuelos and Kulczycki [3] recently.

The asymptotic for the trace of the heat kernel when $\alpha = 2$ (the case of the Laplacian with Dirichlet boundary condition in a domain of \mathbb{R}^d), have been extensively studied by many authors. For Brownian motion Van den Berg [5], proved that under the R -smoothness condition

$$(1.13) \quad \left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \leq \frac{C_d |D| t^{1-d/2}}{R^2}, \quad t > 0.$$

For domains with C^1 boundaries the result

$$(1.14) \quad Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right), \quad \text{as } t \rightarrow 0,$$

was proved by Brossard and Carmona [8], for Brownian motion.

2. PRELIMINARIES

Let the ball in \mathbb{R}^d with center at x and radius r , $\{y : |y - x| < r\}$, be denoted by $B(x, r)$. We will use $\delta_D(x)$ to denote the Euclidean distance between x and the boundary, ∂D , of D . That is, $\delta_D(x) = \text{dist}(x, \partial D)$. Define

$$\psi(\theta) = \int_0^\infty e^{-v} v^{p-1/2} (\theta + v/2)^{p-1/2} dv, \theta \geq 0,$$

where $p = (d + \alpha)/2$. We put $\mathcal{R}(\alpha, d) = \mathcal{A}(-\alpha, d)/\psi(0)$, where $\mathcal{A}(v, d) = (\Gamma((d - v)/2))/(\pi^{d/2} 2^v |\Gamma(v/2)|)$. Let $\nu(x), \tilde{\nu}(x)$ be the densities of the Lévy measures of the relativistic α -stable process and the standard α -stable process, respectively. These densities are given by

$$(2.1) \quad \nu(x) = \frac{\mathcal{R}(\alpha, d)}{|x|^{d+\alpha}} e^{-m^{1/\alpha}|x|} \psi(m^{1/\alpha}|x|),$$

and

$$(2.2) \quad \tilde{\nu}(x) = \frac{\mathcal{A}(-\alpha, d)}{|x|^{d+\alpha}}.$$

We need the following estimate of the transition probabilities of the process X_t which is given in ([14], Lemma 2.2): For any $x, y \in \mathbb{R}^d$ and $t > 0$ there exist constants $c_1 > 0$ and $c_2 > 0$,

$$(2.3) \quad p(t, x, y) \leq c_1 e^{mt} \min \left\{ \frac{t}{|x - y|^{d+\alpha}} e^{-c_2|x-y|}, t^{-d/\alpha} \right\}.$$

We will also use the fact([7], Lemma 6) that if $D \subset \mathbb{R}^d$ is an open bounded set satisfying a uniform outer cone condition, then $P^x(X(\tau_D) \in \partial D) = 0$ for all $x \in D$. For the open bounded set D we will denote by $G_D(x, y)$ the Green function for the set D equal to,

$$G_D(x, y) = \int_0^\infty p_D(t, x, y) dt, \quad x, y \in \mathbb{R}^d.$$

For any such D the expectation of the exit time of the processes X_t from D is given by the integral of the Green function over the domain. That is:

$$E^x(\tau_D) = \int_D G_D(x, y) dy.$$

Lemma 2.1. *Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ we have*

$$r_D(t, x, y) \leq c_1 e^{mt} \left(\frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2 \delta_D(x)} \wedge t^{-d/\alpha} \right).$$

Proof. Using (1.4) and (2.3) we have

$$\begin{aligned}
r_D(t, x, y) &= E^y(p(t - \tau_D, X(\tau_D), x); \tau_D < t) \\
&\leq c_1 e^{mt} E^y \left(\frac{t}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x - X(\tau_D)|} \wedge t^{-d/\alpha} \right) \\
&\leq c_1 e^{mt} \left(\frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2\delta_D(x)} \wedge t^{-d/\alpha} \right).
\end{aligned}$$

□

We need the following result for the proof of Proposition 1.1.

Lemma 2.2.

$$(2.4) \quad \lim_{t \rightarrow 0} p(t, 0) e^{-mt} t^{d/\alpha} = C_1,$$

where

$$C_1 = (4\pi)^{d/2} \int_0^\infty u^{-d/2} \theta_\beta(1, u) du = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}.$$

Proof. By (1.2) we have

$$p(t, x, x) = p(t, 0) = e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-m^{1/\beta} u} \theta_\beta(t, u) du.$$

Now using the scaling of stable subordinator $\theta_\beta(t, u) = t^{-1/\beta} \theta_\beta(1, ut^{-1/\beta})$ and a change of variables we get

$$p(t, 0) = \frac{e^{mt}}{(4\pi)^{d/2} t^{d/\alpha}} \int_0^\infty z^{-d/2} e^{-m^{1/\beta} t^{1/\beta} z} \theta_\beta(1, z) dz = \frac{C_1(t) e^{mt}}{t^{d/\alpha}},$$

then by dominated convergence theorem we obtain

$$\lim_{t \rightarrow 0} p(t, 0) e^{-mt} t^{d/\alpha} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty z^{-d/2} \theta_\beta(1, z) dz,$$

and this last integral is equal to the density of α -stable process at time 1 and $x = 0$ which was calculated in [3] to be

$$\frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}.$$

□

We next give the proof of Proposition 1.1.

Proof of Proposition 1.1. By (1.4) we see that

$$(2.5) \quad \frac{p_D(t, x, x)}{C_1 e^{mt} t^{-d/\alpha}} = \frac{p(t, 0)}{C_1 e^{mt} t^{-d/\alpha}} - \frac{r_D(t, x, x)}{C_1 e^{mt} t^{-d/\alpha}}.$$

Since the first term tend to 1 as $t \rightarrow 0$ by (2.4), in order to prove (1.9), we show that

$$(2.6) \quad \frac{t^{d/\alpha}}{C_1 e^{mt}} \int_D r_D(t, x, x) dx \rightarrow 0, \quad \text{as } t \rightarrow 0.$$

For $q \geq 0$, we define $D_q = \{x \in D : \delta_D(x) \geq q\}$. Then for $0 < t < 1$, consider the subdomain $D_{t^{1/2\alpha}} = \{x \in D : \delta_D(x) \geq t^{1/2\alpha}\}$ and its complement $D_{t^{1/2\alpha}}^C = \{x \in D : \delta_D(x) < t^{1/2\alpha}\}$. Recalling that $|D| < \infty$, by Lebesgue dominated convergence theorem we get $|D_{t^{1/2\alpha}}^C| \rightarrow 0$, as $t \rightarrow 0$. Since $p_D(t, x, x) \leq p(t, x, x)$, by (1.3) we see that

$$\frac{r_D(t, x, x)}{C_1 e^{mt} t^{-d/\alpha}} \leq 1,$$

for all $x \in D$. It follows that

$$(2.7) \quad \frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D_{t^{1/2\alpha}}^C} r_D(t, x, x) dx \rightarrow 0, \quad \text{as } t \rightarrow 0.$$

On the other hand, by Lemma 2.2 in [14] we obtain

$$(2.8) \quad \begin{aligned} \frac{r_D(t, x, x)}{C_1 e^{mt} t^{-d/\alpha}} &= \frac{\mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, x); t \geq \tau_D]}{C_1 e^{mt} t^{-d/\alpha}} \\ &\leq c \mathbb{E}^y \min \left\{ \frac{t^{1+d/\alpha}}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x - X(\tau_D)|}, 1 \right\} \\ &\leq c \min \left\{ \frac{t^{1+d/\alpha}}{\delta_D(x)^{d+\alpha}} e^{-c_2\delta_D(x)}, 1 \right\}. \end{aligned}$$

For $x \in D_{t^{1/2\alpha}}$ and $0 < t < 1$, the right hand side of (2.8) is bounded above by $ct^{d/2\alpha+1/2}$ and hence

$$(2.9) \quad \frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D_t^{1/2\alpha}} r_D(t, x, x) dx \leq ct^{d/2\alpha+1/2}|D|,$$

and this last quantity goes to 0 as $t \rightarrow 0$. \square

For an open set $D \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the distribution $P^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic α -harmonic measure for D . The following Ikeda-Watanabe formula recovers the relativistic α -harmonic measure for the set D from the Green function.

Proposition 2.3 ([14]). *Assume that D is an open, nonempty, bounded subset of \mathbb{R}^d , and A is a Borel set such that $\text{dist}(D, A) > 0$. Then*

$$(2.10) \quad P^x(X(\tau_D) \in A, \tau_D < \infty) = \int_D G_D(x, y) \int_A v(y - z) dz dy, \quad x \in D.$$

Here we need the following generalization already stated and used in [3].

Proposition 2.4. [14, Proposition 2.5] *Assume that D is an open, nonempty, bounded subset of \mathbb{R}^d , and A is a Borel set such that $A \subset D^c \setminus \partial D$ and $0 \leq t_1 < t_2 < \infty, x \in D$. Then we have*

$$P^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s, x, y) ds \int_A v(y - z) dz dy.$$

The following proposition holds for a large class of Lévy processes

Proposition 2.5. [3, Proposition 2.3] *Let D and F be open sets in \mathbb{R}^d such that $D \subset F$. Then for any $x, y \in \mathbb{R}^d$ we have*

$$p_F(t, x, y) - p_D(t, x, y) = E^x(\tau_D < t, X(\tau_D) \in F \setminus D; p_F(t - \tau_D, X(\tau_D), y)).$$

Lemma 2.6. [13, Lemma 5] *Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ and $t > 0$ the following estimates hold;*

$$(2.11) \quad \begin{aligned} p_D(t, x, y) &\leq e^{mt} \tilde{p}_D(t, x, y) \\ r_D(t, x, y) &\leq e^{2mt} \tilde{r}_D(t, x, y). \end{aligned}$$

We need the following lemma given by Van den Berg in [5].

Lemma 2.7. [5, Lemma 5] *Let D be an open bounded set in \mathbb{R}^d with R -smooth boundary ∂D and for $0 \leq q < R$ denote the area of boundary of ∂D_q by $|\partial D_q|$. Then*

$$(2.12) \quad \left(\frac{R-q}{R}\right)^{d-1} |\partial D| \leq |\partial D_q| \leq \left(\frac{R}{R-q}\right)^{d-1} |\partial D|, \quad 0 \leq q < R.$$

Corollary 2.8. ([3], Corollary 2.14) *Let D be an open bounded set in \mathbb{R}^d with R -smooth boundary. For any $0 < q \leq R$ we have*

$$(i) \quad 2^{-d+1} |\partial D| \leq |\partial D_q| \leq 2^{d-1} |\partial D|,$$

$$(ii) \quad |\partial D| \leq \frac{2^d |D|}{R},$$

$$(iii) \quad \left| |\partial D_q| - |\partial D| \right| \leq \frac{2^d dq |\partial D|}{R} \leq \frac{2^{2d} dq |D|}{R^2}.$$

3. PROOF OF MAIN RESULT

Proof of Theorem 1.4. For the case $t^{1/\alpha} > R/2$ the theorem holds trivially. Indeed, by Equation (1.3)

$$Z_D(t) \leq \int_D p(t, x, x) dx \leq \frac{c_1 e^{mt} |D|}{t^{d/\alpha}} \leq \frac{c_1 e^{mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

By Corollary 2.8 and Lemma 2.6 we also have

$$\begin{aligned} C_2(t)|\partial D| &\leq \frac{C_4 e^{2mt} |\partial D| t^{1/\alpha}}{t^{d/\alpha}} \leq \frac{2^d C_4 e^{2mt} |D| t^{1/\alpha}}{R t^{d/\alpha}} \leq \frac{2^{d+1} C_4 e^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}} \\ &\frac{C_1(t) e^{mt} |D|}{t^{d/\alpha}} \leq \frac{C_1 e^{mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}. \end{aligned}$$

Therefore for $t^{1/\alpha} > R/2$ (1.11) holds. Here and in sequel we consider the case $t^{1/\alpha} \leq R/2$. From (1.5) and the fact that $p(t, x, x) = \frac{C_1(t) e^{mt}}{t^{d/\alpha}}$, we have that

$$\begin{aligned} Z_D(t) - \frac{C_1(t) e^{mt} |D|}{t^{d/\alpha}} &= \int_D p_D(t, x, x) dx - \int_D p(t, x, x) dx \\ (3.1) \qquad \qquad \qquad &= - \int_D r_D(t, x, x) dx, \end{aligned}$$

where $C_1(t)$ is as stated in the theorem. Therefore we must estimate (3.1). We break our domain into two pieces, $D_{R/2}$ and its complement $D_{R/2}^C$. We will first consider the contribution of $D_{R/2}$.

Claim 1: For $t^{1/\alpha} \leq R/2$ we have

$$(3.2) \qquad \int_{D_{R/2}} r_D(t, x, x) dx \leq \frac{c e^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

Proof of Claim 1: By Lemma 2.6 we have

$$(3.3) \qquad \int_{D_{R/2}} r_D(t, x, x) dx \leq e^{2mt} \int_{D_{R/2}} \tilde{r}_D(t, x, x) dx,$$

and by scaling of the stable density the right hand side of (3.3) equals

$$(3.4) \qquad \frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \tilde{r}_{D/t^{1/\alpha}}\left(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}\right) dx.$$

For $x \in D_{R/2}$ we have $\delta_{D/t^{1/\alpha}}(x/t^{1/\alpha}) \geq R/(2t^{1/\alpha}) \geq 1$. By [3, Lemma 2.1], we get

$$\tilde{r}_{D/t^{1/\alpha}}\left(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}\right) \leq \frac{c}{\delta_{D/t^{1/\alpha}}^{d+\alpha}(x/t^{1/\alpha})} \leq \frac{c}{\delta_{D/t^{1/\alpha}}^2(x/t^{1/\alpha})} \leq \frac{c t^{2/\alpha}}{R^2}.$$

Using the above inequality, we get

$$\int_{D_{R/2}} r_D(t, x, x) dx \leq \frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \frac{c t^{2/\alpha}}{R^2} dx \leq \frac{c e^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}},$$

which proves (3.2).

Now we will introduce the following notation. Since D has R -smooth boundary, for any point $y \in \partial D$ there are two open balls B_1 and B_2 both of radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D), \partial B_1 \cap \partial B_2 = y$. For any $x \in D_{R/2}$ there exist a

unique point $x_* \in \partial D$ such that $\delta_D(x) = |x - x_*|$. Let $B_1 = B(z_1, R), B_2 = B(z_2, R)$ be inner/outer balls for the point x_* . Let $H(x)$ be the half-space containing B_1 such that $\partial H(x)$ contains x_* and is perpendicular to the segment $\overline{z_1 z_2}$.

We will need the following very important proposition in the proof of Theorem 1.2. Such a proposition has been proved for the stable process in [3, Proposition 3.1].

Proposition 3.1. *Let $D \subset \mathbb{R}^d, d \geq 2$, be an open bounded set with R -smooth boundary ∂D . Then for any $x \in D_{R/2}^C$ and $t > 0$ such that $t^{1/\alpha} \leq R/2$ we have*

$$(3.5) \quad |r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right).$$

Proof. Exactly as in [3], let $x_* \in \partial D$ be a unique point such that $|x - x_*| = \text{dist}(x, \partial D)$ and B_1 and B_2 be balls with radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \setminus (D \cup \partial D), \partial B_1 \cap \partial B_2 = x_*$. Let us also assume that $x_* = 0$ and choose an orthonormal coordinate system (x_1, x_2, \dots, x_d) so that the positive axis $0x_1$ is in the direction of $\overrightarrow{0p}$ where p is the center of the ball B_1 . Note that x lies on the interval $0p$ so $x = (|x|, 0, 0, \dots, 0)$. Note also that $B_1 \subset D \subset (\overline{B_2})^c$ and $B_1 \subset H(x) \subset (\overline{B_2})^c$. For any open sets A_1, A_2 such that $A_1 \subset A_2$ we have $r_{A_1}(t, x, y) \geq r_{A_2}(t, x, y)$ so

$$|r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x).$$

So in order to prove the proposition it suffices to show that

$$r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x) \leq \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right),$$

for any $x = (|x|, 0, \dots, 0), |x| \in (0, R/2]$. Such an estimate was proved for the case $m = 0$ in [3]. In order to complete the proof it is enough to prove that

$$r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x) \leq ce^{2mt} \left\{ \tilde{r}_{B_1}(t, x, x) - \tilde{r}_{(\overline{B_2})^c}(t, x, x) \right\}.$$

To show this given the ball B_2 , we set $U = (\overline{B_2})^c$. Now using the generalized Ikeda-Watanabe formula, Proposition (2.5) and Lemma (2.6) we have

$$\begin{aligned}
& r_{B_1}(t, x, x) - r_U(t, x, x) \\
&= E^x [t > \tau_{B_1}, X(\tau_{B_1}) \in U \setminus B_1; p_U(t - \tau_{B_1}, X(\tau_{B_1}), x)] \\
&= \int_{B_1} \int_0^t p_{B_1}(s, x, y) ds \int_{U \setminus B_1} v(y - z) p_U(t - s, z, x) dz dy \\
&\leq e^{2mt} \int_{B_1} \int_0^t \tilde{p}_{B_1}(s, x, y) ds \int_{U \setminus B_1} \tilde{v}(y - z) \tilde{p}_U(t - s, z, x) dz dy \\
&\leq ce^{2mt} E^x [t > \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}) \in U \setminus B_1; \tilde{p}_U(t - \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}), x)] \\
&= ce^{2mt} (\tilde{r}_{B_1}(t, x, x) - \tilde{r}_U(t, x, x)) \\
&\leq \frac{ce^{2mt} t^{1/\alpha}}{Rt^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right).
\end{aligned}$$

The last inequality follows by Proposition 3.1 in [3]. \square

Now using this proposition we estimate the contribution from $D \setminus D_{R/2}$ to the integral of $r_D(t, x, x)$ in (3.1).

Claim 2: For $t^{1/\alpha} \leq R/2$ we get

$$(3.6) \quad \left| \int_{D \setminus D_{R/2}} r_D(t, x, x) dx - \int_{D \setminus D_{R/2}} r_{H(x)}(t, x, x) dx \right| \leq \frac{ce^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

Proof of Claim 2: By Proposition 3.1 the left hand side of (3.6) is bounded above by

$$\frac{ce^{2mt} t^{1/\alpha}}{Rt^{d/\alpha}} \int_0^{R/2} |\partial D_q| \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right) dq.$$

By Corollary 2.8, (i), the last quantity is smaller than or equal to

$$\frac{ce^{2mt} t^{1/\alpha} |\partial D|}{Rt^{d/\alpha}} \int_0^{R/2} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right) dq.$$

The integral in the last quantity is bounded by $ct^{1/\alpha}$. To see this observe that since $t^{1/\alpha} \leq R/2$ the above integral is equal to

$$\begin{aligned} & \int_0^{t^{1/\alpha}} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right) dq + \int_{t^{1/\alpha}}^{R/2} \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right) dq \\ &= \int_0^{t^{1/\alpha}} 1 dq + \int_{t^{1/\alpha}}^{R/2} \left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} dq \\ &\leq ct^{1/\alpha}. \end{aligned}$$

Using this and Corollary (2.8), (ii), we get (3.6).

Recall that $H = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$. For abbreviation let us denote

$$f_H(t, q) = r_H(t, (q, 0, \dots, 0), (q, 0, \dots, 0)), \quad t, q > 0.$$

Of course we have $r_{H(x)}(t, x, x) = f_H(t, \delta_H(x))$. In the next step we will show that

$$(3.7) \quad \left| \int_{D \setminus D_{R/2}} r_{H(x)}(t, x, x) dx - |\partial D| \int_0^{R/2} f_H(t, q) dq \right| \leq \frac{ce^{2mt} |D| t^{2/\alpha}}{R^2 t^{d/\alpha}}.$$

We have

$$\int_{D \setminus D_{R/2}} r_{H(x)}(t, x, x) dx = \int_0^{R/2} |\partial D_q| f_H(t, q) dq.$$

Hence the left hand side of (3.7) is bounded above by

$$\int_0^{R/2} ||\partial D_q| - |\partial D|| f_H(t, q) dq.$$

By Corollary 2.8, (iii), this is smaller than

$$\begin{aligned} & \frac{c|D|}{R^2} \int_0^{R/2} q f_H(t, q) dq \\ &\leq \frac{c|D|e^{2mt}}{R^2} \int_0^{R/2} q \tilde{f}_H(t, q) dq \\ &= \frac{c|D|e^{2mt}}{R^2} \int_0^{R/2} qt^{-d/\alpha} \tilde{f}_H(1, qt^{-1/\alpha}) dq \\ &= \frac{c|D|e^{2mt}}{R^2 t^{d/\alpha}} \int_0^{R/2 t^{1/\alpha}} qt^{2/\alpha} \tilde{f}_H(1, q) dq \\ &\leq \frac{c|D|e^{2mt} t^{2/\alpha}}{R^2 t^{d/\alpha}} \int_0^\infty q (q^{-d-\alpha} \wedge 1) dq \leq \frac{c|D|e^{2mt} t^{2/\alpha}}{R^2 t^{d/\alpha}}. \end{aligned}$$

This shows (3.7). Finally, we have

$$\begin{aligned}
& \left| |\partial D| \int_0^{R/2} f_H(t, q) dq - |\partial D| \int_0^\infty f_H(t, q) dq \right| \\
& \leq |\partial D| \int_{R/2}^\infty f_H(t, q) dq \\
& \leq \frac{c|D|}{R} \int_{R/2}^\infty f_H(t, q) dq \quad \text{by Corollary 2.8, (ii)} \\
& \leq \frac{c|D|e^{2mt}}{Rt^{d/\alpha}} \int_{R/2}^\infty \tilde{f}_H(1, qt^{-1/\alpha}) dq \\
& = \frac{c|D|e^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \int_{R/2t^{1/\alpha}}^\infty \tilde{f}_H(1, q) dq.
\end{aligned}$$

Since $R/2t^{1/\alpha} \geq 1$, so for $q \geq R/2t^{1/\alpha} \geq 1$ we have $\tilde{f}_H(1, q) \leq cq^{-d-\alpha} \leq cq^{-2}$. Therefore,

$$\int_{R/2t^{1/\alpha}}^\infty \tilde{f}_H(1, q) dq \leq c \int_{R/2t^{1/\alpha}}^\infty \frac{dq}{q^2} \leq \frac{ct^{1/\alpha}}{R}.$$

Hence,

$$(3.8) \quad \left| |\partial D| \int_0^{R/2} f_H(t, q) dq - |\partial D| \int_0^\infty f_H(t, q) dq \right| \leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2t^{d/\alpha}}.$$

Note that the constant $C_2(t)$ which appears in the formulation of Theorem 1.4 satisfies $C_2(t) = \int_0^\infty f_H(t, q) dq$. Now equations (3.1), (3.2), (3.6), (3.7), (3.8) give (1.11). \square

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