MAPS CONJUGATING HOLOMORPHIC MAPS IN $\mathbb{C}^n$

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ABSTRACT. If ψ is a bijection from $\mathbb{C}^n$ onto a complex manifold $\mathcal{M}$, which conjugates every holomorphic map in $\mathbb{C}^n$ to an endomorphism in $\mathcal{M}$, then we prove that ψ is necessarily biholomorphic or antibiholomorphic. This extends a result of A. Hinkkanen to higher dimensions. As a corollary, we prove that if there is an epimorphism from the semigroup of all holomorphic endomorphisms of $\mathbb{C}^n$ to the semigroup of holomorphic endomorphisms in $\mathcal{M}$, or an epimorphism in the opposite direction for a doubly-transitive $\mathcal{M}$, then it is given by conjugation by some biholomorphic or antibiholomorphic map. We show also that there are two unbounded domains in $\mathbb{C}^n$ with isomorphic endomorphism semigroups but which are neither biholomorphically nor antibiholomorphically equivalent.

1. Introduction

The question of determining a mathematical structure of an object from its semigroup of endomorphisms, i.e. the set of all maps from this object into itself with composition as a semigroup operation, goes back to at least C. J. Everett and S. M. Ulam [5], [15]. In the most general form this question can be formulated as follows. Suppose we have two sets A and B with a given structure, whose semigroups of endomorphisms compatible with this structure, are isomorphic. Does there exist a bijective map between A and B, which preserves the structure?

K. D. Magill, L. M. Gluskīn, B. M. Schein, L. B. Šneperman, and I. S. Yoroker studied the question of determining a topological space from its semigroup of continuous endomorphisms. See a survey in [12], [6].

To the best of the authors' knowledge, L. Rubel was the first who raised the question of determining a complex space from its semigroup of holomorphic endomorphisms. In 1993, A. Eremenko proved that every two Riemann surfaces that admit non-constant bounded holomorphic functions, and whose semigroups of holomorphic endomorphisms are isomorphic, are necessarily biholomorphically or antibiholomorphically equivalent. This result was extended by S. Merenkov in [13] to bounded domains in $\mathbb{C}^n$.

On the other hand, A. Hinkkanen [7] proved in 1992 that there exist unbounded domains in $\mathbb{C}$ whose semigroups of holomorphic endomorphisms are isomorphic, but the domains are not even homeomorphic. In the same paper A. Hinkkanen studied another question raised by L. Rubel. Namely, he proved that if ψ is a one-to-one function of the plane onto itself (but with no assumption of continuity), such that $\psi \circ f \circ \psi^{-1}$ is entire whenever f is entire, then ψ has the form $\psi(z) = az + b$, or $\psi(z) = a\bar{z} + b$, where a and b are complex numbers with $a \neq 0$; i.e., ψ is a biholomorphic or antibiholomorphic automorphism.

In higher dimensions, any analog of A. Hinkkanen's theorem must take into account the fact that the automorphism group of $\mathbb{C}^n$ is quite large, since in $\mathbb{C}^2$, for example, there are biholomorphic maps of the form $\psi(z_1, z_2) = (z_1, z_2 + g(z_1))$, where g is an arbitrary entire function. However, one may hope that ψ is still a biholomorphic or antibiholomorphic automorphism. The main theorem, Theorem 1, of the present paper asserts that this is indeed the case.

Note that the set of all holomorphic endomorphisms of a complex manifold $\mathcal{M}$ forms a semigroup (with unit) under composition. We denote this semigroup by $E(\mathcal{M})$. If $\mathcal{M} = \mathbb{C}^n$, we denote the semigroup by E.

Theorem 1. *If ψ is a bijection of $\mathbb{C}^n,\ n \geq 2$ onto a complex manifold $\mathcal{M}$, such that $\psi \circ f \circ \psi^{-1} \in E(\mathcal{M})$ for every map $f \in E$, then ψ is biholomorphic or antibiholomorphic.*

As in the one-dimensional case [7], it is not sufficient to assume that $\psi \circ f \circ \psi^{-1} \in E(\mathcal{M})$ for every polynomial map f in order to conclude that ψ is a homeomorphism. The reason is that there are non-continuous field automorphisms of $\mathbb{C}$ [11]. If ξ is such an automorphism, then we can take $\psi(z_1, \dots, z_n) = (\xi(z_1), \dots, \xi(z_n))$. The conjugation by ψ is an automorphism of semigroups of polynomial maps in $\mathbb{C}^n$, but ψ is not continuous.

We say that a complex manifold $\mathcal{N}$ is *doubly-transitive* if $E(\mathcal{N})$ is doubly-transitive, i.e. if for every pair z_1, z_2 of distinct points in $\mathcal{N}$ and every other pair of points w_1, w_2 in $\mathcal{N}$, there exists $f \in E(\mathcal{N})$, such that $f(z_m) = w_m,\ m = 1, 2$. We say that $\mathcal{N}$ is *weakly doubly-transitive* if in the previous definition we replace the assumption that w_2 arbitrary by requiring that it has to be sufficiently close to w_1. Clearly, every doubly-transitive complex manifold is weakly doubly-transitive, and $\mathbb{C}^n$ is doubly-transitive. As a corollary to Theorem 1, we prove the following

Theorem 2. *If there exists an epimorphism of semigroups $\phi :\ E \to E(\mathcal{M})$, where $\mathcal{M}$ is a complex manifold consisting of more than one point, then*

$$\phi(f) = \psi \circ f \circ \psi^{-1}, \quad \forall f \in E, \tag{1}$$

for some biholomorphic or antibiholomorphic map $\psi :\ \mathbb{C}^n \to \mathcal{M}$.

If there exists an epimorphism of semigroups $\varphi :\ E(\mathcal{M}) \to E$, where $\mathcal{M}$ is a weakly doubly-transitive complex manifold, then

$$\varphi(f) = \eta \circ f \circ \eta^{-1}, \quad \forall f \in E(\mathcal{M}), \tag{2}$$

for some biholomorphic or antibiholomorphic map $\eta :\ \mathcal{M} \to \mathbb{C}^n$.

We note that the converse to this theorem is trivial. If ψ is a biholomorphic or antibiholomorphic map from $\mathbb{C}^n$ to $\mathcal{M}$, then the map $f \mapsto \psi \circ f \circ \psi^{-1}$ is an isomorphism between the semigroups. Similarly, we get an isomorphism of semigroups if there exists an (anti)biholomorphic map $\eta :\ \mathcal{M} \to \mathbb{C}^n$. In particular, we obtain the following corollary, which follows immediately from the previous remarks plus the fact that an antibiholomorphic equivalence from $\mathbb{C}^n$ to $\mathcal{M}$ implies a biholomorphic equivalence simply by composing with the involution $z \mapsto \overline{z}$.

Corollary 1. *Given a complex manifold $\mathcal{M}$, the endomorphism semigroup of $\mathcal{M}$ is isomorphic to the endomorphism semigroup of $\mathbb{C}^n$ if and only if $\mathcal{M}$ is biholomorphic to $\mathbb{C}^n$.*

The first part of Theorem 2 is in some sense quite surprising because, among the complex manifolds of dimension n, $\mathbb{C}^n$ has a large and complicated semigroup of endomorphisms (compare the simple semigroups in Theorem 3 below). Yet the equivalence given above requires only the existence of an epimorphism from the "large" semigroup E onto $E(\mathcal{M})$.

Also, applying methods used by D. Varolin [16], any Stein manifold $\mathcal{M}$ with the (volume) density property is doubly-transitive, and hence can be used in the second part of Theorem 2. Indeed, the fact that the manifold is Stein implies that any single point is a holomorphically convex set. Then for distinct points p_1, p_2, q_1, q_2 in $\mathcal{M}$, Theorem 0.2 of [16] with $K = \{p_2\}$ implies that there is an automorphism, f_1, of $\mathcal{M}$ so that $f_1(p_1) = q_1$ and $f_1(p_2) = p_2$. Likewise, there is an automorphism, f_2, of $\mathcal{M}$ so that $f_2(p_2) = q_2$ and $f_2(q_1) = q_1$. Thus for the function $f = f_2 \circ f_1$ we have $f(p_j) = q_j$. If p_1, p_2 are distinct and q_1, q_2 are arbitrary (and possibly one or both of them is the same as p_1 or p_2), then we can first choose $z_1 \neq z_2$, distinct from the previous four points, map p_j to z_j, and then z_j to q_j (using a constant map if $q_1 = q_2$). Hence $\mathcal{M}$ is doubly-transitive.

We mention also a recent paper by S. G. Krantz [10], where he studies the question of determination of a domain in complex space by its automorphism group. Of course a domain possesses more endomorphisms than automorphisms. Therefore the ability to determine a domain from its automorphism group implies the ability to determine a domain from its endomorphism semigroup. Our Theorem 2 differs from Krantz's result in that, first of all, we assume the existence of an epimorphism between semigroups, rather than an isomorphism. Secondly, the information we assume has a purely algebraic character, i.e. the existence of an algebraic epimorphism, and not a topological one; i.e., we make no a priori assumptions about continuity. To our knowledge it is an open question if the existence of a purely algebraic isomorphism between the automorphism group of $\mathbb{C}^n$ and the automorphism group of $\mathcal{M}$ implies the (anti)biholomorphic equivalence of these manifolds. However, one result along these lines is contained in the work of P. Ahern and W. Rudin [1]. They showed that $\mathrm{Aut}(\mathbb{C}^n)$ is sensitive to the dimension, i.e. if $1 \leq m < n$, then the groups $\mathrm{Aut}(\mathbb{C}^m)$ and $\mathrm{Aut}(\mathbb{C}^n)$ are not algebraically isomorphic.

To complete the analogy with Hinkkanen's results, we show the existence of two unbounded domains in $\mathbb{C}^n$ with isomorphic endomorphism semigroups but which are not (anti)biholomorphically equivalent. This should be compared with Merenkov's result [13], in which it is shown that for two bounded domains in $\mathbb{C}^n$, an isomorphism between the endomorphism semigroups implies the (anti)biholomorphic equivalence between the two domains.

Theorem 3. *There exist unbounded domains D_1 and D_2 in $\mathbb{C}^n$ so that the endomorphism semigroups $E(D_1)$ and $E(D_2)$ are isomorphic but such that there is no biholomorphic or antibiholomorphic map from D_1 onto D_2.*

The paper is organized as follows. In Section 2 we prove that the map ψ in Theorem 1 is a homeomorphism, using the notion of a Fatou-Bieberbach domain and pose a question about Fatou-Bieberbach domains in Stein manifolds with the density property. Section 3 and Section 4 are devoted to the proof that ψ is biholomorphic or antibiholomorphic. In Section 5 we give a proof of Theorem 2, and in Section 6, we prove Theorem 3.

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2. Fatou-Bieberbach domains and continuity of ψ

Below we assume that $n \geq 2$.

Let FB denote the set of Fatou-Bieberbach domains, i.e. proper domains in $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$. A domain from this set will be called an *FB-domain*, and a biholomorphic map from $\mathbb{C}^n$ onto an FB-domain will be called an *FB-map*.

We denote by $\Delta(r)$ the disk in $\mathbb{C}$ centered at 0 and of radius r, and by $\Delta^k(r)$ the k-fold product of $\Delta(r)$. In [3] it was proved that there exists an FB-domain in $\mathbb{C}^n$ which is contained in the union of $\Delta(r^2) \times \Delta^{n-1}(r)$ and the set $S_1 = \{z = (z_1, \dots, z_n) : \ |z_1| \geq r^2 - 3r + ||(z_2, \dots, z_n)||_\infty\}$, for some $r > 4$. This FB-domain is a basin of attraction at 0 of a polynomial map that fixes the origin. Therefore 0 is in the FB-domain. By using rotations, we deduce that there exists an FB-domain which contains the origin and is contained in the union of $\Delta^{k-1}(r) \times \Delta(r^2) \times \Delta^{n-k}(r)$ and the set $S_k = \{z = (z_1, \dots, z_n) : \ |z_k| \geq r^2 - 3r + ||(z_1, \dots, \hat{z}_k, \dots, z_n)||_\infty\}$ for some $r > 4$, $\forall k = 1, \dots, n$, where $\hat{z}_k$ means that z_k is omitted. It follows that in $\mathbb{C}^n$ there are n FB-domains whose intersection is non-empty and bounded. By post-composing the corresponding FB-maps with contractions, and using translations, we conclude that intersections of FB-domains form a base of neighborhoods at each point of $\mathbb{C}^n$.

Now, under the assumptions of Theorem 1, we can prove that ψ is continuous. Let $f_1, \dots, f_n$ be FB-maps as above so that the intersection of their images is bounded. Then, by assumption, $g_i = \psi \circ f_i \circ \psi^{-1}, \ i = 1, \dots, n$ are holomorphic maps in $\mathcal{M}$. Moreover,

$$\psi(f_1(\mathbb{C}^n) \cap \cdots \cap f_n(\mathbb{C}^n)) = \psi(f_1(\mathbb{C}^n)) \cap \cdots \cap \psi(f(\mathbb{C}^n))$$

and since each g_i is an injective holomorphic map, $\psi(f_i(\mathbb{C}^n)) = g_i(\psi(\mathbb{C}^n)) = g_i(\mathcal{M})$ is an open set. It follows that ψ is an open map. Using this plus the fact that ψ is a bijection of $\mathbb{C}^n$ onto a manifold, we see that $\psi^{-1}(K)$ is compact for each compact $K \subset \mathcal{M}$. With a standard argument, we conclude that ψ is a homeomorphism. In particular [8], the dimension of $\mathcal{M}$ must be equal to n.

Note that [16] implies that a Stein manifold $\mathcal{M}$ with the (volume) density property has an injective holomorphic map $F : \ \mathcal{M} \to \mathcal{M}$ with $F(\mathcal{M}) \neq \mathcal{M}$. Since our proof of the continuity of ψ in Theorem 1 is based on the existence of special maps of this form in $\mathbb{C}^n$, it is an interesting open question whether such $\mathcal{M}$ can be shown to have a base of neighborhoods given by finite intersections of injective images of $\mathcal{M}$. If so, then it should be possible to replace $\mathbb{C}^n$ in Theorem 1 by any manifold with these properties.

3. Local linearization of maps

Having shown that ψ is continuous, we proceed as in [13] to prove that ψ is biholomorphic or antibiholomorphic using a simultaneous linearization of certain commuting maps. Let $a \in \mathbb{C}^n$ be an arbitrary point, and $b = \psi(a)$. It is enough to show that ψ is biholomophic or antibiholomorphic in a neighborhood of a.

A set $\mathcal{P} = \{p_i\}_1^n$ will be called a *system of projections* at o in a complex manifold $\mathcal{N}, \ o \in \mathcal{N}$, if it consists of holomorphic maps in $E(\mathcal{N})$ that fix o, and satisfy:

(1) $p_i \neq o, \quad \forall i$;
(2) $p_i^2 = p_i, \quad \forall i$;
(3) $p_i \circ p_j = o, \quad \forall i \neq j$,

where $p_i^2 = p_i \circ p_i$. Let f be a biholomorphic map of $\mathcal{N}$ onto itself, that commutes with all maps of some system of projections $\mathcal{P}$ at o, and fixes o. We also assume that for every neighborhood U of o, and every compact set K, there exists an iterate of

f that brings K into U, i.e. there exists a positive integer l such that $f^l(K) \subset U$. Such a map f clearly exists if $\mathcal{N} = \mathbb{C}^n$, since we can take it to be a contraction at o, and $\{p_i\}$ to be standard projections. Now we introduce a subsemigroup I_f of $E(\mathcal{N})$, consisting of all maps h that satisfy all the properties that f does, with the same system of projections $\mathcal{P}$, and such that h commutes with f. For reasons that will be clear later, we call the triple $\{f, \mathcal{P}, I_f\}$ a *linearizing triple*. It is immediate to verify that all properties listed for a linearizing triple are preserved under conjugation by ψ, i.e. if $\{f, \mathcal{P}, I_f\}$ is a linearizing triple in $\mathbb{C}^n$ at a, then $\{g, \mathcal{Q}, I_g\}$ is a linearizing triple in $\mathcal{M}$ at b, where $g = \psi \circ f \circ \psi^{-1}$, $\mathcal{Q} = \psi \circ \mathcal{P} \circ \psi^{-1}$.

We note here that in general it is impossible to linearize a holomorphic map in a neighborhood of its attracting fixed point due to the presence of resonances among the eigenvalues of its linear part [2], [14]. However, as seen in the following proposition, under the assumption that $h \circ p_i = p_i \circ h$, $\forall i = 1, \dots, n$, the local linearization of $h \in I_f$ is possible.

Proposition 1. *For every linearizing triple $\{f, \mathcal{P}, I_f\}$ in a complex manifold $\mathcal{N}$ at o, there exists a biholomorphic map θ from a neighborhood of o onto a neighborhood of the origin in $\mathbb{C}^n$, such that for every $h \in I_f$, in some neighborhood of o,*

$$\theta \circ h = \Lambda_h \circ \theta, \tag{3}$$

$$\theta \circ p_i = P_i \circ \theta, \quad \forall i, \tag{4}$$

where Λ_h is a diagonal linear map $(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n)$, λ_i, $i = 1, \dots, n$ satisfy $0 < |\lambda_i| < 1$, and are eigenvalues of the linear part of h at o, and P_i is a diagonal matrix similar to the linear part of p_i at o.

The proof of this proposition follows the same arguments as in [13], and therefore we give only an outline here. Because of the property that for every arbitrary compact set and every neighborhood of o, some iterate of f brings the compact set into that neighborhood, it follows that the eigenvalues of the linear part of f at o are smaller than 1 in absolute value. Using the fact that projections are locally linearizable [9], and the commutativity relations $h \circ p_i = p_i \circ h$, $\forall i$, the problem about local linearization reduces to the one-dimensional Schröder equation, which is solved [4]. That all maps h are linearized by the same biholomorphic map θ follows from the uniqueness of the solution to the Schröder equation, and the commutativity relations between f, h, and p_i.

We see in the following lemma that if $\mathcal{N} = \mathbb{C}^n$, then all invertible diagonal linear maps whose entries are smaller than 1 in absolute value appear in (3).

Lemma 1. *The map θ extends to a biholomorphic map on $\mathcal{N}$. Moreover, if $\mathcal{N} = \mathbb{C}^n$, and Λ is a diagonal linear map*

$$(z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n), \quad 0 < |\lambda_i| < 1, \ i = 1, \dots, n,$$

then the extended map θ is a biholomorphism of $\mathbb{C}^n$ onto itself, and there exists $h \in I_f$, such that

$$\theta \circ h = \Lambda \circ \theta. \tag{5}$$

Proof. First we show that the map θ extends to a biholomorphic map on the whole $\mathcal{N}$. This can be seen by using the formula

$$\theta = \Lambda_f^{-k} \circ \theta \circ f^k, \quad k = 1, 2, \dots \tag{6}$$

Because of the property that for every compact subset K of $\mathbb{C}^n$ and every neighborhood U of o, some iterate of f brings K into U, it follows from (6) that θ can be extended to larger and larger sets, until its domain fills the whole $\mathcal{N}$. Since f and Λ_f are biholomorphisms, θ is injective, and hence a biholomorphism on $\mathcal{N}$.

When $\mathcal{N} = \mathbb{C}^n$, the inverse of θ has a representation similar to (6), and therefore θ is onto. Consider a map $h = \theta^{-1} \circ \Lambda \circ \theta \in E$. It is a biholomorphism of $\mathbb{C}^n$ onto itself, and it commutes with every p_i, which follows from (4). Since all entries of Λ are less than 1 in absolute value, it is clear that for every compact set K and a neighborhood U of 0, some iterate of h brings K into U. Using (3), we conclude that h commutes with f, and thus it belongs to I_f. $\square$

4. Matrix equation

Using the results of the previous section, we convert the statement of Theorem 1 to a linearized version, thus reducing the problem to determining the exact form of the solution of a matrix equation ((7) below). By finding this solution, we obtain an explicit expression for a map L defined below, which is conjugate to ψ via biholomorphic maps. This, with some more effort, will lead us to the proof that ψ is either biholomorphic, or antibiholomorphic.

We denote by $\mathcal{D}_0$ the set of invertible diagonal $n \times n$ matrices whose entries are less than 1 in absolute value, and we denote by $\mathcal{D}_n$ the set of all diagonal $n \times n$ matrices. We identify $\mathcal{D}_0$ with the set of diagonal linear maps, and $\mathcal{D}_n$ with a multiplicative semigroup $\mathbb{C}^n$ in the obvious way, and consider a topology on $\mathcal{D}_n$ induced by the standard topology on $\mathbb{C}^n$.

In the previous section, we showed that if $\{f, \mathcal{P}, I_f\}$ is a linearizing triple in $\mathbb{C}^n$, then $\theta : \mathbb{C}^n \to \mathbb{C}^n$ conjugates I_f to the set of diagonal linear maps, which is isomorphic to $\mathcal{D}_0$. Similarly, for a linearizing triple $\{g, \mathcal{Q}, I_g\}$, where $g = \psi \circ f \circ \psi^{-1}$, $\mathcal{Q} = \psi \circ \mathcal{P} \circ \psi^{-1}$, I_g is conjugated by a biholomorphic map $\eta : \mathcal{M} \to \mathbb{C}^n$ to a subset of $\mathcal{D}_0$.

We define a homeomorphism L on $\mathbb{C}^n$ by $L = \eta \circ \psi \circ \theta^{-1}$. For every Λ in $\mathcal{D}_0$ we have

$$\begin{aligned} L \circ \Lambda \circ L^{-1} &= \eta \circ \psi \circ \theta^{-1} \circ \Lambda \circ \theta \circ \psi^{-1} \circ \eta^{-1} \\ &= \eta \circ \psi \circ h \circ \psi^{-1} \circ \eta^{-1} = \eta \circ j \circ \eta^{-1} = M, \end{aligned}$$

where $h = \theta^{-1} \circ \Lambda \circ \theta \in I_f$; $j = \psi \circ h \circ \psi^{-1}$, $M = \eta \circ j \circ \eta^{-1}$, $M \in \mathcal{D}_0$. Therefore the conjugation by L defines an injective map R from $\mathcal{D}_0$ to $\mathcal{D}_0$, $R(\Lambda) = L \circ \Lambda \circ L^{-1}$, which is trivially multiplicative, i.e. $R(\Lambda'\Lambda'') = R(\Lambda')R(\Lambda'')$, $\Lambda', \Lambda'', \Lambda'\Lambda'' \in \mathcal{D}_0$. Since R is continuous, it extends to a multiplicative map, which will also be denoted by R for convenience, from the subset $\overline{\mathcal{D}}_0$ of $\mathcal{D}_n$ that consists of all matrices in $\mathcal{D}_n$ with entries less than or equal to 1 in absolute value, into itself. Indeed, for every matrix Γ in $\overline{\mathcal{D}}_0$, the image $R(\Gamma)$ also belongs to $\overline{\mathcal{D}}_0$, which follows from the continuation process. Now we extend R to all of $\mathcal{D}_n$ as follows. Let Γ be an arbitrary matrix in $\mathcal{D}_n$. We choose $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ in $\mathcal{D}_0$ such that $\sum_{i=1}^n |\lambda_i| \leq 1$ and $\Gamma\Lambda \in \overline{\mathcal{D}}_0$. Define

$$R(\Gamma) = R(\Gamma\Lambda)R(\Lambda)^{-1}.$$

The extended map R is well defined. Indeed, if Λ' is a different matrix with the same properties as Λ, then $R(\Gamma\Lambda)R(\Lambda') = R(\Gamma\Lambda')R(\Lambda)$, and the conclusion follows from the commutativity relations for diagonal matrices. The map R is clearly injective, and multiplicative,

$$R(\Lambda'\Lambda'') = R(\Lambda')R(\Lambda''), \quad \Lambda',\ \Lambda'' \in \mathcal{D}_n. \tag{7}$$

We denote by δ_i the diagonal $n \times n$ matrix which has 1 as its ii'th entry and all other entries 0. The system $\{\delta_i\}_{i=1}^n$ is clearly the only one in $\mathcal{D}_n$ which satisfies $\delta_i \neq 0,\ \delta_i^2 = \delta_i,\ \delta_i\delta_j = 0,\ \forall i \neq j$. Therefore, injectivity of R and (7) imply that $R(\delta_i) = \delta_j,\ \forall i$, where $j = j(i)$ is a permutation. In particular,

$$R(\delta_i\Lambda) = \delta_j R(\Lambda). \tag{8}$$

If we denote the jj'th entry of the diagonal matrix $R(\Lambda)$ by $r_j(\lambda_1, \dots, \lambda_n)$, then (8) implies that r_j depends on λ_i only. For convenience, we write $r_j(\lambda_1, \dots, \lambda_n) = r_j(\lambda_i)$. We can rewrite (7) as

$$r_j(\lambda_i'\lambda_i'') = r_j(\lambda_i')r_j(\lambda_i''), \quad i = 1, \dots, n,\ j = j(i). \tag{9}$$

As in [4], for every $j = j(i)$, the equation (9) has either the constant solution $r_j(\lambda_i) = 1$, or

$$r_j(\lambda_i) = \lambda_i^{\alpha_{ij}}\overline{\lambda}_i^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{C},\ \alpha_{ij} - \beta_{ij} = \pm 1, \tag{10}$$

where the last relation between α_{ij} and β_{ij} is forced by the injectivity of the map R. Using (10), we can obtain an explicit expression for L :

$$\begin{aligned} L(z_1, \dots, z_n) &= \operatorname{diag}(z_{i(1)}^{\alpha_{i(1)1}}\overline{z}_{i(1)}^{\beta_{i(1)1}}, \dots, z_{i(n)}^{\alpha_{i(n)n}}\overline{z}_{i(n)}^{\beta_{i(n)n}})L(1, \dots, 1) \\ &= B(z_1^{\alpha_1}\overline{z}_1^{\beta_1}, \dots, z_n^{\alpha_n}\overline{z}_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1,\ i = 1, \dots, n, \end{aligned} \tag{11}$$

where $i = i(j)$ is an inverse permutation to $j = j(i)$, and B is a constant matrix.

By definition, $\psi = \eta^{-1} \circ L \circ \theta$. From the expression (11) for L we can conclude that ψ is $\mathbb{R}$-differentiable and non-degenerate in $\mathbb{C}^n \setminus \theta^{-1}(A)$, where $A = \cup_{k=1}^n \{(z_1, \dots, z_n) :\ z_k = 0\}$. Since the same conclusion is true for every set obtained by translation of A in $\mathbb{C}^n$, and since θ is a biholomorphism of $\mathbb{C}^n$ onto itself, the map ψ is $\mathbb{R}$-differentiable and non-degenerate everywhere in $\mathbb{C}^n$. However, this is possible if and only if $\alpha_i + \beta_i = 1,\ i = 1, \dots, n$. Combining this with the equation $\alpha_i - \beta_i = \pm 1$, we deduce that either $\alpha_i = 1,\ \beta_i = 0$, or $\alpha_i = 0,\ \beta_i = 1$.

It remains to show that either $\alpha_i = 1,\ \forall i$, or $\alpha_i = 0,\ \forall i$. To get a contradiction, suppose that

$$L(z_1, \dots, z_n) = B(\dots, z_i, \dots, \overline{z}_j, \dots).$$

Then

$$L^{-1}(w_1, \dots, w_n) = (\dots, l_i(w_1, \dots, w_n), \dots, l_j(\overline{w}_1, \dots, \overline{w}_n), \dots),$$

where $l_i,\ l_j$ are nonconstant, linear holomorphic functions. Let $\theta = (\theta_1, \dots, \theta_n)$. We consider a map $h \in E$ in the form

$$h = \theta^{-1}(\dots, \theta_i\theta_j, \dots, \theta_j, \dots)\theta,$$

where $\theta_i\theta_j$ is the i'th coordinate, and θ_j is the j'th coordinate. Using the definition of L, we obtain

$$\begin{aligned} \eta \circ \psi \circ h \circ \psi^{-1} \circ \eta^{-1} &= L \circ \theta \circ h \circ \theta^{-1} \circ L^{-1} \\ &= B'(\dots, l_i(w_1, \dots, w_n)l_j(\overline{w}_1, \dots, \overline{w}_n), \dots, \overline{l_j(\overline{w}_1, \dots, \overline{w}_n)}, \dots) \end{aligned}$$

for some constant matrix B'. This map, and hence $\psi \circ h \circ \psi^{-1}$ is not holomorphic though, which is a contradiction. □

5. Epimorphism between semigroups

In this section we give a proof of Theorem 2.

For a complex manifold $\mathcal{N}$ we denote by $C(\mathcal{N})$ the subsemigroup of $E(\mathcal{N})$ consisting of constant maps. If $\mathcal{N} = \mathbb{C}^n$, we denote $C = C(\mathbb{C}^n)$. In other words,

$$c \in C(\mathcal{N}) \text{ if and only if } \forall f \in E(\mathcal{N}),\ c \circ f = c. \tag{12}$$

There is a natural one-to-one correspondence between the constant maps in $E(\mathcal{N})$ and points of $\mathcal{N}$: for each $z \in \mathcal{N}$ there exists c_z that maps $\mathcal{N}$ to z, and conversely, for each $c \in C(\mathcal{N})$ there exists $z \in \mathcal{N}$, such that $c = c_z$.

Lemma 2. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be complex manifolds, with $\mathcal{N}_1$ being weakly doubly-transitive. Let $\Phi :\ E(\mathcal{N}_1) \to E(\mathcal{N}_2)$ be an epimorphism of semigroups. Then there exists a bijective map $\Psi :\ \mathcal{N}_1 \to \mathcal{N}_2$ such that*

$$\Phi(f) = \Psi \circ f \circ \Psi^{-1}, \quad \forall f \in E(\mathcal{N}_1). \tag{13}$$

Proof. Because of (12), and the assumption that Φ is an epimorphism, for every $c \in C(\mathcal{N}_1)$ we have that $\Phi(c) \in C(\mathcal{N}_2)$. Now we can define a map $\Psi :\ \mathcal{N}_1 \to \mathcal{N}_2$ as follows

$$\Psi(z) = w \text{ if and only if } \Phi(c_z) = c_w.$$

Let f be arbitrary map in $E(\mathcal{N}_1)$. Then

$$f \circ c_z = c_{f(z)}. \tag{14}$$

Applying Φ to both sides of (14), we obtain

$$\Phi(f) \circ c_{\Psi(z)} = c_{\Psi(f(z))},$$

which is equivalent to

$$\Phi(f) \circ \Psi = \Psi \circ f. \tag{15}$$

Equation (15) implies surjectivity of Ψ. Indeed, since Φ is an epimorphism, for every $w \in \mathcal{N}_2$, there exists $f \in E(\mathcal{N}_1)$, such that $\Phi(f) = c_w$. Therefore, by (15), $\Psi \circ f(z) = w,\ \forall z \in \mathcal{N}_1$, which implies that Ψ is onto.

We prove that Ψ is injective by showing that for every $w \in \mathcal{N}_2$ the full preimage $S_w = \Psi^{-1}(w)$ consists of one point. Assume, by contradiction, that S_w consists of more than one point for some w. It cannot be all of $\mathcal{N}_1$, since Ψ is onto. Let z_1 be a point in S_w, such that in arbitrary neighborhood of it there exist a point in S_w, and a point in $\mathcal{N}_1 \setminus S_w$. Let z_2 be arbitrary point in S_w, different from z_1. From our assumption that $\mathcal{N}_1$ is weakly doubly-transitive, it follows that there exists $h \in E(\mathcal{N}_1)$, such that $h(z_1) = w_1 \in S_w$, and $h(z_2) = w_2 \notin S_w$. Evaluating $\Phi(h)$ at w, and applying (15) we have

$$\Phi(h)(w) = \Phi(h) \circ \Psi(z_1) = \Psi \circ h(z_1) = \Psi(w_1) = w,$$
$$\Phi(h)(w) = \Phi(h) \circ \Psi(z_2) = \Psi \circ h(z_2) = \Psi(w_2) \neq w,$$

which is a contradiction. Thus we proved that Ψ is a bijection, and the equation (13) follows from (15). □

The first part of Theorem 2 now follows from Lemma 2 and Theorem 1, if we choose $\mathcal{N}_1 = \mathbb{C}^n$, and $\mathcal{N}_2 = \mathcal{M}$. The second part follows if we take $\mathcal{N}_1 = \mathcal{M}$, $\mathcal{N}_2 = \mathbb{C}^n$, and observe that equation (13) implies that Φ is an isomorphism. □

6. Isomorphic semigroups for inequivalent manifolds

In this section we prove Theorem 3. We construct the domains D_1 and D_2 by taking direct sums of n copies of domains as in Hinkkanen [7]. From [7], we know that there exist unbounded domains U_1, U_2 in $\mathbb{C}$ such that U_1 is neither conformally nor anticonformally equivalent to U_2, and such that $E(U_1)$, and $E(U_2)$ are isomorphic and consist of the constants plus the identity. One such choice of domains is given by $U_1 = \mathbb{C} \setminus \{0, 1, 2\}$, and $U_2 = \mathbb{C} \setminus \{0, 1, 2, \dots\}$. We set $D_1 = U_1 \times \cdots \times U_1$, $D_2 = U_2 \times \cdots \times U_2$, and verify that for these domains the conclusion of Theorem 3 holds.

Let $F \in E(D_m)$, $m = 1, 2$. Then each component f_j of F maps D_m holomorphically into U_m. Therefore, by the choice of U_m, if we fix all z_k, $k = 1, \dots, n, k \neq i$, then the induced map $g_j(z_i)$ is in $E(U_m)$, hence is either a constant map or the identity. Since f_j is a continuous function in a domain, which is a direct sum of domains in $\mathbb{C}$, we conclude that it is identically equal to either a constant, or z_i for some $i = 1, \dots, n$. Using this description of the elements in D_m, we can easily show that $E(D_1)$ and $E(D_2)$ are isomorphic. Let ξ be a bijective map from U_1 onto U_2. If F is an endomorphism of D_1, whose components are $f_1, \dots, f_n$, then we set $\phi(F)$ to be an endomorphism of D_2, whose j'th component is z_i if $f_j = z_i$, and $\xi(c)$ if $f_j = c$, a constant map. It is a simple matter to verify that the map ϕ, so defined, is an isomorphism of semigroups.

To show that D_1 and D_2 are not biholomorphically or antibiholomorphically equivalent, we argue by contradiction. Suppose first that there exists a biholomorphic map F from D_1 onto D_2. Let g be a non-constant restriction of a component of F to a coordinate axis. Such a component exists, since otherwise the map F would be constant. Since g omits more than two points, each of the points $0, 1, 2, \infty$ must be a removable singularity or a pole. Therefore, g extends to a rational map. But this is a contradiction, because g omits infinitely many points. Similarly, we arrive at a contradiction by assuming that there exists an antibiholomorphic map from D_1 onto D_2, and applying the same argument to a conjugate map. □

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