Probability and some problems on the spectral geometry of Schrödinger operators and fractional Laplacian[*]

Rodrigo Bañuelos[†]
Department of Mathematics
banuelos@math.purdue.edu

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The Laplace Operator

- One of the most fundamental operators in the development of several areas of mathematics and its applications is the Laplace operator

  $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

  and its higher dimensional analogues. Called the Laplacian, after the Marquis Pierre–Simon De Laplace (1749–1827), the operator lies at the heart of the mathematical description of heat, light, sound, electricity, magnetism, gravitation, fluid motion, etc., as well as at the heart of modern mathematical analysis. It has been extensively studied by mathematicians and physicists for more than 200 years, often focusing on the geometric properties of its solutions.

- One may surmise that everything worth knowing about this operator would already be known.

- But this “old, classical,” mathematical object does not reveal its secrets lightly.

- **PURPOSE OF THIS TALK:** Explore sharp bounds of geometric nature on some basic quantities associated with Laplace’s operator.

Eigenvalues/Eigenfunctions

Given a region D in the plane there is a sequence of numbers $\{\lambda_n\}$ satisfying

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

and a sequence of functions

$$\varphi_1, \varphi_2, \cdots$$

which solve the boundary value problem (**“Dirichlet boundary” conditions**)

$$\begin{cases} \Delta \varphi_n = -\lambda_n \varphi_n & \text{in } D \\ \varphi_n = 0 & \text{on } \partial D. \end{cases}$$

The functions

$$u_j(t,z) = \varphi_j(z) e^{i\sqrt{\lambda_j}\, t} \qquad z = (x,y)$$

solve the wave equation:

$$\frac{\partial^2 u_j}{\partial t^2} = \Delta u_j = \frac{\partial^2 u_j}{\partial x^2} + \frac{\partial^2 u_j}{\partial y^2}.$$

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- Special attention paid to the **“fundamental”** tone λ_1 and the **“ground”** state φ_1.

Intuition: The smaller λ_1, the larger the **“drum”**. The larger the **“drum”**, the smaller λ_1.

World's Largest Drum!

World's Lowest Fundamental Tone!

World’s Largest Drum!

World’s Lowest Fundamental Tone!

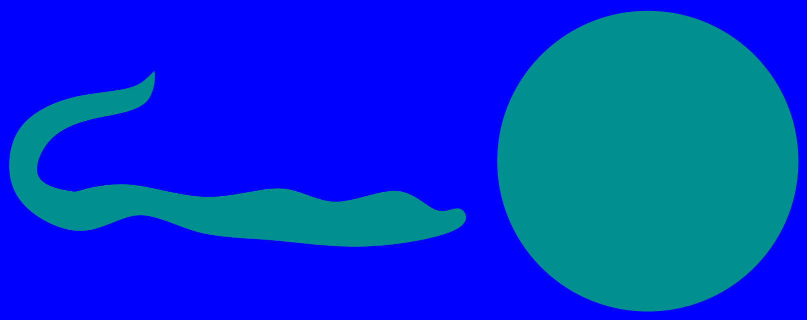
People throughout the world have known for centuries that “big drums” produce low tones and that “small drums” produce high tones.

But, What is “BIG” and what is “SMALL”?

The Isoperimetric Problem

The Greek Philosopher, Proclus, wrote in the fifth century: “The circle (disk), is the first, the most simple, and the most perfect figure.” The “perfect” symmetry of the disk justifies this statement as does the deep property discovered by **Queen Dido** (Deido–the wanderer), a Phoenician princess from the city of Tyre, shortly after her arrival in Africa in 900 B.C.

Dido’s property: Amongst all regions of fixed equal area, the disk has the smallest perimeter.



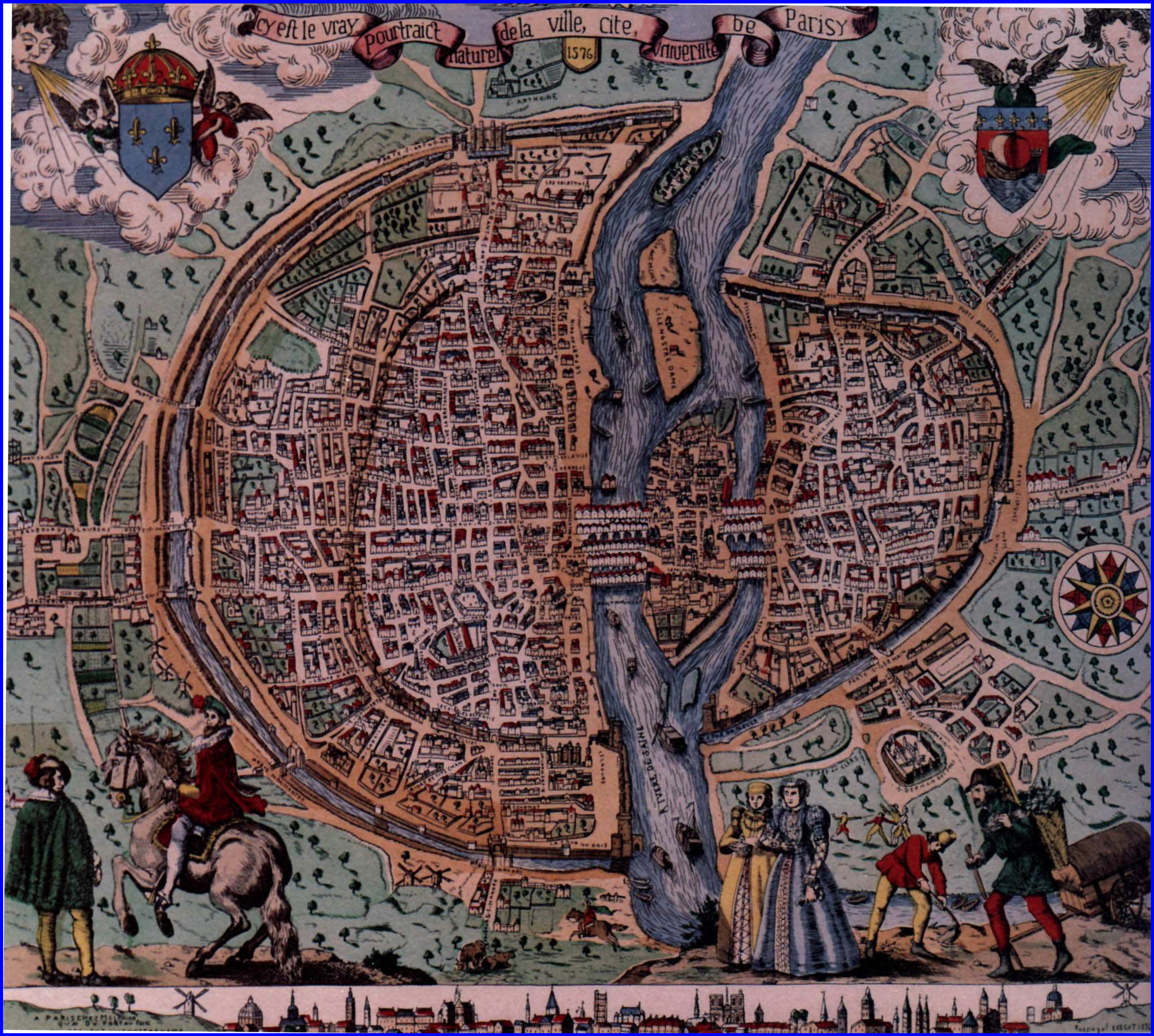
OR: Amongst all figures of equal perimeter, the circle encloses the largest area.

“Mathematically”: Let D be a region in the plane with area $A(D)$ and perimeter $L(D)$. Then

$$A(D) \leq \frac{1}{4\pi} L^2(D),$$

- **Queen Dido in Africa, shortly after her arrival, circa 900 B.C. Engraving by Mathäus Merian, the Elder, 1630. (From: Hilderbrandt–Troma, "Shape and Form of the Natural Universe.")**
- **Dido Purchased land from King Jarvas of Numidia. After some negotiations an agreement was reached. The Queen could only have as much land as she could enclose by the hide of an ox.**
- **Dido had her people cut the hide into strips and enclosed a maximal region. This would have been a semicircle as the city of Carthage was built on the shore.**

Medieval map of Paris



Medieval map of Cologne

George Pólya, Mathematics and Plausible Reasoning (1954):

"The isoperimetric theorem, deeply rooted in our experience and intuition, so easy to conjecture, but not so easy to prove, is an inexhaustible source of inspiration."

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Dido’s property for fundamental tones:

$$\boldsymbol{\lambda}_1(D^*) \le \boldsymbol{\lambda}_1(D)$$

Amongst all “drums” of fixed finite area, the circular one produces the smallest fundamental tone.

Different formulation

$D^* = D^*(0, r)$ be a disk centered at 0 and same area as D:

$$\pi r^2 = \text{Area}(D), \qquad r = \sqrt{\frac{\text{Area}(D)}{\pi}}$$

This is the Celebrated Faber-Krahn inequality proved in 1923–1924. It had been conjectured some 50 years earlier (1877) by the British Physicist, Lord Rayleigh.

Eigenfunction

$$\varphi_1(z) = J_0\left(\frac{j_0|z|}{r}\right),$$

J_0 is a Bessel function of order 0

$$\lambda_1(D^*) = \frac{j_0^2}{r^2}$$

Faber–Krahn: For any planar region D of finite area

$$\frac{\pi j_0^2}{\text{Area}(D)} \leq \lambda_D.$$ "low tone", "large area".

Intervals

For an interval: $I = (0, b)$ of length b,

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{b}\right), \quad \lambda_n = \frac{n^2\pi^2}{b^2}.$$

In particular

$$\lambda_2 = \frac{4\pi^2}{b^2}, \qquad \lambda_1 = \frac{\pi^2}{b^2}$$

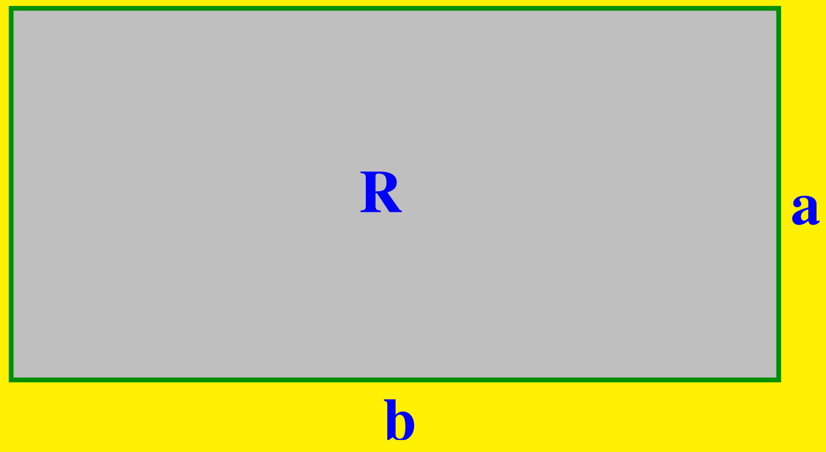
and

$$\lambda_2 - \lambda_1 = \frac{4\pi^2}{b^2} - \frac{\pi^2}{b^2} = \frac{3\pi^2}{b^2}$$

The point: We know everything explicitly for an interval in one dimension!

Rectangles

For the rectangle R $= (0, b) \times (0, a)$ of length b and width a,



(b > a)

$$\varphi_{n,m}(x,y) = \sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{m\pi x}{a}\right), \quad \lambda_{n,m} = \frac{n^2\pi^2}{b^2} + \frac{m^2\pi^2}{a^2}.$$

$$\lambda_2(R) = \frac{4\pi^2}{b^2} + \frac{\pi^2}{a^2}, \quad \lambda_1(R) = \frac{\pi^2}{b^2} + \frac{\pi^2}{a^2}.$$

$$\lambda_2(R) - \lambda_1(R) = \left(\frac{4\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) - \left(\frac{\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) = \frac{3\pi^2}{b^2}$$

Note: For a small, b is essentially the length of the diagonal of R= diameter of R.

Spectral gaps of Schrödinger operators

$H = -\Delta + V$ with Dirichlet conditions in the bounded convex domain $D \subset \mathbb{R}^n$ of finite diameter d_D, $V \geq 0$ is bounded and convex in D.

Conjecture:

$$\lambda_2(\mathrm{D},\mathrm{V}) - \lambda_1(\mathrm{D},\mathrm{V}) > \frac{3\pi^2}{\mathrm{d}_\mathrm{D}^2},$$

with the lower bound approached when $V = 0$ and the domain becomes a thin rectangular box.

- **M. van den Berg 1983,**
- **Ashbaugh–Benguria 1987,**
- **Problem #44 in Yau's 1993 "open problems in geometry"**

Known:

- **False for non-convex domains even with $V = 0$ (Easy!)**
- **A sharp comparison upper bound already exists (Not so easy!).**

Variational Formulas

$$\lambda_1(D,V) = \inf\left\{\frac{\int_D |\nabla u|^2 + u^2 V}{\int_D |u|^2}; u \in C_0^\infty\right\}$$

$$\lambda_2(D,V) = \inf\left\{\frac{\int_D |\nabla u|^2 + u^2 V}{\int_D |u|^2}; u \in C_0^\infty,\ \int_D \varphi_1 u = 0\right\}$$

$$\lambda_2 - \lambda_1 = \inf\left\{\frac{\int_D |\nabla u|^2 \varphi_1^2(x)\,dx}{\int_D |u|^2 \varphi_1(x)\,dx}; u \in C^\infty,\ \int_D \varphi_1^2(x)u(x) = 0\right\}$$

Results Using these characterizations:

- **Singer-Wang-S.T.Yau-S.S.T.Yau (1985): lower bound of $\pi^2/4d_D^2$**
- **R. Lavine (1994): full conjecture for an interval on the real line (only case known).**
- **Many lower bounds including best available R. Smits (1996) π^2/d_D^2.**

R. Bañuelos, Davis, Méndez-Hernández (2004), have proved the conjecture when D and V have some symmetries.

Brownian motion in a region(click here to see Java simulation)

Brownian motion in a region(click here to see Java simulation)

Let τ_D be the first time the Brownian motion hits the boundary of D (also called survival time)

Two Facts:

- $$\lim_{t\to\infty}\frac{1}{t}\log P_z\left(\tau_D > t\right) = -\lambda_1$$
- $$\lim_{t\to\infty} e^{\lambda_1 t} P_z\left(\tau_D > t\right) = \varphi_1(z)\int_D \varphi_1(x)dx$$

Similar formula with non-zero V using “Feynman-Kac’s formula”

Fractional Laplacians/Stable Processes

- **Question:** What is the lowest eigenvalue for rotationally symmetric stable process of order $0 < \alpha < 2$ in the interval $[-1, 1]$?

- **Equivalently:** What is the smallest Dirichlet eigenvalue for the operator $(-\Delta)^{\alpha}$ with Dirichlet boundary conditions in the interval $[-1, 1]$?

For any $D \subset \mathbb{R}^n$,

$$\lambda_1^{\alpha} = A_{\alpha,n} \inf \left\{ \int_D \int_D \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}} \, dx \, dy + 2 \int_D (f(x))^2 \, k_D(x) dx \right\}$$

$$k_D(x) = \int_{D^c} \frac{dy}{|x - y|^{n+\alpha}}, \;\; A_{\alpha,n} = \frac{\alpha \Gamma(n + \alpha/2)}{2^{1-\alpha} \pi^{n/2} \Gamma(1 - \alpha/2)}$$

"$\inf$" taken over all smooth functions f with support in D.

Investigated by: (1) M.Kac-H.Polar (1950), (2) H. Widom (1961), J. Taylor (1967), B. Fristedt (1974), S. Pozin-L. Sakhnovich (1994), J. Bertoin (1998), D. Khoshnevisan-Z. Shi (1998), . . .

Lévy Processes

Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: de Finetti, Kolmogorov, Khintchine, Itô.

- Rich stochastic processes, generalizing several basic processes in probability: Brownian motion, Poisson processes, stable processes, subordinators, . . .
- Regular enough for interesting analysis and applications. Their paths consist of continuous pieces intermingled with jump discontinuities at random times. Probabilistic and analytic properties studied by many.
- Many Developments in Recent Years:
  - ⋆ Applied: Queueing Theory, Math Finance, Control Theory, Porous Media . . .
  - ⋆ Pure: Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes

Definition

A **Lévy Processes** is a stochastic process $\eta = (\eta_t), t \geq 0$ with

- η_t has independent and stationary increments
- Each $\eta_0 = 0$ (with probability 1)
- η is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \longrightarrow s} P\{|\eta_t - \eta_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives “cadlag” paths: Right continuous with left limits.

- **Stationary increments:** $0 \leq s \leq t < \infty$, $A \in \mathbb{R}^n$ Borel

$$P\{\eta_t - \eta_s \in A\} = P\{\eta_{t-s} - \eta_0 \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 \leq t_1 \leq t_2 \cdots \leq t_m < \infty,$$

the random variables

$$\eta_{t_1} - \eta_0,\ \eta_{t_2} - \eta_{t_1}, \ldots, \eta_{t_m} - \eta_{t_{m-1}}$$

are independent.

The characteristic function of $\boldsymbol{\eta_t}$ is

$$\boldsymbol{\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} p_t(dx)}$$

where $\boldsymbol{p_t}$ is the distribution of $\boldsymbol{\eta_t}$.

The characteristic function of η_t is

$$\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t}\right) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} p_t(dx)$$

where p_t is the distribution of η_t.

Lévy-Khintchine Formula: $\varphi_t(\xi) = e^{t\rho(\xi)}$ where

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^n \setminus \{0\}} \left(e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^n$, a non–negative definite symmetric $n \times n$ matrix A and a Borel measure ν on $\mathbb{R}^n \setminus \{0\}$ with

$$\int_{\mathbb{R}^n \setminus \{0\}} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the *symbol* of the process or the *characteristic exponent.* The triple (b, A, ν) is called the *characteristics of the process.*

Converse also true. Given such a triple we can construct a Lévy process.

Examples

1. **Standard Brownian motion:**

   With $(0, I, 0)$, I the identity matrix,

   $$\eta_t = B_t, \quad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**

   $(0, A, 0)$, η_t is "generalized" Brownian motion, mean zero, covariance

   $$E(\eta_s^j \eta_t^i) = a_{ij} \min(s, t)$$

   X_t has the normal distribution (A positive definite here)

   $$\frac{1}{(2\pi t)^{n/2}\sqrt{\det(A)}} \exp\left(-\frac{1}{2t} x \cdot A^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, A, 0)$ get Brownian motion with a drift:

   $$\eta_t = bt + B_t$$

4. **Poisson Process:** The Poisson Process $\eta_t = N_\lambda(t)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

   $$P\{N_\lambda(t) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \ m = 1, 2, \ldots$$

   $N_\lambda(t)$ has continuous paths except for jumps of size 1 at the random times

   $$\tau_m = \inf\{t > 0 : N_\lambda(t) = m\}$$

5. **Compound Poisson Process** obtained by summing iid random variables up to a Poisson Process.

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6. **Relativistic Brownian motion** According to quantum mechanics, a particle of mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

   where c is speed of light. Then $\eta(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

7. **The zeta process:** Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}, \quad z = x + iy \in \mathbb{C}$$

Khintchine: For every fix $x > 1$,

$$\rho_x(y) = \log \left(\frac{\zeta(x + iy)}{\zeta(y)} \right)$$

is the symbol of a Lévy process.

Biane-Pitman-Yor: “Probability laws related to the Jacobi theta and Riemann Zeta functions and Brownian excursions, Bull. Amer math. Soc., 2001.

M Yor: A note about Selberg's integrals with relation with the beta–gamma algebra, 2006.

8. The rotationally invariant Lévy stable processes ("Fractional Diffusions"):

These are self–similar processes in $\mathbb{R}^n$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \quad 0 < \alpha \leq 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = e^{-t|\xi|^\alpha}.$$

$\alpha = 2$ is *Brownian motion.* $\alpha = 1$ is the *Cauchy processes.* $\alpha = 3/2$ is called the *Haltsmark distribution* used to model gravitational fields of stars.

Transition probabilities:

$$P_x\{\eta_t \in A\} = \int_A p_t^\alpha(x-y)dy, \quad \text{any Borel } A \subset \mathbb{R}^n$$

$$p_t^\alpha(x) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}e^{-t|\xi|^\alpha}d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \alpha = 2,$$ **Brownian motion**

$$p_t^1(x) = \frac{C_n\, t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad \alpha = 1,$$ **Cauchy Process**

For any $a > 0$, the two processes

$$\{\eta_{(at)}\,;\ t \geq 0\} \text{ and } \{a^{1/\alpha}\eta_t\,;\ t \geq 0\},$$

have the same finite dimensional distributions *(self-similarity).*

The only processes among these with continuous paths is Brownian motion. BUT, they do have paths which are right continuous with left limits.

Let D be a bounded connected subset of $\mathbb{R}^n$. The first exit time of $\eta(t)$ from D is

$$\tau_D = \inf\{t > 0 : \eta(t) \notin D\}$$

$$\lim_{t\to\infty} \frac{1}{t} \log P_x\{\tau_D > t\} = -\lambda_1$$

More:

$$\lim_{t\to\infty} e^{t\lambda_1} P_x\{\tau_D > t\} = \varphi_1(x) \int_D \varphi_1(y)$$

Remark: These results are obtained from the "heat" semigroup using the formula:

$$P_x\{\tau_D > t\} = \int_D P_t^D(x,y)\,dy = \sum_{j=1}^{\infty} e^{-\lambda_k t} \varphi_k(x) \int_D \varphi_k(y)\,dy$$

Finite Dimensional Distributions (back to probability)

Need to study:

$$\begin{aligned} P_x\{\tau_D > t\} &= P_x\{\eta(s) \in D, \forall\ 0 < s \le t\} \\ &= \lim_{m\to\infty} P_x\{\eta(\frac{jt}{m}) \in D, j = 1, 2, \dots, m\} \end{aligned}$$

More general: Study the finite dimensional distributions:

$$P_x\{\eta(t_1) \in D_1,\ \eta(t_2) \in D_2,\ \dots,\ \eta(t_m) \in D_m\}$$

for any sequence of times and any sequence of domains:

$$0 < t_1 < t_2 < \cdots < t_m; \;\; D_1 \subset \mathbb{R}^n, \dots, D_m \subset \mathbb{R}^n,$$

as a function of $x \in D_1$ and and the domains.

Multiple Integrals (back to analysis)

$$\Phi_m(x, D_1, \cdots, D_m) = \int_{D_1} \cdots \int_{D_m} \prod_{j=1}^{m} P^{\alpha}_{t_j}(x_j - x_{j-1})\, dx_1 \cdots dx_m$$

Study this as a function of $x = x_0$ and $D_1, D_2, \ldots, D_m$.

Set probability and spectral theory aside, place a "do not disturb" sign on your door and, basically, do advance calculus.

Pay Off:

- **New estimates for eigenvalues of the fractional Laplacian and new properties for their eigenfunctions.**
- **Many new "Dido" properties for Brownian motion and other Lévy (including all rotationally symmetric stable) processes.**
- **van en Berg's conjecture for planar regions with two axes of symmetry and symmetric potential $V's$.**
- **Applications to versions of J. Rauche's (1966) "hot–spots" conjecture. ("Holes and Hot Spots in Nature", *Nature,* October 1999.)**

Sample results

- **Eigenvalues for $(-\Delta)^{\alpha/2}$ in the unit ball $B(0,1)$ of $\mathbb{R}^n$.**

$$C_{n,\alpha} = \frac{\Gamma(\frac{n}{2})}{2^{\alpha}\Gamma(1+\frac{n}{2})\Gamma(\frac{n+\alpha}{2})}$$

$$\frac{1}{C_{n,\alpha}} \leq \lambda_\alpha \leq \frac{1}{C_{\alpha,n}} \frac{\beta(\frac{n}{2},\frac{n}{2}+1)}{\beta(\frac{n}{2},\alpha+1}\,)$$

$\alpha = 1$, $B(0,1) = (-1,1)$,

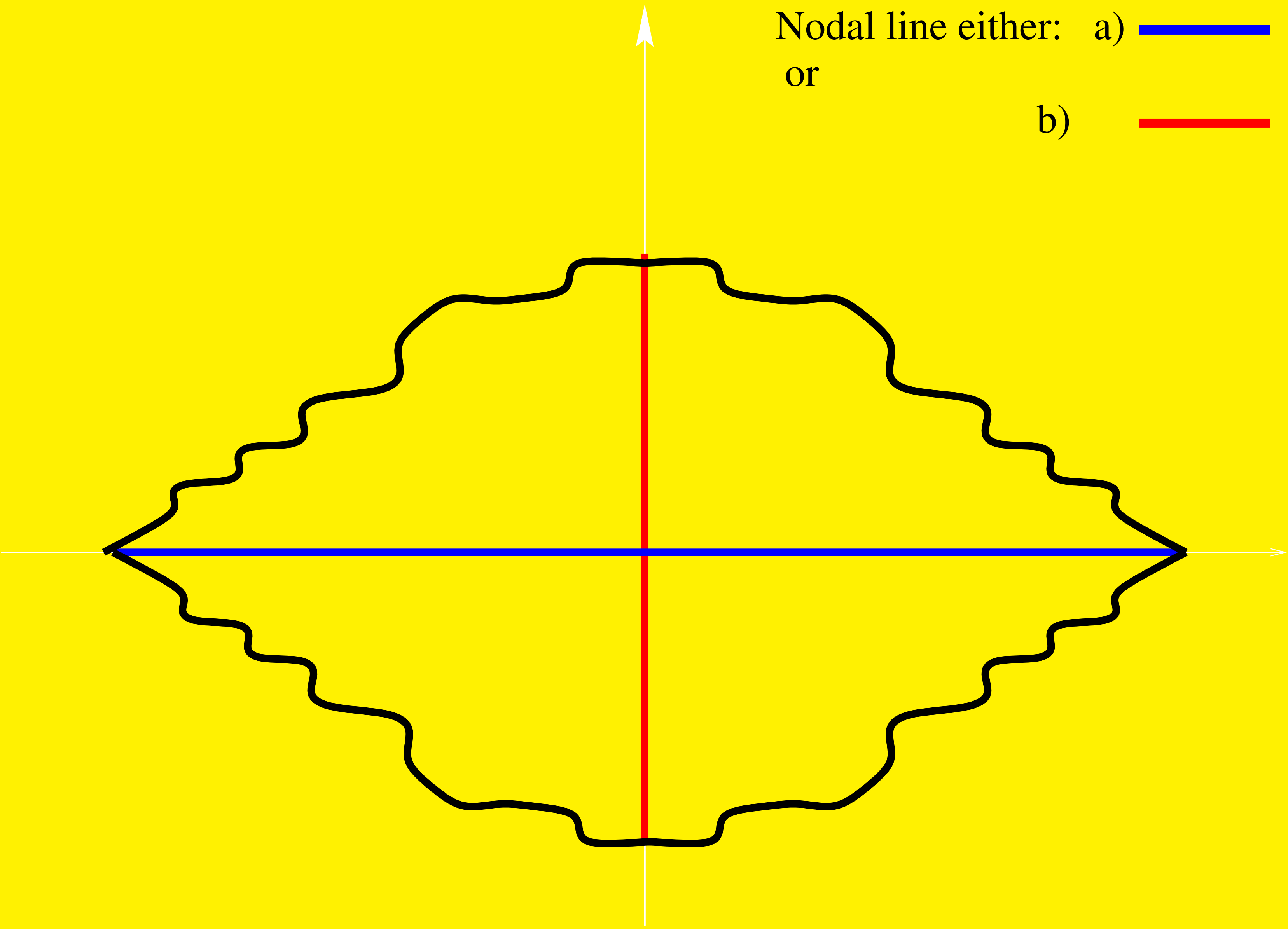
$$1 \leq \lambda_1 \leq \frac{3\pi}{8} \approx 1.178 << (\sqrt{\pi^2/4} = \pi/2)$$

- **Amongst all regions of fixed volume, the ball maximizes the exit time for all symmetric stable processes, including Brownian motion**

- **Amongst all convex regions of fixed inner radius, the infinite strip maximizes the survival time for all symmetric stable processes, including Brownian motion, and the disk minimizes**

- **Similar statements for Eigenvalues**

- **Spectral gap conjecture holds for regions that look like:**



- **"Hot–Spots:" In such regions the ratio φ_2/φ_1 of the second and first Dirichlet eigenfunctions achieves its maximum (and minimum) on the boundary and only on the boundary.**

Credits

- **Collaborators:** P. Méndez-Hernández, D. You, R. Latala, T. Kulczycki, K. Burdzy, M. Pang, M. Pascu.
- **Multiple Integrals:** First used for isoperimetric problems by H. Brascamp, and subsequently by H. Brascamp, E. Lieb, J. Luttinger for log–concavity, etc., and by Crister Borell in (truly) infinite dimensions.