Fractional Diffusions, Lévy Processes*

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  - ⋆ Pure: Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes

Some References

- **General Theory**

1. J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.
2. D. Applebaum. *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2004

- **Applications**

1. B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*, Springer, 2004
2. R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, 2004

- **Recent analytical, probabilistic, geometric, developments:** Work of R. Bass, K. Burdzy, K. Bogdan, Z. Chen, T. Kulczycki, R. Song, J-M. Wu, . . .

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Outline

- Definitions, Structure
- Examples
- Semigroups, generator, . . .
- “Boundary” value problems for rotationally invariant stable processes– the boundary value problems for the ”fractional” Laplacian, Some Questions, Some Results
- Applications to problems for the Laplacian (Brownian motion) which result from the investigations of the more general operators.

Definition

A **Lévy Processes** is a stochastic process $\eta = (\eta_t), t \geq 0$ with

- η has independent and stationary increments
- Each $\eta_0 = 0$ (with probability 1)
- η is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|\eta_t - \eta_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 \le s \le t < \infty$, $A \in \mathbb{R}^n$ Borel

$$P\{\eta_t - \eta_s \in A\} = P\{\eta_{t-s} - \eta_0 \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 \le t_1 \le t_2 \cdots \le t_m < \infty,$$

the random variables

$$\eta_{t_1} - \eta_0,\ \eta_{t_2} - \eta_{t_1}, \ldots, \eta_{t_m} - \eta_{t_{m-1}}$$

are independent.

The characteristic function of η_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} p_t(dx)$$

where p_t is the distribution of η_t.

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Lévy-Khintchine Formula: $\varphi_t(\xi) = e^{t\rho(\xi)}$ where

$$\rho(\xi) = ib\cdot\xi - \frac{1}{2}\xi\cdot A\xi + \int_{\mathbb{R}^n\setminus\{0\}} \left(e^{i\xi\cdot x} - 1 - i\xi\cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^n$, a non–negative definite symmetric $n \times n$ matrix A and a Borel measure ν on $\mathbb{R}^n\setminus\{0\}$ with

$$\int_{\mathbb{R}^n\setminus\{0\}} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the *symbol* of the process or the *characteristic exponent.* The triple (b, A, ν) is called the *characteristics of the process.*

Converse also true. Given such a triple we can construct a Lévy process.

Examples

1. **Standard Brownian motion:**
   With $(0, I, 0)$, I the identity matrix,

$$\eta_t = B_t, \quad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**
   $(0, A, 0)$, η_t is "generalized" Brownian motion, mean zero, covariance

$$E(\eta_s^j \eta_t^i) = a_{ij} \min(s, t)$$

   X_t has the normal distribution

$$\frac{1}{(2\pi t)^{n/2}\sqrt{\det(A)}} \exp\left(-\frac{1}{2t} x \cdot A^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, A, 0)$ get Brownian motion with a drift:

$$\eta_t = bt + B_t$$

4. **Poisson Process:** The Poisson Process $\eta_t = N_\lambda(t)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{N_\lambda(t) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \; m = 1, 2, \ldots$$

$N_\lambda(t)$ has continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : N_\lambda(t) = m\}$$

5. **Compound Poisson Process** obtained by summing iid random variables up to a Poisson Process.

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6. **Relativistic Brownian motion** According to quantum mechanics, a particle of rest mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where c is speed of light. Then $\eta(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

7. **The zeta process:** Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}, \qquad z = x + iy \in \mathbb{C}$$

Khintchine: For every fix $x > 1$,

$$\rho_x(y) = \log\left(\frac{\zeta(x+iy)}{\zeta(y)}\right)$$

is the symbol of a Lévy process.

Biane-Pitman-Yor: “Probability laws related to the Jacobi theta and Riemann Zeta functions and Brownian excursions, Bull. Amer math. Soc., 2001.

M Yor: A note about Selberg’s integrals with relation with the beta–gamma algebra, 2006.

8. **The rotationally invariant Lévy stable processes:** These are self–similar processes in $\mathbb{R}^n$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$ is *Brownian motion.* $\alpha = 1$ is the *Cauchy processes.* $\alpha = 3/2$ is called the *Haltsmark distribution* used to model gravitational fields of stars.

Transition probabilities:

$$P_x\{\eta_t \in A\} = \int_A p_t^\alpha(x-y)dy, \quad \text{any Borel} \quad A \subset \mathbb{R}^n$$

$$p_t^\alpha(x) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} e^{-t|\xi|^\alpha} d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \alpha = 2, \quad \textbf{Brownian motion}$$

$$p_t^1(x) = \frac{C_n\, t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \quad \alpha = 1, \quad \textbf{Cauchy Process}$$

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For any $a > 0$, the two processes

$$\{\eta_{(at)}\,;\; t \geq 0\} \;\text{ and }\; \{a^{1/\alpha}\eta_t\,;\; t \geq 0\},$$

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In the same way, the transition probabilities scale similarly to those for BM:

$$p_t^\alpha(x) = t^{-n/\alpha} p_1^\alpha(t^{-1/\alpha}x)$$

Lévy–Itô decomposition

Lévy processes can be written as

$$\eta(t) = \eta^c(t) + \sum_{s:0\leq s\leq t} \Delta\eta(s)$$

where

$$\Delta\eta(s) = \eta(s) - \eta(s^-) = \text{jump at time s.}$$

The *Poisson random measure, aka "jump measure"* is

$$N(t,U) = N(t,U,\omega) = \sum_{s:0<s\leq t} \chi_U(\Delta\eta(s))$$

Lévy-Itô: The Lévy process $X(t)$ has the representation

$$\eta(t) = bt + B_A(t) + \int_{\{0<|x|<1\}} x\Big[N(t,dx) - td\nu(x)\Big] + \int_{\{|x|\geq|1\}} xN(t,dx)$$

A Lévy stochastic differential equation (Lévy SDE) is of the form

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dB(t) + \int \gamma(t, X(t), z)\tilde{N}(t, dz)$$

(with certain "Lipschitz" assumptions on the coefficients it has a unique solution).

Solutions to Lévy SDE's in the *time homogeneous case*, i.e.,

$$\begin{aligned} \alpha(t,x) &= \alpha(x) \\ \sigma(t,x) &= \sigma(x) \\ \gamma(t,x,z) &= \gamma(x,z) \end{aligned}$$

are called a **"jump diffusion" or "Lévy diffusion"**

The diffusion generator is:

$$
\begin{aligned}
Af(x) &= \lim_{t\to 0}\frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big) \\
&= \sum_{i=1}^{n}\alpha_j(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\
&+ \int\Big[f(x+\gamma(x,z)) - f(x) - \nabla f(x)\cdot\gamma(x,z)\Big]\nu(dz)
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For the Lévy process $\{\eta(t); t \geq 0\}$,

$$T_t f(x) = E[f(\eta(t)|\eta_0 = x] = E_x[f(\eta(t))], \quad f \in \mathcal{S}(\mathbb{R}^n)$$

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$$Af(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \rho(\xi) \hat{f}(\xi) d\xi$$

Using the Lévy–Khintchine formula one can show

$$Af(x) = \sum_{i=1}^{n} b_i \partial_i f(x) + \frac{1}{2} a_{i,j} \partial_i \partial_j f(x) + \int \Big[f(x+y) - f(x) - y \cdot \nabla f(x) \chi_{\{|y|<1\}} \Big] \nu(dy)$$

Examples:

- Standard Brownian motion:

$$Af(x) = \frac{1}{2} \Delta f(x)$$

- Poisson Process of intensity λ:

$$Af(x) = \lambda \Big[f(x+1) - f(x) \Big]$$

- Rotationally Invarient Stable Processes of order $0 < \alpha < 2$, **Fractional Diffusions:**

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &= C_{\alpha,n} \lim_{\varepsilon \to 0} \int_{\{|x-y|>\varepsilon\}} \frac{f(y) - f(x)}{|x-y|^{n+\alpha}} dy \end{aligned}$$

Fractional Diffusions in regions of $\mathbb{R}^n$

From now on $\eta(t)$ is rotationally invariant stable with symbol

$$\rho(\xi) = -|\xi|^{\alpha}, \qquad 0 < \alpha < 2.$$

Let D be a bounded connected subset of $\mathbb{R}^n$. The first exit time of $\eta(t)$ from D is

$$\tau_D = \inf\{t > 0 : \eta(t) \notin D\}$$

Heat Semigroup in D

$$T_t^D f(x) \quad = \quad E_x\Big[f(\eta(t)); \tau_D > t\Big], \quad f \in L^2(D)$$

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$$p_t^D(x,y) = p_t^\alpha(x,y) - E^x(\tau_D < t; p^\alpha_{t-\tau_D}(\eta(\tau_D),\, y)).$$

$p_t^D(x, y)$ is called the **Heat Kernel for the stable process in D.**

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The general theory of heat semigroups gives an orthonormal basis of eigenfunctions

$$\{\varphi_m\}_{m=1}^\infty \quad on \quad L^2(D)$$

with eigenvalues $\{\lambda_m\}$ satisfying

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

That is,

$$T_t^D \varphi_m(x) = e^{-\lambda_m t}\varphi_m(x),\ x \in D.$$

From general heat semigroup theory: λ_1 is simple, φ is strictly positive in D, the spectral gap $\lambda_2 - \lambda_1 > 0$.

Some "basic" problems

- Regularity of heat kernels, eigenfunctions, general solutions of "heat equation", Sobolev, log-Sobolev inequalities, "intrinsic ultracontractivity," . . .
- "Boundary" regularity of solutions
- Estimates on eigenvalues, including the ground state λ_1 and the spectral gap $\lambda_2 - \lambda_1$.
- Number of "nodal" domains (Courant–Hilbert Nodal domain Theorem).
- Geometric properties of eigenfunctions, including a "Brascamp–Lieb" log–concavity type theorem for φ_1.

Many open even when the domain $D = (-1, 1)$.

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Theorem 3: The Faber–Krahn inequality: For any region the eigenvalue is minimized for a ball of same volume.

$$\lambda_1(D^*) \leq \lambda_1(D), \qquad D^* = \text{ball of same volume as D}$$

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Theorem 4: Brascamp–Lieb true for the Cauchy processes ($\alpha = 1$ case) in an interval in one dimension.

Only known case!

The Faber–Krahn inequality (finite dimensional distributions)

Apply the semigroup to the function $f(x) = 1, \; x \in D$

$$T_t^D f(x) \quad = \quad E_x[\chi_D(\eta_t); \tau_D > t] = \int_D p_t(x, y) dy$$

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$$\Rightarrow \qquad \lim_{t\to\infty} \frac{1}{t} \log P_x\{\tau_D > t\} = -\lambda_1$$

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More:

$$
\lim_{t\to\infty} e^{t\lambda_1} P_x\{\tau_D > t\} = \varphi_1(x) \int_D \varphi_1(y)
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More general: Study the finite dimensional distributions:

$$P_x\{\eta(t_1) \in D_1,\ \eta(t_2) \in D_2,\ \ldots,\ \eta(t_m) \in D_m\}$$

for any **sequence of times and any sequence of domains:**

$$0 < t_1 < t_2 < \cdots < t_m; \quad D_1 \subset \mathbb{R}^n, \ldots, D_m \subset \mathbb{R}^n,$$

as a function of $x \in D_1$ and and the domains.

Sample Theorem 1: Take $D_1 = D_2 = \cdots = D_m = D$, $x \in D$.

$$P_x\{\eta(t_1) \in D,\ \eta(t_2) \in D,\ \ldots,\ \eta(t_m) \in D\}$$

$$\leq P_0\{\eta(t_1) \in D^*,\ \eta(t_2) \in D^*,\ \ldots,\ \eta(t_m) \in D^*\}$$

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Proof: From Brascamp–Lieb–Luttinger re-arrangments of multiple integrals.

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Proof: From Brascamp–Lieb–Luttinger re-arrangments of multiple integrals.

Sample Theorem 2: Let $I = (-1, 1)$. For any order sequence of times $t_1, t_2, \ldots t_m$,

$$F(x) = P_x\{\eta(t_1) \in I, \eta(t_2) \in I, \ldots, \eta(t_m) \in I\}, \qquad x \in I$$

is unimodal in I with mode at 0 and "mid–concave"

Conjecture on Dirichlet Spectral gaps of Schrödinger operators

$H = -\Delta + V$ with Dirichlet conditions in the bounded convex domain $D \subset \mathbb{R}^n$ of finite diameter d_D, $V \geq 0$ is bounded and convex in D. We have eigenvalues

$$0 < \lambda_1(D,V) < \lambda_2(D,V) \leq \lambda_3(D,V) \ldots.$$

Conjecture (M. van den Berg 1983, Ashbaugh–Benguria 1987, and Problem #44 in Yau's 1990 "open problems in geometry")

$$gap(D,V) = \lambda_2(D,V) - \lambda_1(D,V) > \frac{3\pi^2}{d_D^2}$$

with the lower bound approached when $V = 0$ and the domain becomes a thin rectangular box.

False for non-convex domains even with $V = 0$.

I.Singer–B.Wang–S.T.Yau–S.S.T.Yau (1985) : Using the "p-function" (Max Principle) techniques of Payne, Philippin, . . .

$$\frac{\pi^2}{4d_D^2} \le \lambda_2(D,V) - \lambda_1(D,V)$$

Many extensions–best lower bound until recently: $\frac{\pi^2}{d_D^2}$

Using inequalities for finite dimensional distributions (which come out of the study of stable processes) leads to the conjecture under some symmetry for D and some potentials:

$$\lambda_2(D,V) - \lambda_1(D,V) \ge \frac{3\pi^2}{d_D^2}$$

Work with P. Médez–Hernández.

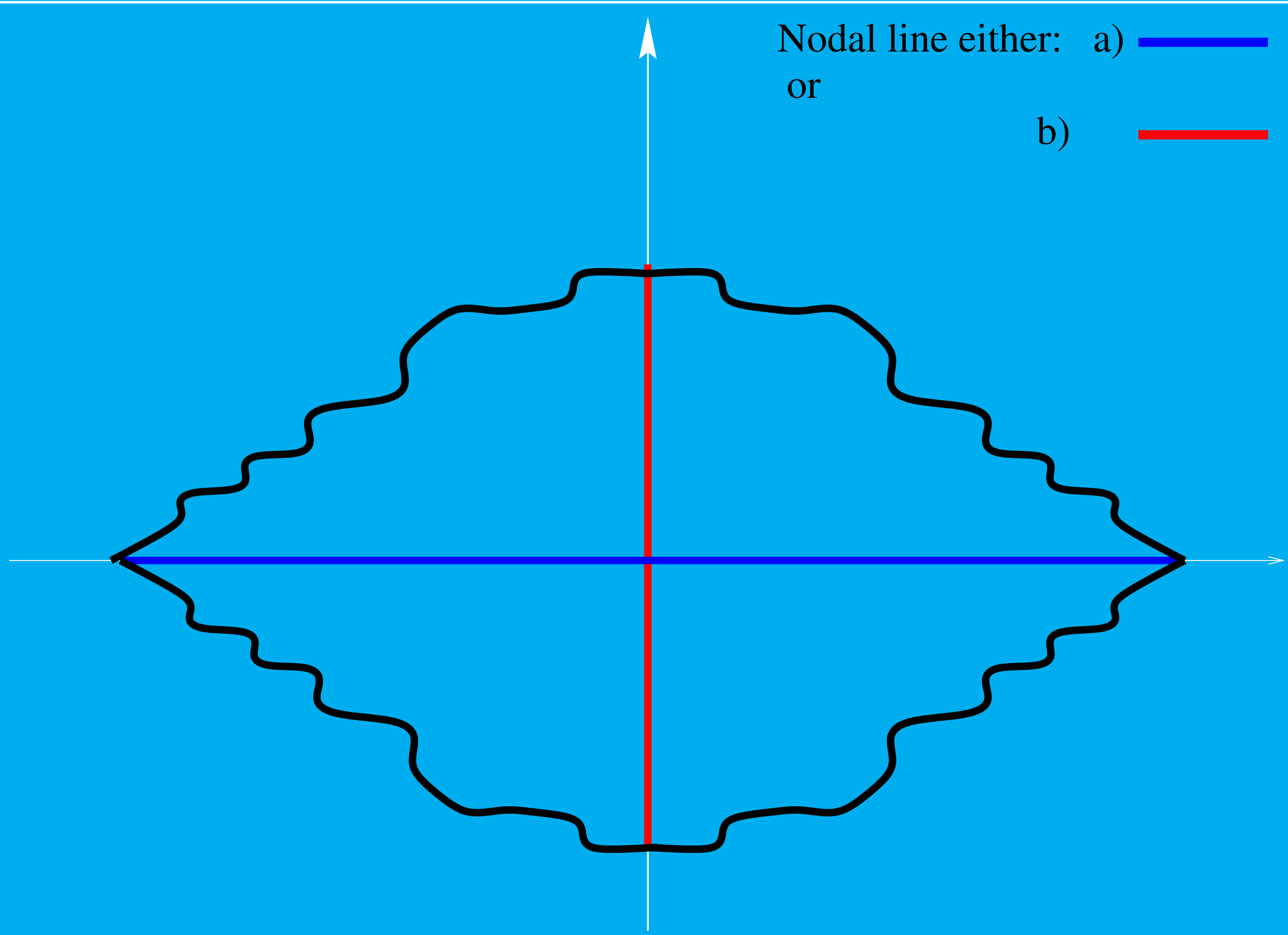
Motivated by the hot–spots conjecture of J. Rauch for the Neumann problem

Conjecture: For any convex domain in $D \subset \mathbb{R}^n$, the function

$$\Psi(x) = \frac{\varphi_2(x)}{\varphi_1(x)}$$

attains it maximum and minimum on the boundary, and only on the boundary, of D.

HERE we are dealing with the eigenfunctions of the “ordinary” Laplacian



$$D^+ = D \cap \{(x,y) \in \mathbb{R}^2 : y > 0\},$$
$$D^- = D \cap \{(x,y) \in \mathbb{R}^2 : y < 0\}.$$

(i) If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then

$$\Psi(z_1) < \Psi(z_2).$$

(ii) If $z_1 = (x, y_1) \in D^-$ and $z_2 = (x, y_2) \in D^-$ with $y_2 < y_1$, then

$$\Psi(z_1) < \Psi(z_2).$$

In particular, Ψ cannot attain a maximum nor a minimum in the interior of D.

(iii) If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then

$$\Psi(z_1) \leq \Psi(z_2).$$

Work with P. Médez–Hernández, 2006.

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Work with P. Médez–Hernández, 2006.

Exact analogue of Neumann case by D. Jerison and N. Nadirashvili.