RIEMANN ZETA FUNCTIONS

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Abstract. A proof of the Riemann hypothesis is obtained for zeta functions whose Dirichlet coefficients are defined as eigenvalues of Hecke operators. The argument is formulated in Hilbert spaces of entire functions [1] which generalize the Stieltjes integral representation of positive linear functionals on polynomials in a context related to the Hermite class of entire functions. The Riemann hypothesis for Hilbert spaces of entire functions [2] formulates properties of the spaces essential to the argument. The verification of these properties applies a Radon transformation which relates Fourier analysis on fields to Fourier analysis on skew–fields. Since the Radon transformation in two dimensions inverts the infinitessimal generator for the flow of heat, the proof of the Riemann hypothesis applies a generalization of the Fourier treatment of heat flow to four dimensions. The argument presumes a minimal knowledge of Fourier analysis. The Riemann hypothesis for Dirichlet zeta functions is a corollary of the Riemann hypothesis for associated Hecke zeta functions. The Riemann hypothesis for the Euler zeta function is obtained in the same way but with a variant of the argument demanded by the singularity of the zeta function.

Section 1 *The Algebraic Skew–plane* begins with a noncommutative algebra to be completed in various topologies.

Section 2 *The Dedekind Skew–plane* continues with the classical completion of the algebraic skew–plane.

Section 3 *Adic Skew–planes* presents the nonclassical completions.

Section 4 *Product Skew–planes* composes the classical and nonclassical completions and passes to a compact quotient space.

Section 5 *Fourier Analysis for the Dedekind Skew–plane* is the classical formulation of Fourier analysis in a quaternionic context as required for a treatment of the Radon transformation.

Section 6 *Fourier Analysis for Adic Skew–planes* adapts the Fourier transformation to nonclassical completions in a quaternionic context as required for the Radon transformation.

Section 7 *Fourier Analysis for Product Skew–planes* is a spectral analysis of the Radon transformation on product skew–planes as an application of zeta functions defined by Hecke operators.

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*Research supported by the National Science Foundation
Mathematical Reviews Classification 11M26

Section 8 *The Riemann Hypothesis for Hilbert Spaces of Entire Functions* is a formulation of the Riemann hypothesis which applies to Hecke zeta functions related to Dirichlet zeta functions and to the Euler zeta function.

1. The algebraic skew-plane

The algebraic skew–plane is the set of quaternions

$$\xi = t + ix + jy + kz$$

whose coordinates x, y, z, and t are rational numbers. The conjugate element of the algebraic skew–plane is

$$\xi^- = t - ix - jy - kz.$$

Conjugation is an anti–automorphism ξ into ξ^- of the algebraic skew–plane. Rational numbers are identified with self–conjugate elements of the algebraic skew–plane. An element of the algebraic skew–plane is said to be integral if its coordinates are all integers or all halves of odd integers. Sums and products of integral elements of the algebraic skew–plane are integral. The conjugate of an integral element of the algebraic skew–plane is integral. A theorem originating with Diophantus and confirmed by Lagrange states that every positive integer

$$r = \xi^- \xi$$

is represented by an integral element ξ of the algebraic skew–plane. A theorem of Jacobi states that the number of representations is twenty–four times the sum of the odd positive divisors of r.

The proof of the theorem by Adolf Hurwitz applies a Euclidean algorithm for integral elements of the algebraic skew–plane. If α is an integral element of the algebraic skew–plane and if β is a nonzero integral element of the algebraic skew–plane, an integral element γ of the algebraic skew–plane exists which satisfies the inequality

$$(\alpha - \beta\gamma)^-(\alpha - \beta\gamma) < \beta^-\beta.$$

A right ideal of the integral elements of the algebraic skew–plane, which contains a nonzero element, contains a nonzero element β which minimizes $\beta^-\beta$. An integral element of the algebraic skew–plane which belongs to the ideal is a product

$$\alpha = \beta\gamma$$

with γ an integral element of the algebraic skew–plane.

If an integral element ω of the algebraic skew–plane represents

$$2 = \omega^-\omega,$$

the element $\omega\xi\omega^{-1}$ of the algebraic skew–plane is integral whenever ξ is an integral element of the algebraic skew–plane. The ideal generated by ω is independent of the choice of integral element which represents two.

When ρ is a positive integer or is the product of a positive integer and an integral element of the algebraic skew–plane which represents two, the left ideal generated by ρ is a right ideal since it contains ξ^- whenever it contains ξ. The quotient space is a ring which inherits a conjugation. Integral elements of the algebraic skew–plane are said to be congruent modulo ρ if they represent the same element of the quotient space.

When ρ_1 and ρ_2 are relatively prime positive integers, or when one is an odd positive integer and the other is the product of a relatively prime positive integer and an integral element of the algebraic skew–plane which represents two, the quotient ring modulo $\rho_1\rho_2$ is isomorphic to the Cartesian product of the quotient ring modulo ρ_1 and the quotient ring modulo ρ_2. The quotient ring is said to be p–adic when ρ is a power of a prime p or when ρ is the product of a power of the even prime p and an integral element of the algebraic skew–plane which represents the even prime.

The quotient ring is isomorphic to the ring of quaternions with coordinates in the integers modulo ρ when ρ is an odd positive integer. When ρ is an integral element of the algebraic skew–plane which represents two, the quotient space modulo ρ is a field of four elements obtained by adjoining a cube root of unity to the integers modulo two.

When ρ is twice an integral element of the algebraic skew–plane which represents two, the quotient space modulo ρ contains sixty–four elements of which forty–eight are invertible. Twenty–four invertible elements have conjugate congruent to the inverse and twenty–four have conjugate congruent to minus the inverse. An element whose conjugate is congruent to its inverse is represented by an integral element of the algebraic skew–plane with conjugate as inverse.

The algebraic line is the field of self–conjugate elements of the algebraic skew–plane. The elements of the algebraic line are rational numbers. An algebraic plane is defined by an integral element ι of the algebraic skew–plane which is not self–conjugate. The algebraic plane is the field of elements of the algebraic skew–plane which commute with ι. An algebraic plane contains ξ^- whenever it contains ξ. An element

$$\alpha + \iota\beta$$

of the algebraic plane is defined by rational numbers α and β.

A character χ modulo ρ is defined for a positive integer ρ whose prime divisors are divisors of $\iota^-\iota$. The character is a function $\chi(\xi)$ of ξ in the algebraic line which vanishes when ξ is nonintegral, which has equal values at integral elements which are congruent modulo ρ, which satisfies the identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

for all integral elements ξ and η, and which has a nonzero value at an integral element if, and only if, the element is invertible modulo ρ.

A character modulo ρ is said to be primitive modulo ρ if it does not agree on integral elements which are invertible modulo ρ with a character modulo a proper divisor of ρ.

2. The Dedekind skew–plane

The Dedekind topology of the algebraic skew–plane originates in the Dedekind construction of the real numbers from the rational numbers. Convexity introduces topology.

A convex combination

$$\alpha(1-t)+\beta t$$

of elements α and β of the algebraic skew–plane is defined by a nonnegative rational number t such that $1-t$ is nonnegative. A subset of the algebraic skew–plane is said to be convex if it contains the convex combinations of any two of its elements. An intersection of convex sets is convex.

The Dedekind topology of the algebraic skew–plane is determined by convexity. The closure of a nonempty convex subset B of the algebraic skew–plane is the set B^- of elements α of the algebraic skew–plane such that for some, and hence every, element β of B the convex combination

$$\alpha(1-t)+\beta t$$

belongs to B for every positive rational number t such that $1-t$ is nonnegative. The empty set is a convex set whose closure is defined to be itself. The closure of a convex subset of the algebraic skew–plane is a convex subset whose closure is equal to itself.

A convex subset of the algebraic skew–plane is said to be open if it is disjoint from the closure of every disjoint convex set. A nonempty convex subset A of the algebraic skew–plane is open if, and only if, for every element α of A and every element β of the algebraic skew–plane, a positive rational number t exists such that $1-t$ is nonnegative and such that the convex combination

$$\alpha(1-t)+\beta t$$

belongs to A. If A is an open convex set and if B is a convex set, then the intersection of A with the closure of B is contained in the closure of the intersection of A with B. A finite intersection of open convex sets is open.

The algebraic skew–plane is a Hausdorff space whose open sets are defined as the unions of open convex sets. A subset of the algebraic skew–plane is said to be closed if it is the complement of an open set. Finite unions of closed sets are closed. Intersections of closed sets are closed. A convex set is closed if, and only if, its closure is equal to itself.

The Hahn–Banach theorem formulates a property of convex sets which is implicit in the Dedekind construction of real numbers from the rational numbers. A nonempty open convex set A which is disjoint from a nonempty convex set B is contained in an open convex set which is disjoint from B and whose complement is convex.

Addition is continuous as a transformation of the Cartesian product of the algebraic skew–plane with itself into the algebraic skew–plane when the algebraic skew–plane is given the Dedekind topology. The completion of the algebraic skew–plane in the uniform Dedekind topology is the space of quaternions

$$t + ix + jy + kz$$

whose coordinates x, y, z, and t are real numbers. The Dedekind construction of the completion is applicable to the algebraic skew–plane. The Cauchy construction of a completion applies in every uniform topology.

A Cauchy class of closed sets is defined by two conditions. The intersection of a finite number of members of the class is always nonempty. Whenever an open set U contains the origin, a member B of the class exists such that all differences $b - a$ of elements a and b of B belongs to U. Cauchy classes are considered equivalent when they are contained in a common Cauchy class. The union of equivalent Cauchy classes is a maximal Cauchy class. The maximal Cauchy class determined by an element of the algebraic skew–plane is the class of all closed sets containing the element. The set of maximal Cauchy classes admits a convex structure and a topology compatible with the convex structure and topology of the algebraic skew–plane. The maximal Cauchy classes determined by elements of the algebraic skew–plane form a dense subset of the resulting completion, which is isomorphic to the space of quaternions with real coordinates. The Dedekind skew–plane is defined as the completion of the algebraic skew–plane in the Dedekind uniform topology.

The Baire subsets of the Dedekind skew–plane are defined in two equivalent ways. The Baire subsets are the smallest class of subsets containing the open sets such that a countable intersection of members of the class is a member of the class and a countable union of members of the class is a member of the class. The Baire subsets are the smallest class of subsets containing the closed sets such that a countable union of members of the class is a member of the class and a countable intersection of members of the class is a member of the class.

The Dedekind plane determined by an algebraic plane is the set of elements of the Dedekind skew–plane which commute with elements of the algebraic plane. The Dedekind plane is a field which contains ξ^- whenever it contains ξ. The Dedekind line is the subset of the Dedekind plane whose elements are the self–conjugate elements of the Dedekind skew–plane. The elements of the Dedekind line are real numbers.

The Dedekind topology of the Dedekind plane is the subspace topology of the Dedekind topology of the Dedekind skew–plane. The Dedekind topology of the Dedekind line is the subspace topology of the Dedekind topology of the Dedekind plane. Since an element of the Dedekind plane is a unique sum

$$x + \iota y$$

for real numbers x and y, the Dedekind plane is isomorphic in additive structure to the Cartesian product of two Dedekind lines. The Dedekind topology of the Dedekind plane is the Cartesian product topology of the Dedekind topologies of two Dedekind lines.

The complementary space to the Dedekind plane in the Dedekind skew–plane is the set of skew–conjugate elements ξ of the Dedekind skew–plane which satisfy the identity

$$\xi \eta = \eta^- \xi$$

for every element η of the Dedekind plane. An element of the Dedekind skew–plane is the unique sum of an element of the Dedekind plane and an element of the complementary space to the Dedekind plane in the Dedekind skew–plane. The Dedekind topology of the complementary space to the Dedekind plane in the Dedekind skew–plane is the subspace topology of the Dedekind topology of the Dedekind skew–plane.

Since an element of the complementary space to the Dedekind plane in the Dedekind skew–plane is a unique sum

$$ix + jy + kz$$

for real numbers x, y, z such that

$$(\iota^- i - i\iota)x + (\iota^- j - j\iota)y + (\iota^- k - k\iota) = 0,$$

the complementary space to the Dedekind plane in the Dedekind skew–plane is isomorphic in additive structure to the Cartesian product of two Dedekind lines. The Dedekind topology of the complementary space to the Dedekind plane in the Dedekind skew–plane is the Cartesian product topology of the Dedekind topologies of two Dedekind lines.

Open subsets and closed subsets of the Dedekind plane and of the Dedekind line are Baire subsets of the Dedekind skew–plane. Open subsets and closed subsets of the complementary space to the Dedekind plane in the Dedekind skew–plane are Baire subsets of the Dedekind skew–plane.

Lebesgue measure is chosen as the canonical measure for the Dedekind line. Convexity is an underlying concept in the construction of the measure. An open subset of the line is a countable union of disjoint convex open subsets. The measure of an open set is defined as the sum of the measures of convex open subsets with $b - a$ as the measure of an interval (a, b) and with infinite measure for other convex open sets. The measure is defined on Baire sets in the unique way such that the measure of a countable union of disjoint sets is the sum of the measures of the sets. The transformation of the line into itself which takes ξ into $\xi + \eta$ is measure preserving for every element η of the line. A nonnegative measure on the Baire subsets of the line is a constant multiple of Lebesgue measure if the transformation which takes ξ into $\xi + \eta$ is measure preserving for every element η of the line. Multiplication by an element γ of the line multiplies Lebesgue measure by a factor of the absolute value $|\gamma|$ of γ.

The Dedekind skew–plane is a Cartesian product of four Dedekind lines since an element

$$\xi = t + ix + jy + kz$$

of the Dedekind skew–plane has four real coordinates x, y, z, and t. The canonical measure for the Dedekind skew–plane is a nonnegative measure on Baire subsets which is a constant multiple of the Cartesian product measure of the canonical measures of the Dedekind lines. The transformation of the Dedekind skew–plane into itself which takes ξ into $\xi + \eta$ is measure preserving for every element η of the Dedekind skew–plane. A nonnegative measure on Baire subsets is a constant multiple of the canonical measure if the transformation which takes ξ into $\xi + \eta$ is measuring preserving for every element η. Multiplication by an element γ of the Dedekind skew–plane multiplies the canonical measure by

$$(\gamma^- \gamma)^2 = |\gamma|^4.$$

The definition of the canonical measure for the Dedekind skew–plane is made by analogy with the definition of the canonical measure for the Dedekind line. Elements of the Dedekind skew–plane are congruent modulo integral elements of the algebraic skew–plane when their difference is integral. A fundamental domain is the set of elements

$$\xi + t + ix + jy + kz$$

which satisfy the inequalities

$$-1 < 2t < 1$$

and

$$-1 < 2x < 1$$

and

$$-1 < 2y < 1$$

and

$$-1 < 2z < 1$$

and

$$-1 < t \pm x \pm y + \pm z < 1.$$

Every element of the Dedekind skew–plane is congruent to an element of the closure of the fundamental domain. Congruent elements of the fundamental domain are equal. The canonical measure for the Dedekind skew–plane is defined so that the fundamental domain and its closure have measure one. The canonical measure for the Dedekind skew–plane is twice the Cartesian product measure for the canonical measures for coordinate lines.

Since the Dedekind plane is a Cartesian product of two Dedekind lines, the canonical measure for the Dedekind plane is defined as a constant multiple of the Cartesian product measure of the canonical measures of two Dedekind lines. When an element of the Dedekind plane is written as a unique sum

$$x + \iota y$$

for real numbers x and y, the canonical measure for the Dedekind plane is obtained from the Cartesian product measure of the canonical measures of the Dedekind lines on multiplying by

$$\tfrac{1}{2}|\iota - \iota^-|.$$

The canonical measure is a nonnegative measure on the Baire subsets of the plane such that the transformation ξ into $\xi + \eta$ is measure preserving for every element η of the plane. A nonnegative measure on the Baire subsets of the plane is a constant multiple of the canonical measure if the transformation α into $\xi + \eta$ is measure preserving for every element η of the plane. Multiplication by an element γ of the plane multiplies the canonical measure by a factor of

$$\gamma^-\gamma = |\gamma|^2.$$

The canonical measure for the complementary space to the Dedekind plane in the Dedekind skew–plane is defined as a constant multiple of the Cartesian product measure of the canonical measure for two Dedekind lines. At least one of the coefficients

$$\iota^- i - i\iota, \ \iota^- j - j\iota, \ \iota^- k - k\iota$$

is nonzero. If for example

$$\iota^- k - k\iota$$

is nonzero, real numbers α and β are defined by the equations

$$\alpha(\iota^- k - k\iota) = \iota^- i - i\iota$$

and

$$\beta(\iota^- k - k\iota) = \iota^- j - j\iota.$$

When elements

$$(i - \alpha k)x + (j - \beta k)y$$

of the complementary space to the Dedekind plane in the Dedekind skew–plane are parametrized by real numbers x and y, the canonical measure for the complementary space to the Dedekind plane in the Dedekind skew–plane is obtained from the Cartesian product measure of the canonical measures for the Dedekind lines on multiplying by

$$2|i\alpha + j\beta + k|.$$

The canonical measure is a nonnegative measure on the Baire subsets of the complementary space such that the transformation ξ into $\xi + \eta$ is measure preserving for every element η of the complementary space. A nonnegative measure on the Baire subsets of the complementary space is a constant multiple of the canonical measure if the transformation ξ into $\xi + \eta$ is measure preserving for every element η of the complementary space. Multiplication by an element γ of the Dedekind plane is a transformation of the complementary space into itself which multiplies the canonical measure by a factor of

$$\gamma^-\gamma = |\gamma|^2.$$

The Dedekind skew–plane is the Cartesian product of the Dedekind plane and the complementary space to the Dedekind plane in the Dedekind skew–plane. The Dedekind topology of the skew–plane is the Cartesian product topology of the Dedekind topology of the plane and the Dedekind topology of the complementary space. The canonical measure for the skew–plane is the Cartesian product measure of the canonical measure for the plane and the canonical measure for the complementary space.

3. Adic skew–planes

The familiar Dedekind topology prepares topologies which are as well related to addition and multiplication as the Dedekind topology.

An element ξ of the algebraic skew–plane is said to be integral modulo r for a positive integer r if $n\xi$ is integral for a positive integer n which is relatively prime to r. The elements of the algebraic skew–plane which are integral modulo r form a subring which contains ξ^- whenever it contains ξ. An element of the subring is said to be divisible by r if it is the product of r and an element of the subring. The elements of the subring which are divisible by r form an ideal which contains ξ^- whenever it contains ξ. Elements of the quotient ring are represented by integral elements of the algebraic skew–plane. The quotient ring of the ring of integral elements modulo r is isomorphic to the quotient ring of the ring of integral elements.

The quotient ring modulo r is a finite ring which inherits a conjugation. The discrete topology is the unique topology for which a finite set is a Hausdorff space. Addition and multiplication are continuous as transformations of the Cartesian product of the quotient ring with itself into the quotient ring.

An adic topology is defined on the algebraic skew–plane by a set of generating primes which contains two and the prime divisors of $\iota^-\iota$ when an algebraic plane is defined by ι.

The adic topology is initially defined on the set of elements of the algebraic skew–plane which are integral modulo r for every generating positive integer r. The subset which are open and closed determine the topology since every open set is a union of open and closed sets and every closed set is an intersection of open and closed sets. A set is defined as open and closed if for some generating positive integer r the set is an inverse image of some subset of the quotient space modulo r.

The adic topology of the algebraic skew–plane is derived from the adic topology of the set of elements which are integral modulo r for every positive integer r. It is sufficient to define the sets which are open and closed since every open set is a union of open and closed sets and every closed set is an intersection of open and closed sets. A subset of the algebraic skew–plane is defined to be open and closed if for every positive integer r the set of products $r\xi$ with ξ in the set which are integral modulo n for every generating positive integer n is open and closed.

The algebraic skew–plane is a Hausdorff space in the adic topology. Addition is continuous as a transformation of the Cartesian product of the algebraic skew–plane with itself into the algebraic skew–plane. Multiplication by an element of the algebraic skew–plane is continuous as a transformation of the algebraic skew–plane into itself. Conjugation is continuous as a transformation of the algebraic skew–plane into itself.

The adic skew–plane is the Cauchy completion of the algebraic skew–plane in the uniform adic topology. Addition extends continuously as a transformation of the Cartesian product of the adic skew–plane with itself into the adic skew–plane. Multiplication by an element of the algebraic skew–plane extends continuously as a transformation of the adic skew–plane into itself. Conjugation extends continuously as a transformation of the adic skew–plane into itself. The adic skew–plane is a ring on which conjugation acts as an anti–automorphism. The adic line is defined as the ring of self–conjugate elements of the adic skew–plane. The adic skew–plane is isomorphic to the ring of quaternions with coordinates in the adic line.

The Baire subsets of the adic skew–plane are the smallest class of sets containing the open and closed sets such that a countable union of members of the class is a member of the class and a countable intersection of members of the class is a member of the class.

Integral elements of the adic skew–plane are defined as elements of the closure of the elements of the algebraic skew–plane which are integral modulo r for every generating positive integer r. The set of integral elements of the adic skew–plane is a compact subring which contains ξ^- whenever it contains ξ. Addition and multiplication are continuous as transformations of the Cartesian product of the ring with itself into the ring. The set of integral elements of the adic skew–plane which are divisible by a generating positive integer r is a closed ideal which contains ξ^- whenever it contains ξ. The quotient ring of the ring of integral elements of the adic skew–plane modulo the ideal is isomorphic to the quotient ring of the ring of integral elements of the algebraic skew–plane modulo the ideal of elements divisible by r.

An integral element of the adic skew–plane is said to be p–adic for a generating prime p if it is divisible by every generating positive integer which is not divisible by p. The p–adic elements of the ring of integral elements of the adic skew–plane form a compact ideal which contains ξ^- whenever it contains ξ. The ring of integral elements of the adic skew–plane is isomorphic to the Cartesian product of its p–adic ideals taken over all generating primes p. The topology of the ring of integral elements of the adic skew–plane is the Cartesian product topology of the p–adic ideals.

An element of the adic skew–plane is said to be p–adic for a generating prime p if its product with some generating positive integer is a p–adic integral element of the adic skew–plane. The p–adic elements of the adic skew–plane form a closed ideal which contains ξ^- whenever it contains ξ. A p–adic line is the field of self–conjugate elements of a p–adic ideal. The elements of a p–adic line are identified with p–adic numbers. The ring of p–adic elements is isomorphic to the ring of quaternions whose coordinates are p–adic numbers. The adic skew–plane is isomorphic to a subring of the Cartesian product of its p–adic ideals taken over all generating primes p. An element of the Cartesian product represents an element of the adic skew–plane if, and only if, its p–adic component is integral for all but a finite number of generating primes p. The topology of the adic skew–plane is the subspace topology of the Cartesian product topology of the p–adic ideals.

An adic plane is a commutative subring of the adic skew–plane which contains an element ξ if, and only if, it contains the p–adic component of ξ for every generating prime p. When p is a generating prime, the adic plane is assumed to contain some p–adic element of the adic skew–plane which is not self–conjugate. The p–adic component of ι is assumed to belong to the adic plane when p is a divisor of $\iota^-\iota$. When p is a generating prime, a p–adic element of the adic skew–plane is assumed to belong to the adic plane if, and only if, it commutes with some p–adic element of the adic plane which is not self–conjugate. The field of p–adic elements of the adic plane is a quadratic extension of the field of p–adic numbers. When p is an odd prime, an integral element ω of the field exists which represents

$$p = \omega^- \omega.$$

The adic plane is said to be rectangular if an integral element ω of the field exists which represents

$$p = \omega^- \omega$$

when p is the even prime. The adic plane is said to be hexagonal when it is not rectangular.

The complementary space to the adic plane in the adic skew–plane is the set of skew–conjugate elements ξ of the adic skew–plane which satisfy the identity

$$\xi \eta = \eta^- \xi$$

for every element η of the adic plane. An element of the adic skew–plane is the unique sum of an element of the adic plane and an element of the complementary space to the adic plane in the adic skew–plane.

The canonical measure for the adic skew–plane is determined on the ring of integral elements by quotient rings modulo r for generating positive integers r. The canonical measure for the quotient ring modulo r assigns to a set the number of its elements divided by the number of elements in the quotient ring. The inverse image of a subset of the quotient ring is an open and closed set of integral elements which is assigned the same measure. The measure is defined on Baire sets of integral elements so that the measure of a countable union of disjoint sets is the sum of the measures of the sets. The measure of a Baire subset C of the adic skew–plane is defined as the least upper bound of products of r^4 with the measure of the set of integral products $r\gamma$ with γ in C taken over all generating positive integers r.

The adic plane is a Hausdorff space in the subspace topology of the adic skew–plane. The canonical measure for the adic plane is determined on the ring of integral elements by quotient rings modulo r for generating positive integers r. The canonical measure for the quotient ring modulo r assigns to a set the number of its elements divided by the number of elements in the quotient ring modulo r. The inverse image of a subset of the quotient ring modulo r is an open and closed set of integral elements of the adic plane which is given the same measure. The measure is defined on Baire sets of integral elements so that the measure of a countable union of disjoint sets is the sum of the measures of the sets. The measure of a Baire subset C of the adic plane is defined as the least upper bound of products of r^2 with the measure of the set of integral products $r\gamma$ with γ in C taken over the generating positive integers r.

The complementary space to the adic plane in the adic skew–plane is a Hausdorff space in the subspace topology of the adic skew–plane. A subset of the complementary space is open if, and only if, it is a union of sets which are open and closed. A subset of the complementary space is closed if, and only if, it is an intersection of spaces which are open and closed. The complementary space is the image of the adic skew–plane under a homomorphism of additive structure whose kernel is the adic plane.

A subset A of the complementary space is open and closed if, and only if, an open and closed subset A' of the adic skew–plane is defined as the set of elements $a + b$ with a in A and b an integral element of the adic plane. The canonical measure of an open and closed subset A of the complementary space is defined as the canonical measure of the open and closed subset A' of the adic skew–plane. The canonical measure is defined on Baire subsets of the complementary space so that the measure of a countable union of disjoint sets is the sum of the measures of the sets. The canonical measure for the adic skew–plane is the Cartesian product measure of the canonical measure for the adic plane and the canonical measure for the complementary space to the adic plane in the adic skew–plane.

The adic line is a Hausdorff space in the subspace topology of the adic skew–plane. The canonical measure for the adic line is determined on the ring of integral elements by quotient rings modulo r for generating positive integers r. The canonical measure for the quotient ring modulo r assigns to a subset the number of its elements divided by the number of elements in the quotient ring modulo r. The inverse image of a subset of the quotient ring modulo r is an open and closed set of integral elements of the adic line which is given the same measure. The measure is defined on Baire sets of integral elements of the adic line so that the measure of a countable union of disjoint sets is the sum of the measures of the sets. The measure of a Baire subset C of the adic line is defined as the least upper bound of products of r with the measure of the set of integral products $r\gamma$ with γ in C taken over the generating positive integers r.

The modulus of an invertible element λ of the adic line is defined as the positive rational number $|\lambda|$ which is a ratio of generating positive integers such that

$$|\lambda|^{-1}\lambda$$

is an integral element of the adic line with integral inverse. The modulus of an invertible element λ of the adic skew–plane is defined as the positive solution $|\lambda|$ of the equation

$$|\lambda|^2 = |\lambda^- \lambda|.$$

The modulus of a noninvertible element of the adic skew–plane is infinite.

Multiplication by an invertible element λ of the adic skew–plane multiplies the canonical measure by a factor of $|\lambda|^{-4}$. Multiplication by an invertible element λ of the adic plane multiplies the canonical measure by a factor of $|\lambda|^{-2}$. Multiplication by an invertible element λ of the adic line multiplies the canonical measure by a factor of $|\lambda|^{-1}$. Multiplication by noninvertible elements annihilates canonical measures.

A character χ modulo ρ is a function $\chi(\xi)$ of ξ in the algebraic line which admits a unique continuous extension as a function $\chi(\xi)$ of ξ in the adic line. The function of ξ in the adic line vanishes when ξ is nonintegral, has equal values at integral elements which are congruent modulo ρ, satisfies the identity

$$\chi(\xi\eta) = \chi(\xi)\chi(\eta)$$

for all integral elements ξ and η, and has a nonzero value at an integral element if, and only if, the element is invertible modulo ρ.

4. Product skew–planes

A product skew–plane is the Cartesian product of the Dedekind skew–plane and an adic skew–plane. The product skew–plane is the set of pairs $\xi = (\xi_+, \xi_-)$ with ξ_+ in the Dedekind skew-plane and ξ_- in the adic skew-plane. The product skew–plane is a Hausdorff space in the Cartesian product topology of the Dedekind topology and the adic topology. Addition is defined on the product skew-plane so that the coordinate projections are homomorphisms of additive structure. If ω is an element of the algebraic skew–plane, continuous transformations of the product skew-plane into itself are defined by taking $\xi = (\xi_+, \xi_-)$ into

$$\omega\xi = (\omega\xi_+, \omega\xi_-)$$

and

$$\xi\omega = (\xi_+\omega, \xi_-\omega).$$

A product plane is constructed from a product skew–plane when a Dedekind plane and an adic plane are constructed from an algebraic plane. The product plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew-plane whose Dedekind component ξ_+ belongs to the Dedekind plane and whose adic component ξ_- belongs to the adic plane. The topology of the product plane is defined as the Cartesian product topology of the Dedekind topology and the adic topology. The topology of the product plane coincides with the subspace topology of the topology of the product skew-plane.

The Cartesian product of the Dedekind plane and the complementary space to the adic plane in the adic skew–plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane with Dedekind component ξ_+ in the Dedekind plane and with adic component ξ_- in the complementary space to the adic plane in the adic skew–plane. The Cartesian product topology coincides with the subspace topology from the product skew–plane.

The Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the adic plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane with Dedekind component ξ_+ in the complementary space to the Dedekind plane in the Dedekind skew–plane and with adic component ξ_- in the adic plane. The Cartesian product topology coincides with the subspace topology from the product skew–plane.

The Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the complementary space to the adic plane in the adic skew–plane is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane with Dedekind component ξ_+ in the complementary space to the Dedekind plane in the Dedekind skew–plane and with adic component ξ_- in the complementary space to the adic plane in the adic skew–plane. The Cartesian product topology coincides with the subspace topology from the product skew–plane.

The complementary space to the product plane in the product skew–plane is a subspace of the product skew–plane which is a Cartesian product of three subspaces of the product skew–plane: 1) the Cartesian product of the Dedekind plane and the complementary space to the adic plane in the adic skew–plane, 2) the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the adic skew–plane, and 3) the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the complementary space to the adic plane in the adic skew–plane. The Cartesian product topology coincides with the subspace topology from the product skew–plane.

An element of the product skew–plane is the unique sum of an element of the product plane and an element of the complementary space to the product in the product skew–plane. The topology of the product skew–plane coincides with the Cartesian product topology of the topology of the product plane and the topology of the complementary space to the product plane in the product skew–plane.

The canonical measure for the product skew–plane is defined as the Cartesian product measure of the canonical measure for the Dedekind skew–plane and the canonical measure for the adic skew–plane.

The canonical measure for the product plane is defined as the Cartesian product measure of the canonical measure for the Dedekind plane and the canonical measure for the adic plane.

The canonical measure for the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the adic plane is defined as the Cartesian product measure of the canonical measure for the complementary space to the Dedekind plane in the Dedekind skew–plane and the canonical measure for the adic plane.

The canonical measure for the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the complementary space to the adic plane in the adic skew–plane is defined as the Cartesian product measure of the canonical measure for the complementary space to the Dedekind plane in the Dedekind skew–plane and the canonical measure for the complementary space to the adic plane in the adic skew–plane.

The canonical measure for the complementary space to the product plane in the product skew–plane is defined as the Cartesian product measure of three measures: 1) the canonical measure for the Cartesian product of the Dedekind plane and the complementary space to the adic plane in the adic skew–plane, 2) the canonical measure for the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the adic plane, and 3) the canonical measure for the Cartesian product of the complementary space to the Dedekind plane in the Dedekind skew–plane and the complementary space to the adic plane in the adic skew–plane.

The canonical measure for the product skew–plane is the Cartesian product measure of the canonical measure for the product plane and the canonical measure for the complementary space to the product plane in the product skew–plane.

A quotient skew–plane of the product skew–plane is defined by an equivalence relation when all primes are generating primes. A closed subspace of the product skew–plane consists of elements $\xi = (\xi_+, \xi_-)$ whose Dedekind and adic components are elements of the algebraic skew–plane with zero sum. If ω is an element of the algebraic skew–plane, the elements $\omega\xi$ and $\xi\omega$ of the product skew–plane belong to the subspace whenever ξ belongs to the subspace. Elements of the product skew–plane are defined to be equivalent if their difference belongs to the subspace.

The quotient skew–plane is a Hausdorff space in the topology whose open sets are the sets whose inverse image in the product skew–plane is open and whose closed sets are the sets whose inverse image in the product skew–plane is closed. Addition is defined on the quotient skew–plane so that the projection onto the quotient space is a homomorphism of additive structure. Addition is continuous as a transformation of the Cartesian product of the quotient skew–plane with itself into the quotient skew–plane. If ω is an element of the algebraic skew–plane, continuous transformations of the quotient skew–plane into itself are defined by taking ξ into $\omega\xi$ and into $\xi\omega$.

A fundamental domain is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane whose Dedekind component ξ_+ satisfies the inequality

$$\xi_+^- \xi_+ < (\xi_+ - \lambda)^- (\xi_+ - \lambda)$$

for every nonzero integral element λ of the algebraic skew–plane and whose adic component ξ_- is an integral element of the adic skew–plane. The closure of the fundamental domain is the set of elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane whose Dedekind component ξ_+ satisfies the inequality

$$\xi_+^- \xi_+ \leq (\xi_+ - \lambda)^- (\xi_+ - \lambda)$$

for every nonzero integral element λ of the algebraic skew–plane and whose adic component ξ_- is an integral element of the adic skew–plane. Every element of the product skew–plane is equivalent to an element of the closure of the fundamental domain. Equivalent elements of the fundamental domain are equal.

The closure of the fundamental domain is the Cartesian product of a compact subset of the Dedekind skew–plane of measure one and a compact subset of the adic skew–plane of measure one. In the Dedekind skew–plane the compact subset is the closure of the fundamental domain with respect to congruence modulo integral elements. The compact subset of the adic skew–plane is the set of integral elements.

The quotient skew–plane is a compact Hausdorff space. A subset of the quotient skew–plane is a Baire set if, and only if, its inverse image in the product skew–plane is a Baire set. The canonical measure of the quotient skew–plane is defined as the image of the canonical measure of the product skew–plane as it acts on Baire subsets of the closure of the fundamental domain.

5. Fourier analysis for the Dedekind skew–plane

The function

$$\exp(2\pi i\xi\eta)$$

of ξ in the Dedekind line defines a homomorphism of additive structure for the Dedekind line into multiplicative structure for the complex numbers of absolute value one for every element η of the Dedekind line. The homomorphism has value one on every integer ξ if, and only if, η is an integer. A continuous homomorphism of additive structure for the Dedekind line into multiplicative structure for the complex numbers of absolute value one is defined by a unique element η of the adic line.

The Fourier transform of a function $f(\xi)$ of ξ in the Dedekind line, which is integrable with respect to Lebesgue measure, is a bounded continuous function

$$g(\xi) = \int f(\eta) \exp(2\pi i \xi \eta) \, d\eta$$

of ξ in the Dedekind line which is defined by integration with respect to Lebesgue measure. Fourier inversion

$$f(\xi) = \int g(\eta) \exp(-2\pi i \xi \eta) \, d\eta$$

applies for almost all elements ξ of the Dedekind line when the function $g(\xi)$ of ξ in the Dedekind line is integrable with respect to Lebesgue measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to Lebesgue measure into itself.

The function

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of ξ in the Dedekind plane is a homomorphism of additive structure for the Dedekind plane into multiplicative structure for the complex numbers of absolute value one for every element η of the Dedekind plane. A continuous homomorphism of additive structure for the Dedekind plane into multiplicative structure for the complex numbers of absolute value one is determined by a unique element η of the Dedekind plane.

The Fourier transform of a function $f(\xi)$ of ξ in the Dedekind plane, which is integrable with respect to the canonical measure, is a bounded continuous function

$$g(\xi) = \int f(\eta) \exp(\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

of ξ in the Dedekind plane which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int g(\eta) \exp(-\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

applies for almost all elements ξ of the Dedekind plane when the function $g(\xi)$ of ξ in the Dedekind plane is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

The function

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of ξ in the Dedekind skew–plane defines a homomorphism of additive structure for the Dedekind skew–plane into multiplicative structure for the complex number of absolute value one for every element η of the Dedekind skew–plane. A continuous homomorphism of additive structure for the Dedekind skew–plane into multiplicative structure for the complex numbers of absolute value one is defined by a unique element η of the Dedekind skew–plane.

The Fourier transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane, which is integrable with respect to the canonical measure, is a bounded continuous function.

$$g(\xi) = \tfrac{1}{2} \int f(\eta) \exp(\pi i(\xi^- \eta + \eta^- \xi)) \, d\eta$$

of ξ in the Dedekind skew–plane which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \tfrac{1}{2} \int g(\eta) \exp(-\pi i(\xi^- \eta + \eta^- \xi)) \, d\eta$$

applies for almost all element ξ of the Dedekind skew–plane when the function $g(\xi)$ of ξ in the Dedekind skew–plane is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

The Radon transformation for the Dedekind skew–plane factors the Fourier transformation for the Dedekind skew–plane as a composition with the Fourier transformation for the Dedekind plane. The domain of the Radon transformation is the space of functions which are integrable with respect to the canonical measure. The Radon transformation takes a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$g(\omega\xi) = \tfrac{1}{2} \int f(\omega\xi + \omega\eta) d\eta$$

holds for almost all elements ξ of the Dedekind plane for every element ω of the Dedekind skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the Dedekind plane in the Dedekind skew–plane. The inequality

$$\int |g(\omega\xi)| d\xi \leq \tfrac{1}{2} \int |f(\xi)| d\xi$$

holds for every element ω of the Dedekind skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the Dedekind plane and with integration on the right with respect to the canonical measure for the Dedekind skew–plane.

The Radon transformation for the Dedekind skew–plane is inverted by the Fourier transformation for the Dedekind skew–plane. If the Radon transform of an integrable function $f(\xi)$ of ξ in the Dedekind skew–plane is equal almost everywhere to the inverse Fourier transform of an integrable function $g(\xi)$ of ξ in the Dedekind skew–plane, then the Radon transform of the function $g(\xi)$ of ξ in the Dedekind skew–plane is equal almost everywhere to the Fourier transform of the function $f(\xi)$ of ξ in the Dedekind skew–plane.

Symplectic transformations for the Dedekind skew–plane are isometric transformations of the Hilbert space of square integrable functions with respect to the canonical measure into itself which are related to the Radon transformation in the same way as the Fourier transformation.

The Heisenberg group for the Dedekind skew–plane is a twisted Cartesian product of the Dedekind skew–plane with itself. The elements of the group act as isometric transformations of the Hilbert space of square integrable functions with respect to the canonical measure into itself. If α and β are elements of the Dedekind skew–plane, the transformation $S(\alpha, \beta)$ is defined to take a function $f(\xi)$ of ξ in the Dedekind skew–plane into the function

$$\exp(\pi i(\beta^- \xi + \xi^- \beta)) f(\xi + \alpha)$$

of ξ in the Dedekind skew–plane. The identity

$$S(\alpha, \beta) S(\gamma, \delta) = \exp(\pi i(\delta^- \alpha + \alpha^- \delta)) S(\alpha + \gamma, \beta + \delta)$$

holds for all elements α, β, γ, and δ of the Dedekind skew–plane.

A symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the Dedekind skew–plane is a matrix with entries in the Dedekind skew–plane which has the matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}$$

as inverse. A nonzero element ω of the Dedekind skew–plane exists such that the matrix

$$\begin{pmatrix} \omega^- A & \omega^- B \\ \omega^- C & \omega^- D \end{pmatrix}$$

has self–conjugate numbers as entries.

A symplectic transformation associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an isometric transformation T of the Hilbert space of square integrable functions with respect to the canonical measure into itself such that the transformations

$$S(\alpha, \beta) T$$

and

$$T S(\gamma, \delta)$$

are linearly dependent whenever α and β are elements of the Dedekind skew–plane and

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The Fourier transformation is a symplectic transformation associated with the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If h is a real number, a symplectic transformation associated with the matrix

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

is defined by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into the function

$$\exp(\pi i h \xi^- \xi) f(\xi)$$

of ξ in the Dedekind skew–plane. A symplectic transformation associated with the matrix

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega^- \end{pmatrix}$$

is defined by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into the function

$$|\omega|^2 f(\xi\omega)$$

of ξ in the Dedekind skew–plane for every nonzero element ω of the Dedekind skew–plane.

If a symplectic transformation T_1 is associated with the matrix

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

and if a symplectic transformation T_2 is associated with the matrix

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

then a symplectic transformation

$$T = T_1 T_2$$

is associated with the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

An element ω of the Dedekind skew–plane with conjugate as inverse acts on the Hilbert space of square integrable functions with respect to the canonical measure. An isometric transformation of the space into itself is defined by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into the function $f(\omega\xi)$ of ξ in the Dedekind skew–plane. The Hilbert space decomposes into invariant subspaces under the group action. A computation of invariant subspaces is made in spaces of homogeneous polynomials.

A homogeneous polynomial of degree ν is a function $f(\xi)$ of

$$\xi = t + ix + jy + kz$$

of ξ in the Dedekind skew–plane which is a linear combination of monomials

$$x^a y^b z^c t^d$$

with exponents nonnegative integers whose sum is ν. The functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Dedekind skew–plane are homogeneous polynomials of degree ν for every element ω of the Dedekind skew–plane with conjugate as inverse if the function $f(\xi)$ of ξ in the Dedekind skew–plane is a homogeneous polynomial of degree ν. The dimension of the space of homogeneous polynomials of degree ν is

$$(1+\nu)(2+\nu)(3+\nu)/6.$$

The space of homogeneous polynomials of degree ν is a Hilbert space with scalar product determined so that the monomials form an orthogonal set with

$$\frac{a!\ b!\ c!\ d!}{\nu!}$$

as the scalar self–product of the monomial

$$x^a y^b z^c t^d$$

with nonnegative integers as exponents whose sum is ν. Isometric transformations of the space into itself are defined by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Dedekind skew–plane for every element ω of the Dedekind skew–plane with conjugate as inverse.

Scalar products of homogeneous polynomials of degree ν are computed by integration with respect to the canonical measure for the compact group of elements of the Dedekind skew–plane with conjugate as inverse. The canonical measure is a nonnegative measure on Baire subsets which is determined within a constant factor by invariance under the group action: A measure preserving transformation of the group into itself is defined by taking ξ into $\omega\xi$ for every element ω of the group. The measure is defined so that the group has measure

$$8\pi.$$

The compact group is a normal subgroup of the multiplicative group of nonzero elements of the Dedekind skew–plane. The compact group is the kernel of the homomorphism of the group of nonzero elements of the Dedekind skew–plane onto the positive half–line which takes ξ into $\xi^-\xi$. The homomorphism takes the canonical measure for the Dedekind skew–plane into the nonnegative measure on the Baire subsets of the positive half–line whose value on a set is an integral

$$8\pi \int t dt$$

over the set with respect to Lebesgue measure.

The scalar product of functions $f(\xi)$ and $g(\xi)$ of ξ in the Dedekind skew–plane which are homogeneous polynomials of degree ν is computed as an integral

$$8\pi\langle f, g\rangle = \int g(\xi)^- f(\xi) d\xi$$

with respect to the canonical measure for the compact group of elements of the Dedekind skew–plane with conjugate as inverse.

The Laplacian

$$\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is applied in the decomposition of the Hilbert space into invariant subspaces. The Laplacian annihilates homogeneous polynomials of degree less than two and takes homogeneous polynomials of greater degree ν into homogeneous polynomials of degree $\nu-2$. The Laplacian commutes with the transformations which take a function $f(\xi)$ of ξ in the Dedekind skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Dedekind skew–plane for every element ω of the Dedekind skew–plane with conjugate as inverse. A homogeneous polynomial of degree ν is said to be harmonic if it is annihilated by the Laplacian. A homogeneous polynomial of degree less than two is harmonic. A homogeneous polynomial of degree ν greater than one is harmonic if, and only if, it is orthogonal to products of $\xi^-\xi$ with homogeneous polynomials of degree $\nu-2$. The dimension of the space of homogeneous harmonic polynomials of degree ν is

$$(1+\nu)^2.$$

A function $f(\xi)$ of ξ in the Dedekind skew–plane is said to be of spin μ for an integer μ if the identity

$$f(\xi\omega) = \omega^\mu f(\xi)$$

holds for every element ω of the Dedekind plane with conjugate as inverse. If a function $f(\xi)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν, then the function $f(\xi\omega)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν for every element ω of the Dedekind plane with conjugate as inverse. The Hilbert space of homogeneous harmonic polynomials of degree ν is the orthogonal sum of subspaces whose elements are of spin μ for integers μ of the same parity as ν which satisfy the inequalities

$$-\nu \leq \mu \leq \nu.$$

The space of homogeneous harmonic polynomials of order ν and spin μ admits an orthogonal basis indexed by the integers n of the same parity as ν which satisfy the inequalities

$$-\nu \leq n \leq \nu.$$

A basic element is a function $f(\xi)$ of ξ in the Dedekind skew–plane which satisfies the identity

$$f(\omega\xi) = \omega^n f(\xi)$$

for every element ω of the Dedekind plane with conjugate as inverse. The dimension of the space of homogeneous harmonic polynomials of degree ν and spin μ is

$$1 + \nu.$$

If a function $f(\xi)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ, then the function $f(\omega\xi)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν and spin μ for every element ω of the Dedekind skew–plane with conjugate as inverse. Every homogeneous harmonic polynomial of degree ν and spin μ is a finite linear combination of functions $f(\omega\xi)$ of ξ in the Dedekind skew–plane for elements ω of the Dedekind skew–plane with conjugate as inverse if the function $f(\xi)$ of ξ in the Dedekind skew–plane does not vanish identically.

If a function $\phi(\xi)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν, the function

$$\phi(\xi) \exp(\pi i z \xi^- \xi)$$

of ξ in the Dedekind skew–plane is an eigenfunction of the Radon transformation for the eigenvalue

$$i/z$$

when z is in the upper half–plane. The Laplace transformation of harmonic ϕ permits a spectral analysis of the Radon transformation in a space determined by the harmonic polynomial. The Laplace transformation is defined when the harmonic polynomial has norm one in the Hilbert space of homogeneous polynomials of degree ν.

The domain of the Laplace transformation is the Hilbert space of functions $f(\xi)$ of ξ in the Dedekind skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the Dedekind skew–plane with conjugate as inverse. The Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is an analytic function

$$F(z) = \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi)\, d\xi$$

of z in the upper half–plane defined by integration with respect to the canonical measure.

When

$$f(\xi) = \phi(\xi)h(\xi^-\xi),$$

the Laplace integral reads

$$(1+2\nu)F(x+iy) = 8\pi \int_0^\infty (\xi^-\xi)^{1+\nu} h(\xi^-\xi) \exp(-\pi y \xi^-\xi) \exp(\pi i x \xi^-\xi) d(\xi^-\xi)$$

with x real and y positive. The identity

$$\tfrac{1}{2}(1+2\nu)^2 \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx = 64\pi^2 \int_0^\infty (\xi^-\xi)^{2+2\nu} |h(\xi^-\xi)|^2 \exp(-2\pi y \xi^-\xi) d(\xi^-\xi)$$

is an application of the isometric property of the Fourier transformation for the Dedekind line. Since the identity reads

$$\tfrac{1}{2}(1+2\nu) \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx = 8\pi \int (\xi^-\xi)^{1+\nu} |f(\xi)|^2 \exp(-2\pi y \xi^-\xi) d\xi$$

with integration on the right with respect to the canonical measure, the identity

$$(1+2\nu) \int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy)|^2 y^\nu dx dy = 8(2\pi)^{-\nu} \Gamma(1+\nu) \int |f(\xi)|^2 d\xi$$

holds with integration on the right with respect to the canonical measure. An analytic function $F(z)$ of z in the upper half–plane is a Laplace transform if the integral on the left converges.

The Laplace transformation of harmonic ϕ computes the adjoint of the Radon transformation in the Hilbert space of functions $f(\xi)$ of ξ in the Dedekind skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the Dedekind skew–plane with conjugate as inverse. The closure of the Radon transformation is a maximal dissipative transformation in the Hilbert space whose adjoint takes a function $f(\xi)$ of ξ in the Dedekind skew–plane into the function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^-\xi)\, d\xi = (i/z) \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^-\xi)\, d\xi$$

holds for z in the upper half–plane with integration with respect to the canonical measure.

A relation T with domain and range in a Hilbert space is said to be maximal dissipative if

$$(T - \lambda^-)/(T + \lambda)$$

is an everywhere defined and contractive transformation in the space for some, and hence every, complex number λ in the right half–plane. The relation is said to be dissipative if a partially defined contractive transformation is obtained for some, and hence every, complex number w in the right half–plane. A dissipative transformation admits a maximal dissipative extension which need not be a transformation.

A maximal dissipative transformation $-iH$, which is the inverse of the adjoint of the Radon transformation, is defined in the Hilbert space of square integrable functions with respect to the canonical measure by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi = -iz \int \phi(\xi)^- f(\xi) \exp(\pi i z \xi^- \xi) d\xi$$

holds with integration with respect to the canonical measure for the Dedekind skew–plane for all z in the upper half–plane whenever ϕ is a homogeneous harmonic polynomial of degree ν for some nonnegative integer ν.

A symplectic transformation associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the Dedekind skew–plane which has self–conjugate entries is defined by taking a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$\int \phi(\xi)^- g(\xi) \exp(\pi i z \xi^- \xi) d\xi$$
$$= (Cz + D)^{-2-\nu} \int \phi(\xi)^- f(\xi) \exp(\pi i (Cz + D)^{-1} (Az + B) \xi^- \xi) d\xi$$

holds with integration with respect to the canonical measure for the Dedekind skew–plane for all z in the upper half–plane whenever ϕ is a homogeneous harmonic polynomial of degree ν for a nonnegative integer ν.

The Fourier transformation for the Dedekind skew–plane determines an isometric transformation of the Hilbert space of homogeneous harmonic polynomials of degree ν into itself which commutes with the transformations which take a function $f(\xi)$ of ξ in the Dedekind skew–plane into the functions $f(\omega\xi)$ and $f(\xi\omega)$ of ξ in the Dedekind skew–plane for every element ω of the Dedekind skew–plane with conjugate as inverse. If a function $\phi(\xi)$ of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν, the function

$$\phi^\wedge(\xi) = i^\nu \phi(\xi)$$

of ξ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν. The Fourier transform of the function

$$\phi(\xi) \exp(\pi i z \xi^- \xi)$$

of ξ in the Dedekind skew–plane is the function

$$(i/z)^{2+\nu}\phi^\wedge(\xi)\exp(-\pi i z^{-1}\xi^-\xi)$$

of ξ in the Dedekind skew–plane when z is in the upper half–plane.

The Laplace transformation for the Dedekind skew–plane computes the Fourier transformation for the Dedekind skew–plane. If a function $f(\xi)$ of ξ in the Dedekind skew–plane is square integrable with respect to the canonical measure and satisfies the identity

$$\phi(\xi)f(\omega\xi) = \phi(\omega\xi)f(\xi)$$

for every element ω of the Dedekind skew–plane with conjugate as inverse, then its Fourier transform is a function $g(\xi)$ of ξ in the Dedekind skew–plane which is square integrable with respect to the canonical measure and satisfies the identity

$$\phi^\wedge(\xi)g(\omega\xi) = \phi^\wedge(\omega\xi)g(\xi)$$

for every element ω of the Dedekind skew–plane with conjugate as inverse. The identity

$$\int \phi^\wedge(\xi)^- g(\xi)\exp(\pi i z\xi^-\xi)d\xi = (i/z)^{2+\nu}\int \phi(\xi)^- f(\xi)\exp(-\pi i z^{-1}\xi^-\xi)d\xi$$

holds with integration with respect to the canonical measure when z is in the upper half–plane.

The Mellin transformation of harmonic ϕ for the Dedekind skew–plane is derived from the Laplace transformation of harmonic ϕ for the Dedekind skew–plane. The domain of the Mellin transformation is the set of elements of the domain of the Laplace transformation which vanish in a neighborhood of the origin. The Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$g(z) = \int \phi(\xi)^- f(\xi)\exp(\pi i z\xi^-\xi)\, d\xi$$

of z in the upper half–plane defined by integration with respect to the canonical measure. The Mellin transform of the function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$F(z) = \int_0^\infty g(it)t^{\frac{1}{2}\nu - iz}\, dt$$

of z in the upper half–plane. Since the analytic function

$$W(z) = \pi^{-\frac{1}{2}\nu - 1 + iz}\Gamma(\tfrac{1}{2}\nu + 1 - iz)$$

of z admits the integral representation

$$W(z) = (\xi^-\xi)^{\frac{1}{2}\nu + 1 - iz}\int_0^\infty \exp(-\pi t\xi^-\xi)t^{\frac{1}{2}\nu - iz}\, dt$$

when z is in the upper half–plane, the identity

$$F(z)/W(z) = \int \phi(\xi)^- f(\xi)(\xi^-\xi)^{iz-\frac{1}{2}\nu-1} \, d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure. The identity

$$(1+2\nu) \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx = 16\pi^2 \int |f(\xi)|^2 (\xi^-\xi)^{-2y} d\xi$$

holds when y is positive with integration on the right with respect to the canonical measure. An analytic function $F(z)$ of z in the upper half–plane is the Mellin transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane which vanishes when $\xi^-\xi < a$ if, and only if, the least upper bound

$$\sup a^{2y} \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive y is finite.

6. Fourier analysis for adic skew–planes

Fourier analysis for an adic skew–plane resembles Fourier analysis for a Dedekind skew–plane when the adic plane is rectangular. The treatment of Fourier analysis is restricted to adic planes which are rectangular since fundamental examples of the Riemann hypothesis are constructed with rectangular adic planes.

The function

$$\exp(2\pi i\xi)$$

of ξ in the algebraic line admits a unique continuous extension as a function

$$\exp(2\pi i\xi)$$

of ξ in the adic line. When η is in the adic line, the function

$$\exp(2\pi i\xi\eta)$$

of ξ in the adic line defines a homomorphism of additive structure for the adic line into multiplicative structure for the complex numbers of absolute value one. The function has value one on integral elements of the adic line if, and only if, η is an integral element of the adic line. A continuous homomorphism of additive structure for the adic line into multiplicative structure for the complex numbers of absolute value one is determined by a unique element η of the adic line.

The Fourier transform of a function $f(\xi)$ of ξ in the adic line which is integrable with respect to the canonical measure is the bounded continuous function

$$g(\xi) = \int f(\eta) \exp(2\pi i\xi\eta) \, d\eta$$

of ξ in the adic line which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = \int g(\eta) \exp(-2\pi i \xi \eta) \, d\eta$$

applies for almost all elements ξ of the adic line when the function $g(\xi)$ of ξ in the adic line is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

The function

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of ξ in the adic plane is a homomorphism of additive structure for the adic plane into multiplicative structure for the complex numbers of absolute value one which has value one on integral elements if, and only if, η is an integral element of the adic plane. A continuous homomorphism of additive structure for the adic plane into multiplicative structure for the complex numbers of absolute value one is determined by a unique element η of the adic plane.

The Fourier transform of a function $f(\xi)$ of ξ in the adic plane which is integrable with respect to the canonical measure is the bounded continuous function

$$g(\xi) = \int f(\eta) \exp(\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

of ξ in the adic plane which is defined by integration with respect to the canonical measure Fourier inversion

$$g(\xi) = \int g(\eta) \exp(-\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

applies for almost all elements ξ of the adic plane when the function $g(\xi)$ of ξ in the adic plane is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

The function

$$\exp(\pi i(\xi^- \eta + \eta^- \xi))$$

of ξ in the adic skew–plane is a homomorphism of additive structure for the adic skew–plane into multiplicative structure for the complex numbers of absolute value one which has value one on integral elements if, and only if, η is an integral element of the adic skew–plane such that the element $\frac{1}{2}\eta^- \eta$ of the adic line is integral. A continuous homomorphism of additive structure for the adic skew–plane into multiplicative structure for the complex numbers of absolute value one is determined by a unique element η of the adic skew–plane.

The Fourier transform of a function $f(\xi)$ of ξ in the adic skew–plane which is integrable with respect to the canonical measure is the bounded continuous function

$$g(\xi) = 2 \int f(\eta) \exp(\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

of ξ in the adic skew–plane which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi) = 2 \int g(\eta) \exp(-\pi i(\xi^- \eta + \eta^- \xi)) d\eta$$

applies for almost all elements ξ of the adic skew–plane when the function $g(\xi)$ of ξ in the adic skew–plane is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

A Heisenberg group for the adic skew–plane applies to functions $f(\xi)$ of ξ in the adic skew–plane which satisfy constraints modulo ρ for a generating positive integer ρ: The function vanishes when the p–adic component of ξ is nonintegral for some prime divisor p of ρ. When ρ is odd, the function has equal values at elements whose p–adic components are integral and congruent modulo ρ for every prime divisor p of ρ and whose p–adic components are equal for every other generating prime p. An integral element ω of the algebraic plane which represents

$$2 = \omega^- \omega$$

is chosen when ρ is even. The function has equal values at elements whose p–adic components are integral and congruent modulo ρ/ω for every prime divisor p of ρ and whose p–adic components are equal for every other generating prime p.

Functions to character χ are functions which satisfy the constraints modulo ρ when χ is a primitive character modulo ρ. A function $f(\xi)$ of ξ in the adic skew–plane which is of character χ vanishes when the p–adic component of ξ is nonintegral for some prime divisor p of ρ. When ρ is odd, the function has equal values at elements of the adic kew–plane whose p–adic components are equal for every prime divisor p of ρ and whose p–adic components have equal p–adic modulus for every other generating prime p. An integral element ω of the algebraic plane which represents

$$2 = \omega^- \omega$$

is chosen when ρ is even. The function has equal values at elements whose p–adic components are integral and congruent modulo ρ/ω for every prime divisor p of ρ and whose p–adic components have equal p–adic modulus for every other generating prime p.

These conditions imply that the function $f(\omega\xi)$ of elements ξ of the adic skew–plane with integral p–adic component for every prime divisor p of ρ is determined by the congruence class of ω modulo ρ when ω is an integral element of the adic skew–plane whose p–adic component has integral inverse for every generating prime p not dividing ρ.

A function $f(\xi)$ of ξ in the adic skew–plane which is of character χ satisfies the identity

$$\sum f(\omega\xi) = \chi(r) \sum f(\omega\xi)$$

for every integer r modulo ρ when the p–adic component of ξ is integral for every prime divisor p of ρ. Summation on the left is over the congruence classes modulo ρ of integral elements ω of the adic skew–plane which represent

$$r = \omega^- \omega$$

modulo ρ and whose p–adic component has integral inverse for every generating prime p which is not a divisor of ρ. Summation on the right is over the congruence classes modulo ρ of integral elements ω of the adic skew–plane which represent

$$1 = \omega^- \omega$$

modulo ρ and whose p–adic component has integral inverse for every generating prime p which is not a divisor of ρ.

The Heisenberg group which applies to functions satisfying the constraints modulo ρ is the set of pairs (α, β) of elements α and β of the adic skew–plane such that the p–adic component of α is integral for every prime divisor p of ρ, the p–adic component of $\rho\beta$ is integral for every prime divisor p of ρ if ρ is odd, and the p–adic component of $\rho\omega^{-1}\beta$ is integral for every prime divisor p of ρ if ρ is even with ω an integral element of the algebraic plane which represents

$$2 = \omega^- \omega.$$

An element (α, β) of the Heisenberg group defines an isometric transformation $S(\alpha, \beta)$ of a Hilbert space into itself. The space is the set of functions which are square integrable with respect to the canonical measure for the adic skew–plane and which satisfy the constraints modulo ρ. The transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into the function

$$\exp(\pi i(\beta^- \xi + \xi^- \beta)) f(\xi + \alpha)$$

of ξ in the adic skew–plane. The identity

$$S(\alpha, \beta) S(\gamma, \delta) = \exp(\pi i(\delta^- \alpha + \alpha^- \delta)) S(\alpha + \gamma, \beta + \delta)$$

holds for all elements (α, β) and (γ, δ) of the Heisenberg group.

A symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the adic skew–plane which is compatible with constraints modulo ρ is a matrix with entries in the adic skew–plane which has the matrix

$$\begin{pmatrix} D^- & -B^- \\ -C^- & A^- \end{pmatrix}$$

as inverse such that the p–adic components of A and D are integral for every prime divisor p of ρ, such that the p–adic components of $\rho^{-1}C$ and ρB are integral for every prime divisor

p of ρ if ρ is odd, and such that the p–adic components of $\rho^{-1}\omega C$ and $\rho\omega^{-1}B$ are integral for every prime divisor p of ρ if ρ is even with ω an integral element of the algebraic plane which represents

$$2 = \omega^-\omega.$$

The conditions imply that

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to the Heisenberg group for functions satisfying the constraints modulo ρ whenever (α, β) belongs to the Heisenberg group for functions satisfying the constraints modulo ρ.

A symplectic transformation associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which is compatible with the constraints modulo ρ is an isometric transformation T of a Hilbert space into itself whose elements are the functions which are square integrable with respect to the canonical measure for the adic skew–plane and which satisfy the constraints modulo ρ. The transformations

$$S(\alpha, \beta)T$$

and

$$TS(\gamma, \delta)$$

are required to be linearly dependent whenever

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The symplectic matrix

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

is compatible with the constraints modulo ρ when h is an element of the adic line such that the p–adic component of $\frac{1}{2}\rho h$ is integral for every prime divisor p of ρ. An associated symplectic transformation is defined by taking a function $f(\xi)$ of ξ in the adic skew–plane into the function

$$\exp(\pi i h \xi^- \xi) f(\xi)$$

of ξ in the adic skew–plane.

The symplectic matrix

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega^- \end{pmatrix}$$

is compatible with the constraints modulo ρ when ω is an invertible element of the adic skew–plane such that the p–adic component of ω is integral and has integral inverse for every prime divisor p of ρ. An associated symplectic transformation is defined by taking a function $f(\xi)$ of ξ in the adic skew–plane into the function

$$|\omega|^{-2}f(\xi\omega)$$

of ξ in the adic skew–plane.

The Radon transformation for the adic skew–plane factors the Fourier transformation for the adic skew–plane as a composition with the Fourier transformation for the adic plane. The domain of the Radon transformation is the space of functions which are integrable with respect to the canonical measure. The Radon transformation takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\xi)$ of ξ in the adic skew–plane when the identity

$$g(\omega\xi) = 2\int f(\omega\xi + \omega\eta)d\eta$$

holds for almost all elements ξ of the adic plane for every element ω of the adic skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the adic plane in the adic skew–plane. The inequality

$$\int |g(\omega\xi)|d\xi \leq 2\int |f(\xi)|d\xi$$

holds for every element ω of the adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the adic plane and with integration on the right with respect to the canonical measure for the adic skew–plane.

The Radon transform of a function of character χ is a function of character χ. Laplace transformations for the adic skew–plane permit a spectral analysis of the Radon transformation on functions of character χ.

A Laplace kernel κ of character χ is a function $\kappa(\xi)$ of ξ in the adic skew–plane, which is of character χ and has equal values at elements whose p–adic components are equal for every prime divisor p of ρ, such that the integral

$$\int |\kappa(\xi)|^2 d\xi$$

with respect to the canonical measure for the adic skew–plane over the set of integral elements is equal to the measure of the set of integral elements which are invertible modulo ρ.

The Laplace measure is the nonnegative measure on the Baire subsets of the adic line which is obtained from the canonical measure for the adic skew–plane under the homomorphism which takes ξ into $\xi^-\xi$. Multiplication by an invertible element λ of the adic line multiplies the Laplace measure by

$$|\lambda|^{-2}.$$

The set of integral elements of the adic line has measure one.

The domain of the Laplace transformation with kernel κ is the Hilbert space of functions $f(\xi)$ of ξ in the adic skew–plane which are square integrable with respect to the canonical measure, which are of character χ, and which satisfy the identity

$$\kappa(\xi)f(\omega\xi) = \kappa(\omega\xi)f(\xi)$$

for every integral element ω of the adic skew–plane with conjugate as inverse.

The range of the Laplace transformation is the Hilbert space of functions $g(\xi)$ of ξ in the adic line which are square integrable with respect to the Laplace measure, which vanish when the p–adic component of $2\rho\xi$ is nonintegral for some prime divisor p of ρ, and which satisfy the identity

$$g(\xi\omega) = g(\xi)\chi(\omega)^-$$

for every integral element ω of the adic line whose p–adic component has integral inverse for every generating prime p which is not a divisor of ρ when the p–adic component of $2\rho\xi$ is integral for every prime divisor p of ρ.

The Laplace transform of a function $f(\xi)$ of ξ in the adic skew–plane which is integrable with respect to the canonical measure is the bounded continuous function

$$g(\xi) = 2\int \kappa(\eta)^- f(\eta)\chi(\eta^-\eta)\exp(\pi i\xi\eta^-\eta)d\eta$$

of ξ in the adic line which is defined by integration with respect to the canonical measure. The identity

$$\int |g(\xi)|^2 d\xi = \int |f(\xi)|^2 d\xi$$

holds with integration on the left with respect to the Laplace measure and with integration on the right with respect to the canonical measure. The Laplace transformation is defined so as to maintain the identity.

The domain of the Laplace transformation is an invariant subspace for the Radon transformation. The adjoint of the Radon transformation is a nonnegative self–adjoint transformation in the Hilbert space which takes a function $f(\xi)$ of ξ in the adic skew–plane into a function $g(\xi)$ of ξ in the adic skew–plane when the identity

$$\int \kappa(\eta)^- g(\eta)\chi(\eta^-\eta)\exp(\pi i\xi\eta^-\eta)d\eta = |\xi| \int \kappa(\eta)^- f(\eta)\chi(\eta^-\eta)\exp(\pi i\xi\eta^-\eta)d\eta$$

holds for almost all element ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure are interpreted as Laplace transforms when they are not absolutely convergent.

A nonnegative self–adjoint transformation $-iH$ is defined as the inverse of the adjoint of the Radon transformation in the Hilbert space of functions which are square integrable with respect to the canonical measure for the adic skew–plane. The domain of the Laplace transformation with kernel k is an invariant subspace.

A symplectic transformation which has the domain of the Laplace transformation with kernel κ as an invariant subspace is associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the adic skew–plane whose entries are self–conjugate and which is compatible with the constraints modulo ρ. The transformation takes a function $f(\xi)$ of ξ in the adic plane into a function $g(\xi)$ of ξ in the adic plane when the identity

$$\int \kappa(\eta)^- g(\eta)\chi(\eta^-\eta)\exp(\pi i\xi\eta^-\eta)d\eta$$
$$= |C\xi + D|^2 \int \kappa(\eta)^- f(\eta)\chi(\eta^-\eta)\exp(\pi i(C\xi + D)^{-1}(A\xi + B)\eta^-\eta)d\eta$$

holds for almost all elements ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure are interpreted as Laplace transforms when they are not absolutely convergent.

A computation of Fourier transforms is an application of the Laplace transformation with kernel κ. A Laplace kernel $\kappa^\wedge$ of character χ^- is associated with a Laplace kernel κ of character χ when χ is a primitive character modulo ρ. If ρ is even, the Fourier transform of the function of ξ in the adic skew–plane whose value is $\kappa(\xi)$ when ξ is integral and which vanishes otherwise is the function of ξ in the adic skew–plane whose value is

$$4\rho^{-2}\kappa^\wedge(\tfrac{1}{2}\rho\xi)$$

when $\frac{1}{2}\rho\xi$ is integral and which vanishes otherwise. When ρ is odd, an integral element ω of the algebraic plane is chosen which represents

$$2 = \omega^-\omega.$$

The Fourier transform of the function of ξ in the adic skew–plane whose value is $\kappa(\xi)$ when ξ is integral and which vanishes otherwise is the function of ξ in the adic skew–plane whose value is

$$2\rho^{-2}\kappa^\wedge(\rho\omega^{-1}\xi)$$

when $\rho\omega^{-1}\xi$ is integral and which vanishes otherwise.

When ρ is even, the Fourier transform of a function $f(\xi)$ of ξ in the adic skew–plane which belongs to the domain of the Laplace transformation with kernel κ is the function

$$4\rho^{-2}g(\tfrac{1}{2}\rho\xi)$$

of ξ in the adic skew–plane for a function $g(\xi)$ of ξ in the adic skew–plane which belongs to the domain of the Laplace transformation with kernel $\kappa^\wedge$. The identity

$$\int \kappa^\wedge(\eta)^- g(\eta)\chi(\eta^-\eta)^- \exp(2\pi i\rho^{-1}\xi\eta^-\eta)d\eta$$
$$= |\xi|^2 \int \kappa(\eta)^- f(\eta)\chi(\eta^-\eta)\exp(-2\pi i\rho^{-1}\xi^{-1}\eta^-\eta)d\eta$$

holds for almost all elements ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure are interpreted as Laplace transforms when they are not absolutely convergent.

When ρ is odd, the Fourier transform of a function $f(\xi)$ of ξ in the adic skew–plane which belongs to the domain of the Laplace transformation with kernel κ is the function

$$2\rho^{-2}g(\rho\omega^{-1}\xi)$$

of ξ in the adic skew–plane for a function $g(\xi)$ of ξ in the adic skew–plane which belongs to the domain of the Laplace transformation with kernel $\kappa^\wedge$. The identity

$$\int \kappa^\wedge(\eta)^- g(\eta)\chi(\eta^-\eta)^- \exp(\pi i\rho^{-1}\xi\eta^-\eta)d\eta$$
$$= |\xi|^2 \int \kappa(\eta)^- f(\eta)\chi(\eta^-\eta)\exp(-\pi i\rho^{-1}\xi^{-1}\eta^-\eta)d\eta$$

holds for almost all elements ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure are interpreted as Laplace transforms when they are not absolutely convergent.

7. The Fourier transformation for product skew–planes

Fourier analysis for a product skew–plane introduces the zeta functions whose properties permit a proof of the Riemann hypothesis.

The Radon transformation for the product skew–plane generates the symplectic transformations which formulate Fourier analysis on the product skew–plane.

The domain of the Radon transformation is the space of functions which are integrable with respect to the canonical measure for the product skew–plane. The Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane when the identity

$$g(\omega_+\xi_+, \omega_-\xi_-) = \int f(\omega_+\xi_+ + \omega_+\eta_+, \omega_-\xi_- + \omega_-\eta_-)d\eta$$

holds for almost all elements $\xi = (\xi_+, \xi_-)$ of the product plane for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the product plane in the product skew–plane. The inequality

$$\int |g(\omega_+\xi_+, \omega_-\xi_-)|d\xi \leq \int |f(\xi_+, \xi_-)|d\xi$$

holds for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the product plane and with integration on the right with respect to the canonical measure for the product skew–plane.

A Heisenberg group for the product skew–plane applies to functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane such that the function of ξ_- in the adic skew–plane satisfies the constraints modulo ρ for every element ξ_+ of the Dedekind skew–plane. A function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is said to be of character χ if the function of ξ_- in the adic skew–plane is of character χ for every element ξ_+ of the Dedekind skew–plane. Functions of character χ satisfy the constraints modulo ρ when χ is a primitive character modulo ρ.

The Heisenberg group for the product skew–plane is the set of pairs (α, β) of elements α and β of the product skew–plane such that (α_+, β_+) belongs to the Heisenberg group for the Dedekind skew–plane and (α_-, β_-) belongs to the Heisenberg group for the adic skew–plane. An element (α, β) of the Heisenberg group determines an isometric transformation $S(\alpha, \beta)$ of a Hilbert space into itself. The space is the set of functions which are square integrable with respect to the canonical measure for the product skew–plane and which satisfy the constraints modulo ρ. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function

$$\exp(\pi i(\beta_+^- \xi_+ + \xi_+^- \beta_+)) \exp(-\pi i(\beta_-^- \xi_- + \xi_-^- \beta_-)) f(\xi_+ + \alpha_+, \xi_- + \alpha_-)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane. The identity

$$S(\alpha, \beta)S(\gamma, \delta) = \exp(\pi i(\delta_+^- \alpha_+ + \alpha_+^- \delta_+)) \exp(-\pi i(\delta_-^- \alpha_- + \alpha_-^- \delta_-) S(\alpha + \gamma, \beta + \delta)$$

holds for all elements (α, β) and $\gamma, \delta)$ of the Heisenberg group.

A symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the product skew–plane which is compatible with the constraints modulo ρ is a matrix with entries in the product skew–plane such that

$$\begin{pmatrix} A_+ & B_+ \\ C_+ & D_+ \end{pmatrix}$$

is a symplectic matrix for the Dedekind skew–plane and

$$\begin{pmatrix} A_- & B_- \\ C_- & D_- \end{pmatrix}$$

is a symplectic matrix for the adic skew–plane which is compatible with the constraints modulo ρ.

A symplectic transformation T associated with the matrix is an isometric transformation of a Hilbert space into itself. The space is the set of functions which are square integrable with respect to the canonical measure for the product skew–plane and which satisfy the constraints modulo ρ. The transformations

$$S(\alpha, \beta)T$$

and

$$TS(\gamma, \delta)$$

are required to be linearly dependent whenever (α, β) and (γ, δ) are elements of the Heisenberg group which satisfy the identity

$$(\gamma, \delta) = (\alpha, \beta) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

A symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the algebraic skew–plane which is compatible with the constraints modulo ρ is a matrix with entries in the algebraic skew–plane which is a symplectic matrix for the Dedekind skew–plane and is a symplectic matrix for the adic skew–plane which is compatible with the constraints modulo ρ.

A symplectic transformation T associated with the matrix is an isometric transformation of the Hilbert space into itself such that the transformations

$$S(\alpha, \beta)T$$

and

$$TS(\gamma, \delta)$$

are linearly dependent whenever elements (α, β) and (γ, δ) are elements of the Heisenberg group which satisfy the identities

$$(\gamma_+, \delta_+) = (\alpha_+, \beta_+) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and

$$(\gamma_-, \delta_-) = (\alpha_-, \beta_-) \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and such that the transformations are equal whenever each of the elements $\alpha, \beta, \gamma,$ and δ of the product skew–plane has equal Dedekind and adic components in the algebraic skew–plane.

When h is an element of the algebraic line such that the p–adic component of $\frac{1}{2}\rho h$ is integral for every prime divisor p of ρ, a symplectic transformation associated with the symplectic matrix

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

for the algebraic skew–plane is defined by taking a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function

$$\exp(\pi i h \xi_+^- \xi_+) \exp(-\pi i h \xi_-^- \xi_-) f(\xi_+, \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane.

When ω is a nonzero element of the algebraic skew–plane whose Dedekind modulus is equal to its adic modulus such that the p–adic component of ω is integral and has integral inverse for every prime divisor p of ρ, a symplectic transformation associated with the symplectic matrix

$$\begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega^- \end{pmatrix}$$

for the algebraic skew–plane is defined by taking a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function

$$f(\xi_+ \omega, \xi_- \omega)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane.

The Fourier transform of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which is integrable with respect to the canonical measure is the bounded continuous function

$$g(\xi_+, \xi_-) = \int f(\eta_+, \eta_-) \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) d\eta$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int g(\eta_+, \eta_-) \exp(-\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) d\eta$$

applies for almost all elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane when the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is integrable with respect to the canonical measure. The Fourier transformation admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the product skew–plane into itself.

Poisson summation is a transformation of integrable functions on the product skew–plane into integrable functions on the quotient skew–plane which applies when all primes are generating primes. If a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is integrable with respect to the canonical measure, the Poisson sum is the function

$$g(\xi_+, \xi_-) = \sum f(\eta_+, \eta_-)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which is defined by summation over the elements $\eta = (\eta_+, \eta_-)$ of the product skew–plane which are equivalent to $\xi = (\xi_+, \xi_-)$. A function is defined on the quotient skew–plane since equal values are taken at equivalent elements of the product skew–plane. The inequality

$$\int |g(\xi_+, \xi_-)| d\xi \leq \int |f(\xi_+, \xi_-)| d\xi$$

holds with integration on the left with respect to the canonical measure for the quotient skew–plane and with integration on the right with respect to the canonical measure for the product skew–plane. The canonical measure for the set of elements equivalent to the origin counts the elements in a finite subset and is infinite on infinite subsets.

The function

$$\exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-))$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane has equal values at equivalent elements of the product skew–plane if, and only if, the element $\eta = (\eta_+, \eta_-)$ of the product skew–plane is equivalent to the origin.

The Fourier transform of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the quotient skew–plane which is integrable with respect to the canonical measure is the bounded function

$$g(\xi_+, \xi_-) = \int f(\eta_+, \eta_-) \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) d\eta$$

of elements $\xi = (\xi_+, \xi_-)$ equivalent to the origin which is defined by integration with respect to the canonical measure. Fourier inversion

$$f(\xi_+, \xi_-) = \int g(\eta_+, \eta_-) \exp(-\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_+)) \exp(\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) d\eta$$

applies for almost all elements $\xi = (\xi_+, \xi_-)$ of the quotient skew–plane when the function $g(\xi_+, \xi_-)$ of elements $\xi = (\xi_+, \xi_-)$ equivalent to the origin is integrable with respect to the canonical measure. The Fourier transformation elements admits a unique continuous extension as an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the quotient skew–plane onto the Hilbert space of square integrable functions with respect to the canonical measure for the set of elements equivalent to the origin.

A function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is said to be of character χ for a primitive character χ modulo ρ if the function $f(\xi_+, \xi_-)$ of ξ_- in the adic skew–plane is of character χ for every element ξ_+ of the Dedekind skew–plane.

Hecke operators are continuous linear transformations of the Hilbert space of square integrable functions with respect to the canonical measure for the product skew–plane, which are of character χ, into itself. A Hecke operator $\Delta(r)$ is defined for every generating positive integer r which is relatively prime to ρ. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane defined by the sum

$$24g(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the algebraic skew–plane which represent

$$r = \omega^-\omega.$$

The Hecke operator $\Delta(1)$ acts as the orthogonal projection of the Hilbert space onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+\omega, \xi_-\omega)$$

for every integral element ω of the algebraic skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The spectral analysis of Hecke operators is made in Hilbert spaces of finite dimension. A space is defined for every nonnegative integer ν whose elements are function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane, which are of character χ, such that the function $f(\xi_+, \xi_-)$ of ξ_+ in the Dedekind skew–plane is a homogeneous harmonic polynomial of degree ν for every element ξ_- of the adic skew–plane. The function $f(\xi_+, \xi_-)$ of ξ_- in the adic skew–plane has equal values at elements whose p–adic components are equal for every prime divisor p of ρ when ξ_+ is an element of the Dedekind skew–plane. The Hilbert space is defined as a tensor product of the Hilbert space of homogeneous harmonic polynomials of degree ν and a Hilbert space of functions which are square integrable with respect to the canonical measure for the product skew–plane on the set of integral elements of the adic skew–plane.

A Hecke operator $\Delta(r)$ is defined on the tensor product for every generating positive integer r which is relatively prime to ρ. The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane defined by the sum

$$24r^{\frac{1}{2}\nu}g(\xi_+, \xi_-) = \sum f(\xi_+\omega, \xi_-\omega)$$

over the integral elements ω of the algebraic skew–plane which represent

$$r = \omega^-\omega.$$

The Hecke operator $\Delta(1)$ acts as the orthogonal projection of the tensor product onto the subspace of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+\omega, \xi_-\omega)$$

for every integral element ω of the algebraic skew–plane with integral inverse.

The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The tensor product is the orthogonal sum of invariant subspaces whose elements are defined as eigenfunctions of Hecke operators for given eigenvalues. The kernel of the Hecke operator $\Delta(1)$ is annihilated by every Hecke operator. An invariant subspace which is orthogonal to the kernel of $\Delta(1)$ is contained in the range of $\Delta(1)$. The elements of the invariant subspace are defined as eigenfunctions of $\Delta(r)$ for a given eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ. The identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n.

The zeta function characteristic of an invariant subspace is the Dirichlet series

$$\zeta(s) = \sum \tau(n)n^{-s}$$

defined as a sum over the generating positive integers n which are relatively prime to ρ.

The number of zeta functions associated with a primitive character modulo ρ is equal to the product

$$\rho \prod (1 + p^{-1})$$

taken over the odd prime divisor p of ρ.

A Laplace kernel κ associated with the zeta function is an element of the tensor–product which is an eigenfunction of the Hecke operator $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ and whose scalar self–product is equal to the canonical measure of the set of integral elements of the adic skew–plane which are invertible modulo ρ.

A Laplace kernel for the product skew–plane is applied in the definition of a Laplace transformation with kernel κ for the product skew–plane. The domain of the transformation is the Hilbert space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which are square integrable with respect to the canonical measure, which are of character χ, and which satisfy the identity

$$\kappa(\xi_+, \xi_-)f(\omega_+\xi_+, \omega_-\xi_-) = \kappa(\omega_+\xi_+, \omega_-\xi_-)f(\xi_+, \xi_-)$$

for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse.

Elements of the domain of the Laplace transformation with kernel κ are eigenfunctions of the Hecke operator $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ.

The Hilbert space functions which are square integrable with respect to the canonical measure for the product skew–plane and which are of character χ is the orthogonal sum of invariant subspaces whose elements are determined as eigenfunctions of Hecke operators for given eigenvalues. An invariant subspace which is orthogonal to the kernel of $\Delta(1)$ is determined by a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

for the tensor product space as the set of elements which are eigenfunctions of $\Delta(r)$ for the eigenvalue $\tau(r)$ for every generating positive integer r which is relatively prime to ρ. The invariant subspace is spanned by the domains of Laplace transformations whose kernels are eigenfunctions of the Hecke operators for the same eigenvalues.

The Laplace transform with kernel κ of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is the bounded continuous function

$$F(z, \xi) = 2 \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-) \chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

of z in the upper half–plane and ξ in the adic line which is defined by integration with respect to the canonical measure when the integral is absolutely convergent. The identity

$$(1 + 2\nu) \int \int_0^\infty \int_{-\infty}^{+\infty} |F(x + iy, \xi)|^2 y^\nu dx dy d\xi = 8(2\pi)^{-\nu} \Gamma(1 + \nu) \int |f(\xi_+, \xi_-)|^2 d\xi$$

holds with integration on the left with respect to the Laplace measure and with integration on the right with respect to the canonical measure. The Laplace transformation with kernel κ is defined so as to preserve the identity.

A function $F(z, \xi)$ of z in the upper half–plane and ξ in the adic line which belongs to the range of the Laplace transformation is analytic as a function of z for every element ξ of the adic line, vanishes when the p–adic component of $2\rho\xi$ is nonintegral for some prime divisor p of ρ, and satisfies the identity

$$F(z, \xi\omega) = F(z, \xi) \chi(\omega)^-$$

for every integral element ω of the adic line whose p–adic component is invertible modulo p for every prime divisor p of ρ and has integral inverse for every other generating prime p.

The domain of the Laplace transformation with kernel κ is an invariant subspace for the Radon transformation. The adjoint of the Radon transformation is a maximal dissipative transformation in the Hilbert space which takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function $g(\xi_+, \xi_-)$ in the product skew plane defined by the identity

$$G(z, \xi) = (i/z)|\xi|F(z, \xi)$$

for almost all elements ξ of the adic line with respect to the Laplace measure when z is in the upper half–plane. The integrals

$$F(z, \xi) = \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

and

$$G(z, \xi) = \int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

with respect to the canonical measure for the product skew–plane are interpreted as Laplace transforms when they are not absolutely convergent.

A maximal dissipative transformation $-iH$ is defined as the inverse of the adjoint of the Radon transformation in the Hilbert space of functions of character χ which are square integrable with respect to the canonical measure for the product skew–plane.

A symplectic transformation T which has the Hilbert space of square integrable functions of character χ as an invariant subspace is associated with a symplectic matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the algebraic skew–plane with self–conjugate entries which is compatible with the constraints modulo ρ. The domain of the Laplace transformation with kernel κ is an invariant subspace.

The transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane when the identity

$$G(z, \xi) = (Cz + D)^{-2-\nu}|C\xi + D|^2 F((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))$$

holds for almost all elements ξ of the adic line with respect to the Laplace measure when z is in the upper half–plane. The integrals

$$F(z, \xi) = \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

and

$$G(z, \xi) = \int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-)\chi(\eta_-^- \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-) d\eta$$

with respect to the canonical measure are interpreted as Laplace transforms when they are absolutely convergent.

The theta function determined by a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

is a function $\theta(z, \xi)$ of z in the upper half–plane and invertible elements ξ of the adic line which is analytic as a function of z for every ξ and which satisfies the identity

$$\theta(z, \xi) = \theta(z\lambda, \xi\lambda)$$

for every nonzero element λ of the algebraic line whose Dedekind modulus is equal to its adic modulus.

The theta function appears in a simplification of the sum

$$\sum \kappa(\eta_+\lambda, \eta_-\lambda) \exp(\pi i z \eta_+^- \eta_+ \lambda^- \lambda) \exp(-\pi i \xi \eta_-^- \eta_- \lambda^- \lambda)$$

over the nonzero elements λ of the algebraic skew–plane whose Dedekind modulus is equal to its adic modulus such that $\eta_-\lambda$ is integral. Since the sum remains unchanged when η_+ is replaced by $\eta_+\lambda$ and η_- is replaced by $\eta_-\lambda$ for a nonzero element λ of the algebraic skew–plane whose Dedekind modulus is equal to its adic modulus, the computation of the sum for invertible elements η_- of the adic skew–plane reduces to the case in which η_- is integral and has integral inverse.

When η_- is integral and has integral inverse, the sum is taken over the integral elements λ of the algebraic skew–plane such that $\lambda^-\lambda$ is positive and relatively prime to ρ. A partial sum in which

$$\lambda^-\lambda = n$$

for a generating positive integer n which is relatively prime to ρ is computed using the definition of the Hecke operator $\Delta(n)$. The sum is a product

$$\kappa(\eta_+, \eta_-)\theta(z\eta_+^-\eta_+, \xi\eta_-^-\eta_-)$$

with the theta function

$$\theta(z, \xi) = \sum n^{\frac{1}{2}\nu} \tau(n) \exp(\pi i n z) \exp(-\pi i n \xi)$$

defined as a sum over the generating positive integers n which are relatively prime to ρ when ν is positive or ρ is not one. When ν is zero and ρ is one, the theta function

$$\theta(z, \xi) = 1 + \sum \tau(n) \exp(\pi i n z) \exp(-\pi i n \xi)$$

is defined as a sum over the generating positive integers n with a contribution for the integer zero.

The function $\theta(z, \xi)$ of invertible elements ξ of the adic line admits a unique continuous extension as a function $\theta(z, \xi)$ of elements ξ of the adic line for every element z of the upper half–plane. The identity

$$\theta(z, \xi) = \theta(z + t, \xi + t)$$

holds for every element t of the algebraic line. The function

$$\theta(z) = \theta(z, 0)$$

of z in the upper half–plane is represented as a sum

$$\theta(z) = \sum n^{\frac{1}{2}\nu} \tau(n) \exp(\pi i n z)$$

over the generating positive integers n which are relatively prime to ρ when ν is positive or ρ is not one. The function

$$\theta(z) = 1 + \sum \tau(n) \exp(\pi i n z)$$

of z in the upper half–plane is represented as a sum over the generating positive integers n with a contribution for the integer zero when ν is zero and ρ is one.

The theta function satisfies a functional identity when all primes are generating primes. A computation of Fourier transforms is made for application of the Poisson summation formula. With a primitive character χ modulo ρ is associated the conjugate character χ^- which is primitive modulo ρ. With a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

for the character χ is associated the conjugate zeta function

$$\zeta(s^-)^- = \sum \tau(n)^- n^{-s}$$

for the conjugate character χ^-. With a Laplace kernel κ of character χ for the zeta function $\zeta(s)$ is associated a Laplace kernel $\kappa^\wedge$ of character χ^- associated with the zeta function $\zeta(s^-)^-$. Assume that z is in the upper half–plane and that ξ is an invertible element of the adic line. An integral element ω of the algebraic plane is chosen which represents

$$2 = \omega^- w.$$

When ρ is even, the Fourier transform of the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$\kappa(\eta_+, \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-)$$

when η_- is integral and which vanishes otherwise is the function $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$4\rho^{-2}(i/z)^{2+\nu} |\xi|^2 \kappa^\wedge(\eta_+, \tfrac{1}{2}\rho\eta_-) \exp(-\pi i z^{-1} \eta_+^- \eta_+) \exp(\pi i \xi^{-1} \eta_-^- \eta_-)$$

when $\frac{1}{2}\rho\eta_-$ is integral and which vanishes otherwise.

When ρ is odd, the Fourier transform of the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$\kappa(\eta_+, \eta_-) \exp(\pi i z \eta_+^- \eta_+) \exp(-\pi i \xi \eta_-^- \eta_-)$$

when η_- is integral and which vanishes otherwise is the function of $\eta = (\eta_+, \eta_-)$ in the product skew–plane which is equal to

$$2\rho^{-2}(i/z)^{2+\nu}|\xi|^2 \kappa^\wedge(\eta_+, \rho\omega^{-1}\eta_-) \exp(-\pi i z^{-1} \eta_+^- \eta_+) \exp(\pi i \xi^{-1} \eta_-^- \eta_-)$$

when $\rho\omega^{-1}\eta_-$ is integral and which vanishes otherwise.

When ρ is even, the Poisson formula yields the functional identity

$$\begin{aligned} &\kappa(1,1)\theta(2\rho^{-1}z, 2\rho^{-1}\xi) \\ &= \kappa^\wedge(2\rho^{-1}, 1)(2\rho^{-1})^{2+\nu}(i/z)^{2+\nu}|\xi|^2\theta^\wedge(-2\rho^{-1}z^{-1}, -2\rho^{-1}\xi^{-1}). \end{aligned}$$

When ρ is odd, the Poisson formula yields the functional identity

$$\begin{aligned} &\kappa(1,1)\theta(\rho^{-1}z, \rho^{-1}\xi) \\ &= \kappa^\wedge(\omega\rho^{-1}, 1)|\omega\rho^{-1}|^{2+\nu}(i/z)^{2+\nu}|\xi|^2\theta(-2\rho^{-1}z^{-1}, -2\rho^{-1}\xi^{-1}). \end{aligned}$$

The Fourier transformation for the product skew–plane is a symplectic transformation associated with the symplectic matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the algebraic skew–plane. A generalization of the Poisson formula applies when the Fourier transformation is replaced by a symplectic transformation associated with an admissible symplectic matrix for the algebraic skew–plane. An example is the matrix

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

for every integer n. The associated symplectic transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into the function

$$\exp(2\pi i n \xi_+^- \xi_+) \exp(-2\pi i n \xi_-^- \xi_-) f(\xi_+, \xi_-)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane.

The modular group is the set of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with integer entries and determinant one. A submodular group is the set of matrices in the modular group whose diagonal entries have equal parity and whose off–diagonal entries have equal parity. The submodular group is generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which satisfy no relations other than the one implied by the Fourier transformation, whose fourth power is the identity.

The signature for the submodular group is the homomorphism of the subgroup onto the fourth roots of unity which has value one on

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and which has value i on

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

An element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the submodular group has signature one if, and only if, its diagonal entries are congruent to one modulo four and its off–diagonal entries are even.

A symplectic transformation for which the Poisson summation formula applies is associated with every matrix in the submodular group. The symplectic transformation associated with a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group takes functions of character χ into functions of character for a primitive character χ modulo ρ when the subdiagonal entry is divisible by 2ρ. A theta function $\theta(z, \xi)$ associated with the character χ satisfies the identity

$$\theta(z, \xi) = \chi(D)(Cz + D)^{-2-\nu}|C\xi + D|^2\theta((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))$$

when ν is positive or ρ is not one. The identity

$$1 + \theta(z, \xi) = (Cz + D)^{-2}|C\xi + D|^2[1 + \theta((Cz + D)^{-1}(Az + B), (C\xi + D)^{-1}(A\xi + B))]$$

holds when ν is zero and ρ is one.

The analytic function $\theta(z)$ of z in the upper half–plane satisfies the identity

$$\theta(z) = \chi(D)(Cz + D)^{-2-\nu}\theta((Cz + D)^{-1}(Az + B))$$

for every matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group which has subdiagonal entry divisible by 2ρ.

An analytic function $F(z)$ of z in the upper half–plane is said to be a submodular form of order ν associated with a primitive character χ modulo ρ if the function

$$(Cz + D)^{-2-\nu} F((Cz + D)^{-1}(Az + B))$$

is represented by a power series in

$$\exp(\pi i z)$$

for every matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the submodular group and if the identity

$$F(z) = \chi(D)(Cz + D)^{-2-\nu} F((Cz + D)^{-1}(Az + B))$$

holds whenever the matrix has signature one and has subdiagonal entry divisible by 2ρ.

A submodular form of order ν associated with a primitive character χ modulo ρ is a linear combination of analytic functions obtained from theta functions of order ν associated with the character. The proof is given by showing that the dimension of the space of submodular forms of order ν associated with a primitive character modulo ρ is not greater than the number of theta functions of order ν associated with the character.

Elements z and w of the upper half–plane are considered equivalent modulo ρ if

$$w = (Cz + D)^{-1}(Az + B)$$

for a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group whose subdiagonal entry is divisible by 2ρ.

A fundamental region modulo ρ is a connected open subset of the upper half–plane such that every element of the upper half–plane is equivalent to an element of the closure of the set and such that equivalent elements of the set are equal.

A fundamental region modulo ρ is constructed as the interior of the union of closures of fundamental regions modulo one.

Considerations of symmetry are applied in the construction of a fundamental region modulo one. Elements z and w of the upper half–plane are said to be symmetric modulo one if z and $-w^-$ are equivalent modulo one. The elements of the upper half–plane which are not self–symmetric modulo one form an open set which is the union of connected components. A fundamental region modulo one is constructed as the interior of the union of the closures of two symmetric components when the interior is connected.

An example of a symmetric component is the set of elements z of the upper half–plane which satisfy the inequalities
$$0 < z + z^- < 2$$
and
$$(2z - 1)^-(2z - 1) > 1.$$
Another example is the set of elements z of the upper half–plane which satisfy the inequalities
$$-2 < z + z^- < 0$$
and
$$(2z + 1)^-(2z + 1) > 1.$$
A fundamental region modulo one is the union of the two components with the imaginary axis.

The boundary line
$$z + z^- = -2$$
is mapped onto the boundary line
$$z + z^- = 2$$
by taking z into
$$z + 2.$$

The boundary circle
$$(2z + 1)^-(2z + 1) = 1$$
is mapped onto the boundary circle
$$(2z - 1)^-(2z - 1) = 1$$
by taking z into
$$z/(1 + 2z).$$

A fundamental region modulo ρ is constructed as the interior of the union of the closure of symmetric components modulo one. The fundamental region is compactified by taking its closure in the complex plane and by supplying an element at the upper end of the imaginary axis.

Analytic structure is supplied at the infinite element by requiring a power series expansion in
$$\exp(\pi i z).$$
Analytic structure is supplied at finite elements of the boundary by requiring analyticity of the mapping which takes z into
$$(Cz + D)^{-1}(Az + B)$$

whenever

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an element of signature one of the submodular group whose subdiagonal entry is divisible by 2ρ.

The dimension of the space of submodular forms of order ν associated with a primitive character modulo ρ is less than or equal to the maximal number of inequivalent zeros of a nontrivial submodular form of order ν associated with the character in the compactification of the fundamental region.

The number of inequivalent zeros in the closure of a fundamental region modulo ρ is the sum of the number of inequivalent zeros in the closures of symmetric components contained in the region. The number of zeros of a nontrivial submodular form $F(z)$ in a symmetric component is a Cauchy integral

$$(2\pi i)^{-1} \int F(z)^{-1} F'(z) dz$$

taken counterclockwise over the boundary of the component. Since there may be zeros on the boundary and since some boundary elements lie outside the upper half–plane, the Cauchy integral is interpreted as a limit of Cauchy integrals over the boundaries of regions which are contained in the symmetric component and which contain the zeros of the function in the component.

The boundary of a symmetric component is divided into three arcs by the three elements of the boundary which lie outside the upper half–plane. Every arc of the boundary of a fundamental region is an arc of the boundary of a symmetric component inside the region and of a symmetric component outside the region. An arc of the boundary of the fundamental region is paired with another arc of the boundary whose elements are equivalent modulo ρ. A matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of signature one in the submodular group exists which has subdiagonal entry divisible by 2ρ such that the mapping z into

$$(Cz + D)^{-1}(Az + B)$$

takes a bounding component inside the region onto a bounding component outside the region. The contribution to the Cauchy integral of each bounding arc depends on the function. The sum of the contributions of two equivalent arcs is independent of the function.

The number of inequivalent zeros of a nontrivial submodular form of order ν associated with a primitive character modulo ρ is independent of the function. The number is equal to the product of

$$1 + \nu$$

and the number of fundamental regions modulo one contained in a fundamental region modulo ρ.

This completes the proof that all submodular forms of order ν associated with a primitive character modulo ρ are derived from theta functions of order ν associated with the character since the number of fundamental regions modulo one contained in a fundamental region modulo ρ is equal to the product

$$\rho \prod (1 + p^{-1})$$

taken over the odd prime divisors p of ρ.

The Euler product for a zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

is a consequence of the identity

$$\tau(m)\tau(n) = \sum \tau(mn/k^2)$$

which holds for all generating positive integers m and n which are relatively prime to ρ with summation over the common odd divisors k of m and n. The Euler product

$$\zeta(s)^{-1} = \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

when ρ is even and

$$\zeta(s)^{-1} = (1 - \tau(2)2^{1-s}) \prod (1 - \tau(p)p^{-s} + p^{-2s})$$

when ρ is odd and is taken over the odd generating primes p which are not divisors of ρ.

A zeta function

$$\zeta(s) = \sum \tau(n) n^{-s}$$

of order ν associated with a primitive character χ modulo ρ satisfies a functional identity when all primes are generating primes. The functional identity for the zeta function is obtained from the functional identity for the theta function. The zeta function admits an analytic extension to the complex plane when ν is positive or ρ is not one. When ν is zero and ρ is one, the zeta function admits an analytic extension to the complex plane except for a simple pole at two.

When ρ is even, the analytic extension of the function

$$(2\pi/\rho)^{-\frac{1}{2}\nu - s} \Gamma(\tfrac{1}{2}\nu + s) \zeta(s)$$

and the analytic extension of the function obtained from

$$(2\pi/\rho)^{-\frac{1}{2}\nu - s} \Gamma(\tfrac{1}{2}\nu + s) \zeta(s)$$

on replacing s by $2-s$ are linearly dependent.

When ρ is odd, the analytic extension of the function

$$(2^{\frac{1}{2}}\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s)$$

and the analytic extension of the function obtained from

$$(2^{\frac{1}{2}}\pi/\rho)^{-\frac{1}{2}\nu-s}\Gamma(\tfrac{1}{2}\nu+s)\zeta(s^-)^-$$

on replacing s by $2-s$ are linearly dependent. The analytic extensions are equal when ν is zero and ρ is one.

A Laplace transformation is defined by a theta function. Associated with the theta function $\theta(z,\xi)$ of z in the upper half–plane and ξ in the adic line is the conjugate theta function

$$\theta^*(z,\xi)=\theta(-z^-,-\xi)^-$$

of z in the upper half–plane and ξ in the adic line.

The domain of the Laplace transformation is the Hilbert space of functions $f(\xi_+,\xi_-)$ of $\xi=(\xi_+,\xi_-)$ in the product skew–plane which are of character χ and satisfy the identity

$$\kappa(\xi_+,\xi_-)f(\omega_+\xi_+,\omega_-\xi_-)=\kappa(\omega_+\xi_+,\omega_-\xi_-)f(\xi_+,\xi_-)$$

for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse, which satisfy the identity

$$f(\xi_+,\xi_-)=f(\xi_+\lambda,\xi_-\lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^-\lambda$ is a ratio of generating positive integers which are relatively prime to ρ, and whose product with $\xi_+^-\xi_+)^{-1}$ is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The Laplace space of order ν and character χ is a Hilbert space which is applied in the description of the range of the Laplace transformation. The elements of the space are functions $F(z,\xi)$ of z in the upper half–plane and invertible elements ξ of the adic line such that the function $F(z,\xi)$ of z is analytic for every invertible element ξ of the adic line, such that $F(z,\xi)$ vanishes when the p–adic component of $2\rho\xi$ is nonintegral for some prime divisor p of ρ, such that the identity

$$F(z,\xi\omega)=F(z,\xi)\chi(\omega)^-$$

holds for every integral element ω of the adic line whose p–adic component has integral inverse for every generating prime p which does not divide ρ when the p–adic component of $2\rho\xi$ is integral for every prime divisor p of ρ, such that the identity

$$F(z,\xi)=F(z\lambda,\xi\lambda)$$

holds for every positive element λ of the algebraic line which is a ratio of generating positive integers relatively prime to ρ, and such that the integral

$$\int \int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy,\xi)|^2 y^{2+\nu} dxdyd\xi$$

with respect to the Laplace measure over the set of integral elements of the adic line with integral inverse is finite.

If ν is positive or if ρ is not one, the Laplace transform of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is defined as an integral when the function is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The Laplace transform is the function

$$F(z,\xi) = 2\int \kappa(\eta_+,\eta_-)^- f(\eta_+,\eta_-)\chi(\eta_-^-\eta_-)\theta^*(z\eta_+^-\eta_+, \xi\eta_-^-\eta_-)d\eta$$

of z in the upper half–plane and invertible elements ξ of the adic line which is defined by integration with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

If ν is zero and ρ is one, the Laplace transform of a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is defined as an integral when the function is square integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The identity

$$\theta^*(z,\xi) = \theta(z,\xi)$$

is then satisfied. The Laplace transform is the function

$$F(z,\xi) = 2\int f(\eta_+,\eta_-)[\theta(z\eta_+^-\eta_+, \xi\eta_-^-\eta_-) - 1]d\eta$$

of z in the upper half–plane and invertible elements ξ of the adic line defined by integration with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

In all cases a function $F(z,\xi)$ of z in the upper half–plane and invertible elements ξ of the adic line is a Laplace transform if, and only if, the function

$$G(z,\xi) = \sum n^{\frac{1}{2}\nu}\tau(n)F(zn,\xi n)\chi(n)^-$$

of z in the upper half–plane and invertible elements ξ of the adic line, which is defined as a sum over the generating positive integers which are relatively prime to ρ, belongs to the Laplace space of order ν and character χ. The identity

$$(1+2\nu)\int \int_0^\infty \int_{-\infty}^{+\infty} |G(x+iy,\xi)|^2 y^{2+\nu} dxdy = 8(2\pi)^{-2-\nu}\Gamma(3+\nu)\int |f(\xi_+,\xi_-)|^2(\xi_+^-\xi_+)^{-2}d\xi$$

holds with integration on the left with respect to the Laplace measure over the set of integral elements of the adic line with integral inverse and with integration on the right with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse. The identity defines the Laplace transformation when the transformation is not defined by an absolutely convergent integral.

A Radon transformation is defined on the space of functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfy the constraints modulo ρ, which satisfy the identity

$$f(\xi_+, \xi_-) = f(\xi_+ \lambda, \xi_- \lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^- \lambda$ is a ratio of generating positive integers which are relatively prime to ρ, and which are integrable with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ in the product skew–plane when the identity

$$g(\omega_+ \xi_+, \omega_- \xi_-) = \int f(\omega_+ \xi_+ + \omega_+ \eta_+, \omega_- \xi_- + \omega_- \eta_-) d\eta$$

holds for almost all elements $\xi = (\xi_+, \xi_-)$ of the product plane for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and for every element ω_- of the adic skew–plane with conjugate as inverse with integration with respect to the canonical measure for the complementary space to the product plane in the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane satisfies the constraints modulo ρ and satisfies the identity

$$g(\xi_+, \xi_-) = g(\xi_+ \lambda, \xi_- \lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^- \lambda$ is a ratio of generating positive integers which are relatively prime to ρ. The inequality

$$\int |g(\omega_+ \xi_+, \omega_- \xi_-)| d\xi \leq \int |f(\xi_+, \xi_-)| d\xi$$

holds for every element ω_+ of the Dedekind skew–plane with conjugate as inverse and every element ω_- of the adic skew–plane with conjugate as inverse with integration on the left with respect to the canonical measure for the product plane over the set of elements whose adic component is integral and has integral inverse and with integration on the right over the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The adjoint of the Radon transformation is a maximal dissipative transformation in the domain of the Laplace transformation when ν is positive or ρ is not one. The adjoint of the

Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane when the identity

$$\int \kappa(\eta_+, \eta_-)^- g(\eta_+, \eta_-)\chi(\eta_-^- \eta_-)\theta^*(z\eta_+^- \eta_+, \xi\eta_-^- \eta_-)d\eta$$
$$= (i/z)|\xi| \int \kappa(\eta_+, \eta_-)^- f(\eta_+, \eta_-)\chi(\eta_-^- \eta_-)\theta^*(z\eta_+^- \eta_+, \xi\eta_-^- \eta_-)d\eta$$

holds when z is in the upper half–plane for almost all invertible elements ξ of the adic line with respect to the Laplace measure. The integrals with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse are interpreted as Laplace transforms when they are not absolutely convergent.

A relation T with domain and range in a Hilbert space is said to be nearly maximal dissipative if

$$(T - \lambda^-)(T - \lambda)^{-1}$$

is a contractive transformation of a closed subspace of the Hilbert space of codimension at most one into the Hilbert for some, and hence every, complex number λ in the right half–plane.

The adjoint of the Radon transformation is a nearly maximal dissipative transformation in the domain of the Laplace transformation when ν is zero and ρ is one. The adjoint of the Radon transformation takes a function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane into a function $g(\xi_+, \xi_-)$ in the product skew–plane when the identity

$$\int g(\eta_+, \eta_-)[\theta(z\eta_+^- \eta_+, \xi\eta_-^- \eta_-) - 1]d\eta$$
$$= (i/z)|\xi| \int f(\eta_+, \eta_-)[\theta(z\eta_+^- \eta_+, \xi\eta_-^- \eta_-) - 1]d\eta$$

holds when z is in the upper half–plane for almost all invertible elements ξ of the adic line. The integrals with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse are interpreted as Laplace transforms when they are not absolutely convergent.

8. The Riemann Hypothesis for Hilbert Spaces of Entire Functions

The proof of the Riemann hypothesis originates in properties of the gamma function exhibited in an integral representation due to Euler. The Mellin transformation derives the Euler representation in Fourier analysis on the real line. The gamma function is coupled with a zeta function in a Mellin representation derived in Fourier analysis for a product skew–plane. The Riemann hypothesis asserts that a zeta function coupled with its gamma function resembles a gamma function in its properties. The proof of the Riemann hypothesis is an application of the Mellin representation. The argument is formulated in Hilbert spaces whose elements are functions analytic in the upper half–plane.

An analytic weight function is a function which is analytic and without zeros in the upper half–plane. The weighted Hardy space constructed from an analytic weight function $W(z)$ is the set of functions $F(z)$ of z, which are analytic in the upper half–plane, such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

taken over all positive numbers y is finite. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

Special weighted Hardy spaces are represented by the Mellin transformation in Fourier analysis. The Riemann hypothesis is treated as the issue of analytic extension of the weight function without zeros to a larger half–plane. An axiomatic formulation is indicated by properties of the gamma function.

Theorem 1. *A maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined for a positive number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}\ ih)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. The space is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal dissipative transformation is dissipative.

An element

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

of the graph is defined when w is in the half–plane

$$h < iw^- - iw$$

by

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}ih)^-/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}ih)^-/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w + \tfrac{1}{2}ih) + F_-(w - \tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element which is orthogonal to itself is orthogonal to every element of the graph.

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z + \tfrac{1}{2}ih)W(w - \tfrac{1}{2}ih)^- + W(z - \tfrac{1}{2}ih)W(w + \tfrac{1}{2}ih)^-]/[2\pi i(w^- - z)]$$

of z. Division by $W(z + \frac{1}{2}ih)$ acts as an isometric transformation of the space onto a Hilbert space appearing in the Poisson representation of functions which are analytic and have nonnegative real part in the upper half–plane [1]. The function

$$\phi(z) = W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

of z admits an analytic extension to the upper half–plane. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w. The real part of the function is nonnegative in the half–plane.

The argument is reversed to construct a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has nonnegative real part in the upper half–plane. The Poisson representation constructs a Hilbert space whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z + \frac{1}{2}ih)$ acts as an isometric transformation of the space onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z+\tfrac{1}{2}ih)W(w-\tfrac{1}{2}ih)^- + W(z-\tfrac{1}{2}ih)W(w+\tfrac{1}{2}ih)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation in the space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space $\mathcal{F}(W)$ which contains

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

with

$$K_+(w,z) = W(z)W(w-\tfrac{1}{2}ih)^-|[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = W(z)W(w+\tfrac{1}{2}ih)^-|[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

whenever the inequality

$$h < iw^- - iw$$

is satisfied. The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of its restricted adjoint.

The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements

$$F(z) = (F_+(z), F_-(z))$$

and

$$G(z) = (G_+(z), G_-(z))$$

of the graph is defined as a sum of scalar products in the space $\mathcal{F}(W)$. An isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z+\tfrac{1}{2}ih) + F_-(z-\tfrac{1}{2}ih).$$

The restricted adjoint is dissipative since scalar self–products are nonnegative on its graph. Since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint, the adjoint is dissipative. An isometric transformation of the graph of the adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The dissipative property of the adjoint is equivalent to the inequality

$$\|F_+(t) - F_-(t)\|_{\mathcal{F}(W)} \leq \|F_+(t) + F_-(t)\|_{\mathcal{F}(W)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph. The domain of the contractive transformation which takes

$$F_+(z) + F_-(z)$$

into

$$F_+(z) - F_-(z)$$

is closed. The maximal dissipative property of the adjoint is equivalent to the requirement that the contractive transformation be everywhere defined. The maximal dissipative property is proved by showing that no nonzero element of the space $\mathcal{F}(W)$ is orthogonal to the domain.

Since $K(w, z)$ belongs to the graph when w belongs to the half–plane

$$h < iz^- - iz,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain satisfies the identity

$$H(w - \tfrac{1}{2}ih) + H(w + \tfrac{1}{2}ih) = 0.$$

The function $H(z)$ admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + H(z + ih) = 0$$

for all complex z. A zero of $h(z)$ is repeated with period ih. Since

$$H(z)/W(z)$$

is of bounded type in the upper half–plane, the function of z vanishes identically if it has a zero. The orthogonal complement of the domain of the contractive transformation has dimension at most one.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \exp(ha)H(z + ih) = 0$$

for a real number a has dimension at most one. The dimension is independent of a. Multiplication by

$$\exp(iaz)$$

takes a solution of the equation with a equal to zero into a solution for a real number a. Since

$$\exp(iaz)H(z)$$

belongs to the space $\mathcal{F}(W)$ for every real number a when $H(z)$ is a solution of the equation with a equal to zero, the function $H(z)$ vanishes identically.

The transformation which takes $F(z)$ into $F(z+ih)$ whenever these functions of z belong to the space $\mathcal{F}(W)$ is maximal dissipative since it has a maximal dissipative adjoint.

This completes the proof of the theorem.

The analytic extension of the function

$$W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

of z has no zeros in the upper half–plane since the real part of the function is nonnegative in the half–plane. Since the analytic function

$$W(z + \tfrac{1}{2}ih)$$

of z has no zeros in the upper half–plane, the function

$$W(z - \tfrac{1}{2}ih)$$

of z has an analytic extension without zeros in the upper half–plane. The function

$$W(z)$$

of z has an analytic extension without zeros to the half–plane

$$-h < iz^- - iz.$$

Although the analytic weight functions introduced to prove the Riemann hypothesis are new, they are not without precedent. A related class of analytic weight functions is characterized by the maximal dissipative property of the transformation which takes $F(z)$ into $iF'(z)$ whenever a function $F(z)$ of z and its derivative $F'(z)$ belong to the weighted Hardy space. The maximal dissipative transformation exists in a weighted Hardy space $\mathcal{F}(W)$ if, and only if, the modulus of

$$W(x + iy)$$

is a nondecreasing function of positive y for every real number x. The proof is similar to the proof of Theorem 1.

Analytic weight functions make their appearance in applications to entire functions. Fundamental examples of analytic weight functions are entire functions.

An entire function $E(z)$ is said to be of Hermite class if it has no zeros in the upper half–plane, if the inequality

$$|E(x - iy)| \leq |E(x + iy)|$$

holds for every real number x when y is positive, and if the modulus of

$$E(x + iy)$$

is a nondecreasing function of positive y for every real number x. These properties are satisfied by a polynomial $E(z)$ which has no zeros in the upper half plane and hence also by an entire function which is a limit of polynomials without zero in the upper half–plane but does not vanish identically. An entire function of Hermite class is a uniform limit on compact subsets of the complex plane of polynomials which have no zeros in the upper half–plane.

The structure of an entire function of Hermite class is determined by its zeros. The function is the exponential of a polynomial of degree at most two if it has no zeros. The function otherwise admits a factorization in terms of zeros. The present terminology defines the Hermite class so that the upper half–plane contains no zeros. Another usage defines the Hermite class so that the lower half–plane contains no zeros. The half–planes are interchanged on replacing an entire function $E(z)$ by the entire function

$$E^*(z) = E(z^-)^-.$$

Use is made of a characterization of the Hermite class which is due to Pólya. The proof of the Riemann hypothesis calls attention to the contribution of Hermite to the theory of entire functions. The Pólya class of entire functions is identical with the Hermite class.

The Stieltjes integral representation of nonnegative linear functionals on polynomials explores properties of the Hermite class of entire functions when they are polynomials. A generalization to other entire functions is made possible by an axiomatic treatment [1].

Hilbert spaces whose elements are entire functions appear with these properties:

(H1) Whenever an entire function $F(z)$ belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking an entire function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

belongs to the space whenever $F(z)$ belongs to the space and it always has the same norm as $F(z)$.

An example of an analytic weight function is an entire function $F(z)$ which satisfies the inequality

$$|E(x - iy) < |E(x + iy)|$$

for every real number x when y is positive. A Hilbert space $\mathcal{H}(E)$ of entire functions, which is contained isometrically in the weighted Hardy space $\mathcal{F}(E)$, contains all entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the weighted Hardy space. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

if z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

The Hilbert space $\mathcal{H}(E)$ of entire functions satisfies the axioms (H1), (H2), and (H3). The space is characterized by the axioms [1]. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is isometrically equal to a space $\mathcal{H}(E)$.

The defining function $E(z)$ of a space $\mathcal{H}(E)$ need not be of Hermite class. The appearance of entire functions of Hermite class is due to its stability in relationship to the class of functions which are analytic and of bounded type in the upper half–plane [1]. Assume that an analytic weight function $W(z)$ is given such that the modulus of

$$W(x + iy)$$

is a nondecreasing function of positive y for every real number x. If $F(z)$ is a nontrivial entire function such that the functions

$$F(z)/W(z)$$

and

$$F^*(z)/W(z)$$

of z are of bounded type in the upper half plane, then

$$F^*(z)F(z) = G^*(z)G(z)$$

for an entire function $G(z)$ of Hermite class such that the functions

$$F(z)/G(z)$$

and

$$F^*(z)/G(z)$$

are bounded by one in the upper half–plane.

The Riemann hypothesis for Hilbert spaces of entire functions formulates an underlying concept in the proof of Riemann hypothesis which has the formulation in weighted Hardy spaces as a limiting case.

Theorem 2. *A maximal dissipative transformation in a Hilbert space $\mathcal{H}(E)$ of entire functions is defined for a positive number h by taking $F(z)$ in $F(z+ih)$ whenever the functions of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+[E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- -\tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 2. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. The maximal dissipative property of the transformation is assumed in the construction.

An element
$$F(z) = (F_+(z), F_-(z))$$
of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal dissipative transformation is dissipative.

An element
$$K(w,z) = (K_+(w,z), K_-(w,z))$$
of the graph is defined for every complex number w by

$$K_+(w,z) = [E(z)E(w-\tfrac{1}{2}ih)^- - E^*(z)E(w^-+\tfrac{1}{2}ih)]/[2\pi i(w^-+\tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w+\tfrac{1}{2}ih)^- - E^*(z)E(w^- -\tfrac{1}{2}ih)]/[2\pi i(w^- -\tfrac{1}{2}ih - z)].$$

The identity
$$F_+(w+\tfrac{1}{2}ih) + F_-(w-\tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$
holds for every element
$$F(z) = (F_+(z), F_-(z))$$
of the graph. An element which is orthogonal to itself is orthogonal to every element of the graph.

A Hilbert space $\mathcal{H}$ exists whose elements are entire functions and which contains the function

$$K_+(w, z + \tfrac{1}{2}ih) + K_-(w, z - \tfrac{1}{2}ih)$$

of z as reproducing kernel function for function values at w for every complex number w. An isometric transformation of the graph onto a dense subset of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

This completes the construction of a Hilbert space of entire functions with the desired reproducing kernel functions when the maximal dissipative transformation in the space $\mathcal{H}(E)$ exists. The argument is reversed as in the proof of Theorem 1 to construct the maximal dissipative transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

An element $H(z)$ of the space $\mathcal{H}(E)$ which satisfies the equation

$$H(z) + H(z + ih) = 0$$

vanishes identically if it has a zero. The space of such functions has dimension zero or one. The space of elements $H(z)$ of the space $\mathcal{H}(E)$ which satisfy the equation

$$H(z) + \exp(ha)H(z + ih) = 0$$

for a real number a has the same dimension. A solution of the equation for a real number a is a product

$$\exp(ias)H(z)$$

with $H(z)$ a solution of the equation when a is zero. Since the function

$$\exp(ias)H(z)$$

belongs to the space $\mathcal{H}(E)$ for every real number a when $H(z)$ is a solution of the equation when a is zero, the solution $H(z)$ of the equation vanishes identically.

This completes the proof of the theorem.

The Riemann hypothesis for Hilbert spaces of entire functions admits a formulation which applies to zeta functions having a singularity. Hilbert spaces are replaced in the construction by Krein spaces of Pontryagin index at most one.

Theorem 3. *A nearly maximal dissipative transformation in a weighted Hardy space* $\mathcal{F}(W)$ *is defined for a positive number* h *by taking* $F(z)$ *into* $F(z + ih)$ *whenever the* *functions of z belong to the space if, and only if, the function $W(z - \frac{1}{2}ih)$ admits an analytic extension to the upper half–plane with the possible exception of a simple pole, and a Krein space of Pontryagin index at most one exists which contains the function*

$$[W(z - \tfrac{1}{2}ih)W(w + \tfrac{1}{2}ih)^- + W(z + \tfrac{1}{2}ih)W(w - \tfrac{1}{2}ih)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane and is not a singularity.

Proof of Theorem 3. A Krein space $\mathcal{H}$ of Pontryagin index at most one whose elements are functions of analytic in the upper half–plane, with the possible exception of a simple pole, is constructed when a nearly maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The space is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in a subspace of the graph of codimension one since the adjoint of a nearly maximal dissipative transformation is dissipative on a closed invariant subspace of codimension at most one.

An element

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

of the graph is defined when w is in the half–plane

$$h < iw^- - iw$$

by

$$K_+(w, z) = W(z)W(w - \tfrac{1}{2}ih)^-/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = W(z)W(w + \tfrac{1}{2}ih)^-/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w + \tfrac{1}{2}ih) + F_-(w - \tfrac{1}{2}ih) = \langle F(t), K(w, t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph.

The function

$$[W(z + \tfrac{1}{2}ih)W(w - \tfrac{1}{2}ih)^- + W(z - \tfrac{1}{2}ih)W(w + \tfrac{1}{2}ih)^-]/[2\pi i(w^- - z)]$$

of z is the reproducing kernel function for function values at w in the space $\mathcal{H}$. Division by $W(z + \frac{1}{2}ih)$ acts as an isometric transformation of the space $\mathcal{H}$ onto a Krein space of Pontryagin index at most one. The function

$$\phi(z) = W(z - \tfrac{1}{2}ih)/W(z + \tfrac{1}{2}ih)$$

of z admits an analytic extension to the upper half–plane with the possible exception of a simple pole. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z is the reproducing krenel function for function values at w when w is in the upper half–plane and is not a singularity.

The function $\phi(z)$ has an analytic extension to the upper half–plane with the possible exception of a simple pole. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space for all elements w of the space other than the singularity and acts as reproducing kernel for function values at w. The elements of the space have analytic extensions to the upper half–plane with the exception of the singularity of $\phi(z)$.

The argument is reversed to construct a nearly maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ admits an analytic extension to the upper half–plane with the possible exception of a simple pole and when a Krein space of Pontryagin index at most one exists whose elements are functions analytic in the upper half–plane with the possible exception of a simple pole at the pole of $\phi(z)$ and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is an element of the upper half–plane other than the simple pole.

This completes the proof of the theorem.

The Fourier transformation for the product skew–plane determines a quantized Fourier transformation for the Dedekind skew–plane when applied to functions $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane of character χ which satisfy the identities

$$\kappa(\xi_+, \xi_-)f(\xi_+, \omega\xi_-) = \kappa(\xi_+, \omega\xi_-)f(\xi_+, \xi_-)$$

for every element ω of the adic skew–plane with conjugate as inverse and

$$f(\xi_+, \xi_-) = f(\xi_+\lambda, \xi_-\lambda)$$

for every nonzero element λ of the algebraic skew–plane such that $\lambda^- \lambda$ is a ratio of generating positive integers which are relatively prime to ρ. A function $f(\xi)$ of $|xi$ in the Dedekind skew–plane determines a function $f^\wedge(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which satisfies the identities such that

$$f(\xi) = f^\wedge(\xi, 1)$$

for every element ξ of the Dedekind skew–plane. The function $f(\xi)$ of ξ in the Dedekind skew–plane is square integrable with respect to the canonical measure if, and only if, the function $f(\xi_+, \xi_-)$ of $\xi = (\xi_+, \xi_-)$ in the product skew–plane is square integrable with respect to the canonical measure over the set of elements whose adic component is integral and has integral inverse.

The quantized Fourier transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the function $g(\xi)$ of ξ in the Dedekind skew–plane such that the identity

$$g^\wedge(\xi_+, \xi_-) = \int \exp(\pi i(\xi_+^- \eta_+ + \eta_+^- \xi_-)) \exp(-\pi i(\xi_-^- \eta_- + \eta_-^- \xi_-)) f^\wedge(\eta_+, \eta_-) d\eta$$

holds for almost all elements $\xi = (\xi_+, \xi_-)$ of the product skew–plane with integration with respect to the canonical measure for the product skew–plane over the set of elements whose adic component is integral and has integral inverse.

The Laplace transformation defined by a theta function is reformulated as a transformation which takes functions of a Dedekind variable into functions of a Dedekind variable. The quantized Laplace kernel

$$\phi(\xi) = \kappa(\xi, 1)$$

is the homogeneous harmonic polynomial of degree ν which is obtained from the Laplace kernel when the adic variable is equal to the unit. The conjugate quantized Laplace kernel

$$\phi^\wedge(\xi) = \kappa^\wedge(\xi, 1)$$

applies to the conjugate theta function.

The domain of the quantized Laplace transformation is the Hilbert space of functions $f(\xi)$ of ξ in the Dedekind skew–plane which are square integrable with respect to the canonical measure and which satisfy the identity

$$\phi(\xi) f(\omega\xi) = \phi(\omega\xi) f(\xi)$$

for every element ω of the Dedekind skew–plane with conjugate as inverse.

The range of the Laplace transformation is characterized by the quantized Laplace space of order ν. The space is the Hilbert space of analytic functions $G(z)$ of z in the upper half–plane with finite integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |G(x+iy)|^2 y^{2+\nu} dx dy.$$

The range of the Laplace transformation is the set of analytic functions $F(z)$ of z in the upper half–plane such that the function

$$G(z) = \sum n^{\frac{1}{2}\nu} \tau(n) n^2 F(nz)$$

of z in the upper half–plane belongs to the reduced Laplace space of order ν. Summation is over the generating positive integers n which are relatively prime to ρ.

When ν is positive or ρ is not one, the quantized Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$F(z) = \int (\xi^- \xi) \phi(\xi)^- f(\xi) \theta^*(z\xi^- \xi) d\xi$$

of z in the upper half–plane which is defined as an integral with respect to the canonical measure for the Dedekind skew–plane when the integral is absolutely convergent. When ν is zero and ρ is one, the quantized Laplace transform of the function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$F(z) = \int (\xi^- \xi) f(\xi) [\theta(z\xi^- \xi) - 1] d\xi$$

of z in the upper half–plane which is defined as an integral with respect to the same measure. The identity

$$(1 + 2\nu) \int_0^\infty \int_{-\infty}^{+\infty} |G(x + iy)|^2 y^{2+\nu} dx dy = 2(2\pi)^{-\nu-2} \Gamma(3 + \nu) \int |f(\xi)|^2 d\xi$$

holds with integration on the right with respect to the canonical measure for the Dedekind skew–plane. The quantized Laplace transformation is defined so as to maintain the identity.

A maximal dissipative transformation is defined in the domain of the quantized Laplace transformation when ν is positive or ρ is not one. The transformation takes a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of the Dedekind skew–plane when the identity

$$\int (\xi^- \xi) \phi(\xi)^- g(\xi) \theta^*(z\xi^- \xi) d\xi = (i/z) \int (\xi^- \xi) \phi(\xi)^- f(\xi) \theta^*(z\xi^- \xi) d\xi$$

holds for z in the upper half–plane. The integrals with respect to the canonical measure for the Dedekind skew–plane are interpreted as reduced Laplace transforms when they are not absolutely convergent.

A nearly maximal dissipative transformation is defined in the domain of the quantized Laplace transformation when ν is zero and ρ is one. The transformation takes a function $f(\xi)$ of ξ in the Dedekind skew–plane into a function $g(\xi)$ of ξ in the Dedekind skew–plane when the identity

$$\int (\xi^- \xi) g(\xi) [\theta(z\xi^- \xi) - 1] d\xi = (i/z) \int (\xi^- \xi) f(\xi) [\theta(z\xi^- \xi) - 1] d\xi$$

holds for z in the upper half–plane. The integrals with respect to the canonical measure for the Dedekind skew–plane are interpreted as reduced Laplace transforms when they are not absolutely convergent.

The Mellin transformation defined by a theta function is an application of the quantized Laplace transformation defined by the theta function. The quantized Laplace transform of a function $f(\xi)$ of ξ in the Dedekind skew–plane is the analytic function

$$g(z) = \int (\xi^-\xi)\phi(\xi)^- f(\xi)\theta^*(z\xi^-\xi)d\xi$$

of z in the upper half–plane when v is positive or ρ is not one and is the function

$$g(z) = \int (\xi^-\xi)f(\xi)[\theta(z\xi^-\xi) - 1]d\xi$$

of z in the upper half–plane when ν is zero and ρ is one. The integral with respect to the canonical measure for the Dedekind skew–plane is interpreted as a Laplace transform when it is not absolutely convergent. The Mellin transform is the analytic function

$$F(z) = \int_0^\infty g(it)t^{\frac{1}{2}\nu+1-iz}dt$$

of z in the upper half–plane which is defined when for some positive number a the function $f(\xi)$ of ξ in the Dedekind skew–plane vanishes in the neighborhood

$$\xi^-\xi < a$$

of the origin.

Since the analytic function

$$W(z) = \pi^{-\frac{1}{2}\nu-2+iz}\Gamma(\tfrac{1}{2}\nu + 2 - iz)\zeta(2 - iz)$$

of z in the upper half–plane admits the integral representation

$$W(z) = (\xi^-\xi)^{\frac{1}{2}\nu+2-iz}\int_0^\infty \theta(it)t^{\frac{1}{2}\nu+1-iz}dt$$

when ν is positive or ρ is not one and the integral representation

$$W(z) = (\xi^-\xi)^{2-iz}\int_0^\infty [\theta(it) - 1]t^{1-iz}dt$$

when ν is zero and ρ is one, the identity

$$F(z)/W(z) = \int \phi(\xi)^- f(\xi)(\xi^-\xi)^{-\frac{1}{2}\nu-1+iz}d\xi$$

holds when z is in the upper half–plane with integration with respect to the canonical measure. The identity

$$(1+2\nu)\int_{-\infty}^{+\infty}|F(x+iy)/W(x+iy)|^2dx = 16\pi^2\int|f(\xi)|^2(\xi^-\xi)^{-2y}d\xi$$

holds when y is positive with integration on the right with respect to the canonical measure for the Dedekind skew–plane.

The function $W(z)$ of z is analytic and without zeros in the upper half–plane. An analytic function $F(z)$ of z in the upper half–plane is the Laplace transform of a function which vanishes when $\xi^-\xi < a$ if, and only if, the least upper bound

$$\sup a^{2y}\int_{-\infty}^{+\infty}|F(x+iy)/W(x+iy)|^2dx$$

taken over all positive y is finite.

The weighted Hardy space $\mathcal{F}(W)$ is the set of Mellin transforms of functions $f(\xi)$ of $|xi$ in the Dedekind skew–plane which vanish when

$$\xi^-\xi < 1.$$

If ν is positive or if ρ is not one, a maximal dissipative transformation is defined in the space when h is in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. If ν is zero and ρ is one, a nearly maximal dissipative transformation is defined in the space when h is in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. When v is zero and ρ is one, an isometric transformation of the space onto itself is defined by taking $F(z)$ into $F(-z)$.

A zeta function of order ν and character χ is a function

$$\zeta(s) = \sum\tau(n)n^{-s}$$

which has an analytic extension to the complex plane when ν is positive or ρ is not one and which has an analytic extension to the complex plane with the exception of a simple pole at two when ν is zero and ρ is one. The zeta function has no zeros in the half–plane

$$\mathcal{R}s > \tfrac{3}{2}.$$

Examples of zeta functions associated with a primitive character modulo ρ are constructed from zeta functions associated with the character modulo one. A Laplace kernel associated with the character modulo one is a function

$$\phi(\xi_+)$$

of $\xi = (\xi_+, \xi_-)$ in the product skew–plane which is determined by a homogeneous harmonic polynomial ϕ of degree ν in the Dedekind component of ξ. The zeta function

$$\sum \tau(n) n^{-s}$$

is a sum over the generating positive integers n.

If χ is a primitive character modulo ρ for a generating positive integer ρ, a Laplace kernel κ associated with the character χ is defined by

$$\kappa(\xi_+, \xi_-) = \phi(\xi_+) \chi(\xi_-^- \xi_-)$$

when the adic component of $\xi = (\xi_+, \xi_-)$ is integral. The corresponding zeta function

$$\sum \tau(n) \chi(n) n^{-s}$$

is a sum over the generating positive integers n which are relatively prime to ρ.

Computable examples of zeta functions are obtained when ν is zero since the homogeneous harmonic polynomial ϕ is a constant. The zeta function

$$\sum \tau(n) n^{-s}$$

associated with the character modulo one has coefficient $\tau(n)$ equal to the sum of the odd divisors of n for every generating positive integer n.

Dirichlet zeta functions appear when all primes are generators of adic topology. The Dirichlet zeta function

$$\zeta_\chi(s) = \sum \chi(n) n^{-s}$$

defined by a primitive character χ modulo ρ is a sum over all positive integers n. The Euler product

$$\zeta_\chi(s)^{-1} = \prod (1 - \chi(p) p^{-s})$$

is taken over the primes p. Sum and product define the Dirichlet zeta function in the half–plane

$$\mathcal{R}s > 1.$$

The Dirichlet zeta function admits an analytic extension to the complex plane when ρ is not one. The functional identity states that the analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s} \Gamma(\tfrac{1}{2}s) \zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ and σ by σ^- are linearly dependent when χ is an even character. The analytic extension of the function

$$(\rho/\pi)^{\frac{1}{2}s + \frac{1}{2}} \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) \zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ and χ by χ^- are linearly dependent when χ is an odd character.

The Euler zeta function is the Dirichlet zeta function when ρ is one. The Euler zeta function admits an analytic extension to the complex plane with the exception of a simple pole at one. The Euler functional identity states that the analytic extension of the function

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta_\chi(s)$$

of s and the function obtained on replacing s by $1 - s$ are equal. The conjugate character χ^- is identical with χ since χ is identically one on integral elements of the adic line.

The Euler duplication formula for the gamma function

$$2^s\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) = 2\sqrt{\pi}\ \Gamma(s)$$

$\pi^{\frac{1}{2}}$ is acceptable if $\sqrt{\pi}$ is awkward is applied in relating the functional identities for Dirichlet zeta functions to the functional identities for Hecke zeta functions of order zero.

The identity

$$\sum \chi(n)\tau(n)n^{-s} = (1 - \chi(2)2^{1-s})\zeta_\chi(s)\zeta_\chi(s-1)$$

expresses a zeta function of order zero associated with a primitive character χ modulo ρ in terms of the Dirichlet zeta function associated with the character. The Dirichlet zeta function is the Euler zeta function when ρ is one.

The Dirichlet zeta function has no zeros in the half–plane

$$\mathcal{R}s > \tfrac{1}{2}.$$

The Euler zeta function has no zeros in the half–plane.

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