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Cnoidal wave solutions to Boussinesq systems

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Abstract
In this paper, two different techniques will be employed to study the cnoidal wave solutions of the Boussinesq systems. First, the existence of periodic travelling-wave solutions for a large family of systems is established by using a topological method. Although this result guarantees the existence of cnoidal wave solutions in a parameter region in the period and phase speed plane, it does not provide the uniqueness nor the non-existence of such solutions in other parameter regions. The explicit solutions are then found by using the Jacobi elliptic function series. Some of these explicit solutions fall in the parameter region where the cnoidal wave solutions are proved to exist, and others do not; so the method with Jacobi elliptic functions provides additional cnoidal wave solutions. In addition, the explicit solutions can be used in many ways, such as in testing numerical code and in testing the stability of these waves.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper, the existence of periodic travelling-wave solutions to a restricted four-parameter family of Boussinesq systems

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{aligned} \tag{1.1}$$

that was put forward by Bona, Chen and Saut (see [4, 5]) to approximate the motion of small-amplitude long waves on the surface of an ideal fluid under the force of gravity and in situations where the motion is sensibly two-dimensional, will be discussed. The independent variable x is proportional to distance in the direction of propagation while t is proportional to elapsed time, with time scale $\sqrt{h_0/g}$, where g is the gravitational force and h_0 (scaled to 1) the undisturbed water depth. The dependent variables η and u have the following physical interpretation. The quantity $\eta(x,t)$ is deviation relative to the undisturbed surface, so $\eta(x,t)+h_0$ corresponds to the total depth of the liquid at (x,t) while $u(x,t)$ is the horizontal velocity field at the height θh_0, where $0 \leqslant \theta \leqslant 1$. From the derivation of (1.1), the parameters a, b, c, d are not independently specified but must obey the consistency conditions

$$a+b=\frac{1}{2}\left(\theta^2-\frac{1}{3}\right) \qquad \text{and} \qquad c+d=\frac{1}{2}(1-\theta^2)\geqslant 0. \tag{1.2}$$

If a_0 connotes a typical wave amplitude and μ a typical wavelength, the condition of 'small amplitude and long wavelength' just mentioned amounts to

$$\alpha=\frac{a_0}{h_0}\ll 1, \qquad \beta=\frac{h_0^2}{\mu^2}\ll 1, \qquad \frac{\alpha}{\beta}=\frac{a_0\mu^2}{h_0^3}\approx 1.$$

As with one-way models, there are potentially many different but formally equivalent Boussinesq systems. The plethora of possibilities is owed in the main to the choice of the dependent variable u at different water depths and to the fact that the lower-order relations can be used systematically to alter the higher-order terms without disturbing the formal level of approximation. Systems in (1.1) are first order approximations in α and β to Euler's equation, justified rigorously by Bona, Colin and Lannes (cf [6]). We refer the reader to the papers [4, 5, 12] for a further discussion about the derivation and well-posedness of these systems.

In this paper, we extend the results obtained in [7] (see also [1, 2]) for a single equation to systems of equations which are suitable for more general physical situations (namely, the waves are no longer assumed to be uni-directional). It is noted later in remark 3.2 that, even though we will be able to transform the system of ordinary differential equations in the travelling frame to a single equation, the technique used in [7] still does not apply. We also employ a completely different approach, namely the Jacobi elliptic function series, to find explicit solutions. These two approaches are then compared at the end and they do complement each other.

In this paper, attention will be specially given to the coupled Benjamin, Bona and Mahony (BBM)-system:

$$\begin{aligned}\eta_t+u_x+(\eta u)_x-\frac{1}{6}\eta_{xxt}&=0,\\ u_t+\eta_x+uu_x-\frac{1}{6}u_{xxt}&=0,\end{aligned} \tag{1.3}$$

which is when $a=c=0$ and $b=d=1/6$. This system is well-posed and with nice properties, such as the presence of the operator $1-\frac{1}{6}\partial_x^2$, the existence of Hamiltonian and well-developed numerical schemes (see [3–5]).

The paper is organized as follows. Section 2 recalls definitions that will be used and gives a brief review of the topological degree theory for positive operators. In section 3, the theory is first applied to the coupled BBM-system to show the existence of periodic travelling-wave solutions $(\eta(x,t), u(x,t))$ of the form

$$\begin{aligned}\eta(x,t)=\eta(x-\omega t)&=\sum_{n=-\infty}^{\infty}\eta_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)},\\ u(x,t)=u(x-\omega t)&=\sum_{n=-\infty}^{\infty}u_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)},\end{aligned} \tag{1.4}$$

where l and ω connote the half-period and the phase speed, respectively. It is proved that for any $|\omega| > 1$ and for any large enough l, there exists an infinitely smooth non-trivial solution in the form of (1.4). The section ends with general results on systems (1.1) and (1.2), where $b, d > 0$ and $a, c \leqslant 0$. It is proved that for any $|\omega|^2 > \max\{1, \frac{ac}{bd}\}$ or $|\omega|^2 < \min\{1, -\frac{a+c}{b+d}, \frac{ac}{bd}\}$, and for any large enough l, there exists an infinitely smooth non-trivial solution in the form of (1.4).

In section 4, attention will be directed to explicit periodic travelling-wave solutions of the coupled BBM-system. The explicit series solutions in terms of Jacobi elliptic functions are found for $(\omega, l(\omega))$, where ω is in $(0, 0.3219)$ or in $(\frac{5}{2}, \infty)$. The solutions in this form also exist for other systems in (1.1) and details will appear elsewhere.

2. Preliminaries and notation

In this section, we recall definitions that will be used and give a brief review of the topological degree theory for positive operators.

For $1 \leqslant p < +\infty$ and Ω an open set in $\mathbb{R}$, let $L^p(\Omega)$ be the usual Banach space of real or complex-valued, Lebesgue measurable functions defined on Ω with the norm

$$\|f\|^p_{L^p(\Omega)} = \int_\Omega |f|^p \, \mathrm{d}x$$

and $L^\infty(\Omega)$ be the space of measurable, essentially bounded functions with the norm

$$\|f\|_{L^\infty} = \text{ess} \sup_{x\in\Omega} |f(x)|.$$

When it introduces no confusion, $L^p(\Omega)$ is simply written as L^p. Similarly, let $\mathbb{C}$ denote the complex field and l_p be the usual Banach space

$$l_p \equiv \left\{ \boldsymbol{u} = \{u_n\}_{n=-\infty}^{\infty} : u_n \in \mathbb{C}, \sum_{n=-\infty}^{\infty} |u_n|^p < \infty \right\}$$

with the norm

$$\|\boldsymbol{u}\|_p^p = \sum_{n=-\infty}^{\infty} |u_n|^p,$$

whereas l_∞ is defined as

$$l_\infty \equiv \left\{ \boldsymbol{u} = \{u_n\}_{n=-\infty}^{\infty} : u_n \in \mathbb{C}, \sup_{-\infty<n<\infty} |u_n| < \infty \right\}$$

with its usual norm

$$\|\boldsymbol{u}\|_\infty = \sup_{-\infty<n<\infty} |u_n|.$$

The following elementary facts from analysis are recalled. Any $\boldsymbol{f} = \{f_n\}_{n=-\infty}^{\infty} \in l_2$ defines a periodic function f of period $2l$, where

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{\mathrm{i}\frac{n\pi x}{l}}. \tag{2.1}$$

Vice versa, if $f \in L^2(-l, l)$, then f can be expanded almost everywhere as a series in the form (2.1), with $f_n = \frac{1}{2l} \int_{-l}^{l} f(x)\mathrm{e}^{-\mathrm{i}(n\pi x/l)} \, \mathrm{d}x$. In this sense, one can identify $f \in L^2(-l, l)$ with the sequence of its Fourier coefficients $\boldsymbol{f} = \{f_n\}_{n=-\infty}^{\infty}$. Moreover, $\|f\|_{L^2} = (2l)^{\frac{1}{2}} \|\boldsymbol{f}\|_2$. For any $\boldsymbol{u}$ and $\boldsymbol{v}$ in l_2, the convolution $\boldsymbol{u} \times \boldsymbol{v}$ is defined as

$$\boldsymbol{u} \times \boldsymbol{v} = \{(\boldsymbol{u} \times \boldsymbol{v})_n\}_{n=-\infty}^{\infty},$$

where $(\boldsymbol{u} \times \boldsymbol{v})_n = \sum_{k=-\infty}^{\infty} u_{n-k} v_k$. Since $\|\boldsymbol{u} \times \boldsymbol{v}\|_\infty \leqslant \|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_2$, it follows that $\boldsymbol{u} \times \boldsymbol{v} \in l_\infty$.

For the convenience of the reader, a brief review of the topological degree theory for positive operators on Banach spaces is given here and we refer the reader to the works of Krasnosel'skii [10, 11], Granas [9] and Benjamin *et al* [1] for details.

Let X be a Banach space equipped with the norm $\|\cdot\|_X$. We define a closed subset $K \subset X$ as a cone, if the following conditions are satisfied:

(i) $\lambda K \equiv \{\lambda f : f \in K\} \subset K$ for all $\lambda \geqslant 0$,
(ii) $K + K \equiv \{f + g : f, g \in K\} \subset K$,
(iii) $K \cap \{-K\} \equiv K \cap \{-f : f \in K\} = \{0\}$.

For any $0 < r < R < \infty$, denote

$$B_r = \{f \in X : \|f\|_X < r\}, \qquad \partial B_r = \{f \in X : \|f\|_X = r\},$$
$$K_r = K \cap B_r, \qquad \partial K_r = K \cap \partial B_r$$
$$\text{and } K_r^R = \{f \in K : r < \|f\|_X < R\}.$$

An operator $\mathcal{A}$ defined on K is said to be positive if $\mathcal{A}K \subset K$. A positive operator $\mathcal{A}$ is compact if $\mathcal{A}(K_r)$ has a compact closure. Note that the operator $\mathcal{A}$ is not necessarily linear. In fact, for the remaining of our paper $\mathcal{A}$ will be nonlinear.

A triple $(K, \mathcal{A}, U)$ is called admissible if

(i) K is a convex subset of X,
(ii) $U \subset K$ is open in the relative topology on K,
(iii) $\mathcal{A} : K \to K$ is continuous and $\mathcal{A}(U)$ is a subset of a compact set in K and
(iv) $\mathcal{A}$ has no fixed point on ∂U, the boundary of the open set U in the relative topology on K.

Denote the set of all admissible triples by $\mathcal{T}$. Let $(K, \mathcal{A}, U) \in \mathcal{T}$ and $\mathcal{A}$ be a constant mapping on K, namely there is a point $a \in K$ such that $\mathcal{A}u = a$ for every $u \in K$. The fixed point index of the positive operator $\mathcal{A}$ on U is defined as

$$\mathrm{i}(K, \mathcal{A}, U) = \begin{cases} 1 & \text{if } a \in U, \\ 0 & \text{if } a \notin U. \end{cases}$$

We mention here, among the many properties of $\mathrm{i}(K, \mathcal{A}, U)$, the three that will be of use in our current problem.

(a) (Homotopy invariance) If two triples $(K, \mathcal{A}, U)$ and $(K, \mathcal{B}, U) \in \mathcal{T}$ and $\mathcal{A}$ is homotopic to $\mathcal{B}$ on U, then $\mathrm{i}(K, \mathcal{A}, U) = \mathrm{i}(K, \mathcal{B}, U)$.
(b) (Fixed point property) If $(K, \mathcal{A}, U) \in \mathcal{T}$ and $\mathrm{i}(K, \mathcal{A}, U) \neq 0$, then $\mathcal{A}$ has at least one fixed point in U.
(c) (Additivity) If $(K, \mathcal{A}, U) \in \mathcal{T}$ and $U_1, U_2, \ldots, U_n$ is a collection of mutually disjoint open subsets of U such that $\mathcal{A}u \neq u$ for all $u \in U \setminus \cup_{j=1}^n U_j$, then $\mathrm{i}(K, \mathcal{A}, U) = \sum_{j=1}^n \mathrm{i}(K, \mathcal{A}, U_j)$.

The following three lemmas are taken directly from [1] in which K is a cone, the operator $\mathcal{A}$ is positive, continuous and compact on K.

Lemma 2.1. *Suppose that* $0 < \rho < \infty$ *and that either*

(a) $\mathcal{A}x - x \notin K$ *for all* $x \in \partial K_\rho$ *or*
(b) $t\mathcal{A}x \neq x$ *for all* $x \in \partial K_\rho$ *and all* $t \in [0, 1]$.

Then $\mathrm{i}(K, \mathcal{A}, K_\rho) = 1$.

Lemma 2.2. *Suppose that* $0 < \rho < \infty$ *and that either*

($\tilde{a}$) $x - \mathcal{A}x \notin K$ *for all* $x \in \partial K_\rho$ *or*
($\tilde{b}$) *there exists a non-zero* $\tilde{x} \in K$ *such that* $x - \mathcal{A}x \neq \lambda\tilde{x}$ *for all* $x \in \partial K_\rho$ *and all* $\lambda \geqslant 0$.

Then $\mathrm{i}(K, \mathcal{A}, K_\rho) = 0$.

Lemma 2.3. *Let* $(K, \mathcal{A}, U)$ *be admissible. If there exists a non-zero* $\tilde{x} \in K$ *such that* $x - \mathcal{A}x \neq \lambda\tilde{x}$ *for all* $x \in \partial U$ *and all* $\lambda \geqslant 0$, *then* $i(K, \mathcal{A}, U) = 0$.

The following theorem is an immediate consequence of the first two lemmas.

Theorem 2.4. *Suppose that either (a) or (b) holds for an* r *satisfying* $0 < r < \infty$ *and that either* ($\tilde{a}$) *or* ($\tilde{b}$) *holds for an* R *satisfying* $r < R < \infty$. *Then* $\mathcal{A}$ *has at least one fixed point in* $K_r^R \equiv \{f \in K, r < \|f\|_X < R\}$. *Moreover,* $\mathrm{i}(K, \mathcal{A}, K_r^R) = -1$.

The theory described above will be utilized in section 3 to establish the existence of cnoidal wave solutions for (1.1) as follows. By substituting (1.4) into systems (1.1) and equating the Fourier coefficients, one obtains an infinite system which can be posed as a fixed point problem on a certain cone. Using the theory above, the index of the operator associated with this fixed point problem is shown to be non-zero (hence, there must exist at least one solution in the cone). The analysis is complicated a bit by the fact that the trivial (constant) solution lies in the cone. By choosing the half-period l large enough, however, one can then exclude this trivial solution.

3. Existence theorem

3.1. The coupled BBM-system

Let $\eta(x, t) = \eta(x - \omega t)$, $u(x, t) = u(x - \omega t)$ be the travelling-wave solution. Substituting it into (1.3) yields

$$\begin{aligned} &-\omega\eta' + u' + (\eta u)' + \frac{1}{6}\omega\eta''' = 0,\\ &-\omega u' + \eta' + \frac{1}{2}(u^2)' + \frac{1}{6}\omega u''' = 0, \end{aligned} \tag{3.1}$$

where the primes denote the derivatives with respect to the moving frame $\xi = x - \omega t$. By integrating once and letting the constants of integration be C_1 and C_2, the system becomes

$$\begin{aligned} &-\omega\eta + u + \eta u + \frac{1}{6}\omega\eta'' = C_1,\\ &-\omega u + \eta + \frac{1}{2}u^2 + \frac{1}{6}\omega u'' = C_2. \end{aligned}$$

Assuming (f, g) is a constant solution, we see that

$$\begin{aligned} &-\omega f + g + fg - C_1 = 0,\\ &-\omega g + f + \frac{1}{2}g^2 - C_2 = 0. \end{aligned}$$

Solving f from the second equation and substituting it into the first, one sees that g satisfies a cubic equation which is for sure to have a real root. Hence there is at least one constant solution.

By introducing the new dependent variables $\bar{\eta} = \eta - f$ and $\bar{u} = u - g$, the system in $\bar{\eta}$ and $\bar{u}$ then reads as

$$\begin{aligned} &-\omega\bar{\eta} + \bar{u} + (\bar{\eta}\bar{u}) + \frac{1}{6}\omega\bar{\eta}'' + f\bar{u} + g\bar{\eta} = 0, \\ &-\omega\bar{u} + \bar{\eta} + \frac{1}{2}\bar{u}^2 + \frac{1}{6}\omega\bar{u}'' + g\bar{u} = 0. \end{aligned} \tag{3.2}$$

For simplicity in notation, we will study the case with $f = g = 0$, namely the case where $C_1 = C_2 = 0$. The system then becomes

$$\begin{aligned} &-\omega\eta + u + (\eta u) + \frac{1}{6}\omega\eta'' = 0, \\ &-\omega u + \eta + \frac{1}{2}u^2 + \frac{1}{6}\omega u'' = 0. \end{aligned} \tag{3.3}$$

It will become clear later that the method used in (3.3) can be extended to some other cases with non-zero C_1 and C_2.

Remark 3.1. *Notice that the result obtained here will only be an existence theory due to several factors, such as the method of index theory and the integrating constants being taken to be zero. It is easy to see that there are other solutions. For example,* $\eta = c_1$, $u = c_2$ *with* c_1 *and* c_2 *being any constants are solutions to (3.1), but (3.3) only admits three constant solutions, namely* $(0, 0)$, $((\omega^2 - 4 \pm \omega\sqrt{\omega^2 + 8})/4, (3\omega \mp \sqrt{\omega^2 + 8})/2)$, *for any real* ω.

The study of the existence of periodic solutions to (3.3) is carried out as follows. Substituting (1.4) into (3.3) and equating the Fourier coefficients yield the following infinite system:

$$\begin{aligned} &-\omega\eta_n + u_n - \frac{\omega}{6}\left(\frac{n\pi}{l}\right)^2 \eta_n = -(\boldsymbol{\eta} \times \boldsymbol{u})_n, \\ &-\omega u_n + \eta_n - \frac{\omega}{6}\left(\frac{n\pi}{l}\right)^2 u_n = -\frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n, \end{aligned} \tag{3.4}$$

where $\boldsymbol{\eta} = \{\eta_n\}$, $\boldsymbol{u} = \{u_n\}$ and $-\infty < n < \infty$.

The system (3.4) can be put into a more convenient matrix form

$$T_n \begin{bmatrix} \eta_n \\ u_n \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\eta} \times \boldsymbol{u})_n \\ \frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \end{bmatrix}, \tag{3.5}$$

where

$$T_n = \begin{bmatrix} \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right) & -1 \\ -1 & \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right) \end{bmatrix}. \tag{3.6}$$

For the phase speed $|\omega| > 1$, T_n is invertible for all n with

$$T_n^{-1} = \frac{1}{\omega^2\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)^2 - 1} \begin{bmatrix} \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right) & 1 \\ 1 & \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right) \end{bmatrix}. \tag{3.7}$$

The l^∞−norm of T_n^{-1} is defined as

$$\|T_n^{-1}\|_{l^\infty} = \max_{1\leqslant j\leqslant 2} \sum_{i=1}^{2} |T_n^{-1}(i,j)| = \frac{1}{\omega\left(1+\frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)-1},$$

where $T_n^{-1}(i,j)$ is the (i,j)th entry of the matrix T_n^{-1}.

To set up the problem as a fixed point problem, a set $K \subset l_2 \times l_2 = X$ is defined by

$$K = \{(\boldsymbol{\eta}, \boldsymbol{u}) = \{(\eta_n, u_n)\} \in X : (\eta_n, u_n) = (\eta_{-n}, u_{-n}),$$
$$\eta_0 \geqslant \eta_1 \geqslant \cdots \geqslant 0,\ u_0 \geqslant u_1 \geqslant \cdots \geqslant 0\}.$$

One can easily verify that K is indeed a cone in X equipped with the norm

$$\|(\boldsymbol{\eta}, \boldsymbol{u})\|_X^2 = \sum_{n=-\infty}^{\infty} (|\eta_n|^2 + |u_n|^2).$$

An operator $\mathcal{A}$ on K is now defined as follows: for any $\boldsymbol{w} \equiv (\boldsymbol{\eta}, \boldsymbol{u}) = \{(\eta_n, u_n)\} \in K$, $\mathcal{A}\boldsymbol{w} = \{(\mathcal{A}\boldsymbol{w})_n\}$, where

$$(\mathcal{A}\boldsymbol{w})_n = T_n{}^{-1} \begin{bmatrix} (\boldsymbol{\eta} \times \boldsymbol{u})_n \\ \frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \end{bmatrix}. \tag{3.8}$$

Thus (3.5) can be written in the form $\boldsymbol{w} = \mathcal{A}\boldsymbol{w}$ and the fixed points of operator $\mathcal{A}$ in the cone K are solutions of (3.5).

Remark 3.2. *Since (3.3) can be transformed into a single equation, by solving for η from the second equation and substituting it into the first equation, it is natural to wonder if the existing theorems for the single equation such as the ones in [7] can be applied directly. Unfortunately, this attempt does not succeed since the nonlinear terms in the resulting equation*

$$\omega^2 u^{(4)} + (u-\omega)(12\omega u'' + 18u^2 - 36wu) + 6\omega(u')^2 - 36u = 0 \tag{3.9}$$

are not of one sign.

Remark 3.3. *The invertibility of T_n^{-1} hinges on the condition that $|\omega| > 1$. Thus the existence result we are aiming to establish is only for the case when $|\omega| > 1$. Notice that if $(\eta(x-\omega t), u(x-\omega t))$ is a solution, $(\eta(x+\omega t), -u(x+\omega t))$ is also a solution; thus the existence of a cnoidal wave solution for $\omega > 1$ will imply the existence of a cnoidal wave solution for $\omega < -1$. For the rest of the section 3.1, $\omega > 1$ will be assumed.*

Lemma 3.4. *For $\omega > 1$, $\mathcal{A}$ is a continuous, positive and compact operator on the cone K.*

Proof. The proof follows the same lines as in [7] and is reproduced here for the reader's convenience.

(a) $\mathcal{A}$ is a positive operator on K; i.e. $\mathcal{A}$ maps K into itself.
For any $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in K$, let $\tau_n \equiv (\boldsymbol{\eta} \times \boldsymbol{u})_n = \sum_{k=-\infty}^{\infty} \eta_{n-k} u_k$. It is easy to verify that for all $n \geqslant 0$,

$$\tau_{-n} = \sum_{k=-\infty}^{\infty} \eta_{-n-k} u_k = \sum_{k=-\infty}^{\infty} \eta_{n+k} u_k = \sum_{k=-\infty}^{\infty} \eta_{n-(-k)} u_{-k} = \sum_{m=-\infty}^{\infty} \eta_{n-m} u_m = \tau_n$$

and

$$\tau_n - \tau_{n+1} = \sum_{k=-\infty}^{\infty} \eta_{n-k}u_k - \sum_{k=-\infty}^{\infty} \eta_{n+1-k}u_k = \sum_{k=0}^{\infty} \eta_{n-k}u_k + \sum_{k=0}^{\infty} \eta_{n+k+1}u_{k+1}$$
$$- \sum_{k=0}^{\infty} \eta_{n+1+k}u_k - \sum_{k=0}^{\infty} \eta_{n-k}u_{k+1} = \sum_{k=0}^{\infty} (\eta_{n-k} - \eta_{n+1+k})(u_k - u_{k+1}) \geqslant 0.$$

Therefore, τ_n is a decreasing function of $|n|$ and

$$0 \leqslant \tau_n \leqslant \tau_0 = (\boldsymbol{\eta} \times \boldsymbol{u})_0 \leqslant \|\boldsymbol{\eta}\|_2 \|\boldsymbol{u}\|_2 \leqslant \|\boldsymbol{w}\|_X^2. \tag{3.10}$$

Since each entry $T_n^{-1}(i, j)$ of T_n^{-1} is positive and even in n, decreasing with respect to $|n|$, the sequence $\{\|T_n^{-1}\|_{l^\infty}\}$ is square-summable, i.e. $\sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_{l^\infty}^2 < \infty$; it follows immediately that $\mathcal{A}K \subset K$.

(b) $\mathcal{A}$ is continuous.

Let $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u})$ and $\bar{\boldsymbol{w}} = (\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{u}})$ be two arbitrary elements in K. For all n, the difference $(\mathcal{A}\boldsymbol{w})_n - (\mathcal{A}\bar{\boldsymbol{w}})_n$ can be bounded componentwise, namely,

$$|(\boldsymbol{\eta} \times \boldsymbol{u})_n - (\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n| \leqslant \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_2 \|\boldsymbol{\eta}\|_2 + \|\bar{\boldsymbol{u}}\|_2 \|\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}\|_2,$$
$$|(\boldsymbol{u} \times \boldsymbol{u})_n - (\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n| \leqslant \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_2 (\|\bar{\boldsymbol{u}}\|_2 + \|\boldsymbol{u}\|_2).$$

Hence, it follows that

$$\begin{aligned}
\|\mathcal{A}\boldsymbol{w} - \mathcal{A}\bar{\boldsymbol{w}}\|_X^2 &\leqslant 2\gamma^2 [\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_2 (\|\boldsymbol{\eta}\|_2 + \|\bar{\boldsymbol{u}}\|_2 + \|\boldsymbol{u}\|_2) + \|\bar{\boldsymbol{u}}\|_2 \|\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}\|_2]^2 \\
&\leqslant 2\gamma^2 (\|\boldsymbol{\eta}\|_2 + \|\boldsymbol{u}\|_2 + \|\bar{\boldsymbol{\eta}}\|_2 + \|\bar{\boldsymbol{u}}\|_2)^2 (\|\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}\|_2 + \|\boldsymbol{u} - \bar{\boldsymbol{u}}\|_2)^2 \\
&\leqslant 4\gamma^2 (\|\boldsymbol{\eta}\|_2 + \|\boldsymbol{u}\|_2 + \|\bar{\boldsymbol{\eta}}\|_2 + \|\bar{\boldsymbol{u}}\|_2)^2 \|\boldsymbol{w} - \bar{\boldsymbol{w}}\|_X^2 \leqslant \gamma^2 D^2 \|\boldsymbol{w} - \bar{\boldsymbol{w}}\|_X^2,
\end{aligned}$$

where

$$\gamma = \left[\sum_{n=-\infty}^{\infty} \|T_n^{-1}\|_{l^\infty}^2 \right]^{\frac{1}{2}} = \left[\sum_{n=-\infty}^{\infty} \left(\frac{1}{\omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right) - 1} \right)^2 \right]^{\frac{1}{2}} \tag{3.11}$$

and $D = 2(\|\boldsymbol{\eta}\|_2 + \|\boldsymbol{u}\|_2 + \|\bar{\boldsymbol{\eta}}\|_2 + \|\bar{\boldsymbol{u}}\|_2)$. The operator $\mathcal{A}$ is now readily seen to be continuous from K into itself.

(c) $\mathcal{A}$ is compact.

Consider a bounded set M in X, say $M \subset \{\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in X : \|\boldsymbol{w}\|_X \leqslant B\}$. For each N, a cut-off operator $\mathcal{A}_N$ is defined as follows:

$$(\mathcal{A}_N \boldsymbol{w})_n = \begin{cases} (\mathcal{A}\boldsymbol{w})_n, & \text{for } -N \leqslant n \leqslant N, \\ 0, & \text{otherwise.} \end{cases} \tag{3.12}$$

Then $\mathcal{A}_N$ is a compact operator having a rank of $(2N + 1)$ as $\mathcal{A}$ is continuous. Now, for $\boldsymbol{w} \in M$,

$$|(\mathcal{A}\boldsymbol{w})_n| \leqslant \|T_n^{-1}\|_{l^\infty} \begin{bmatrix} \|\boldsymbol{\eta}\|_2 \|\boldsymbol{u}\|_2 + \frac{1}{2}\|\boldsymbol{u}\|_2^2 \\ \|\boldsymbol{\eta}\|_2 \|\boldsymbol{u}\|_2 + \frac{1}{2}\|\boldsymbol{u}\|_2^2 \end{bmatrix}. \tag{3.13}$$

Thus,

$$\|\mathcal{A}_N \boldsymbol{w} - \mathcal{A}\boldsymbol{w}\|_X^2 \leqslant 4B^4 \gamma_N^2 \leqslant 4B^4 \gamma^2,$$

where $\gamma_N = \left[\sum_{|n|\geqslant N} \|T_n^{-1}\|_{l^\infty}^2\right]^{\frac{1}{2}}$. Consequently, $\sup_{\boldsymbol{w}\in M} \|\mathcal{A}_N \boldsymbol{w} - \mathcal{A}\boldsymbol{w}\|_X^2 \to 0$ as $N \to \infty$. Thus $\mathcal{A}$ is compact as it is the uniform limit of compact operators on bounded sets.

□

Attention is now turned to the fixed points of $\mathcal{A}$. A fixed point $(\boldsymbol{p}, \boldsymbol{q}) = \{(p_n, q_n)\}$ is said to be trivial if $p_n = q_n = 0$ for all $n \neq 0$. It is fairly straightforward to check that for a trivial fixed point of (3.5), $p_0 = 0$ if and only if $q_0 = 0$. A trivial solution with $p_0 = q_0 = 0$ corresponds to the origin while $p_0, q_0 \neq 0$ corresponds to a non-zero constant solution.

Remark 3.5. *As mentioned in remark 3.1 the operator $\mathcal{A}$ has three trivial fixed points, but only two of them are in K. They are the origin and the constant travelling-wave solution $(\boldsymbol{p}^*, \boldsymbol{q}^*)$, where*

$$\boldsymbol{p}^* = (\cdots, 0, p_0, 0, \cdots) \qquad \text{and} \qquad \boldsymbol{q}^* = (\cdots, 0, q_0, 0, \cdots)$$

with

$$p_0 = \frac{1}{4}(\omega^2 - 4 + \omega\sqrt{\omega^2 + 8}) \qquad \text{and} \qquad q_0 = \frac{3\omega - \sqrt{\omega^2 + 8}}{2}.$$

The other does not belong to K as $\frac{1}{4}(\omega^2 - 4 - \omega\sqrt{\omega^2 + 8}) < 0$.

Proposition 3.6. *Let γ be as defined in (3.11). Then for any r satisfying*

$$0 < r < r_0 \equiv \min\left\{\frac{1}{2\gamma}, \|(\boldsymbol{p}^*, \boldsymbol{q}^*)\|_X\right\}, \tag{3.14}$$

$\boldsymbol{w} \neq t\mathcal{A}\boldsymbol{w}$ for all $\boldsymbol{w} \in \partial K_r$ and for all $t \in [0, 1]$.

Proof. Suppose there exist a $\boldsymbol{w} \in \partial K_r$ and a $t \in [0, 1]$ such that $\boldsymbol{w} = t\mathcal{A}\boldsymbol{w}$. Then using (3.13) on all $(\mathcal{A}\boldsymbol{w})_n$ yields

$$\|\boldsymbol{w}\|_X^2 = r^2 = t\sum_{n=-\infty}^{\infty} ((\mathcal{A}\boldsymbol{w})_n)^2 \leqslant 4r^4\gamma^2, \tag{3.15}$$

which implies that $r \geqslant \frac{1}{2\gamma}$, a contradiction. □

Proposition 3.7. *For any*

$$R > R_0 \equiv \max\left\{\frac{2(\omega^2 - 1)}{\omega}(\gamma(\omega^2 - 1) + \sqrt{3}), \|(\boldsymbol{p}^*, \boldsymbol{q}^*)\|_X\right\}, \tag{3.16}$$

there exists a non-zero $\tilde{\boldsymbol{w}} \in K$ such that $\boldsymbol{w} - \mathcal{A}\boldsymbol{w} \neq \lambda\tilde{\boldsymbol{w}}$, for all $\boldsymbol{w} \in \partial K_R$ and all $\lambda \geqslant 0$.

Proof. Following the same idea as in [7], let $\tilde{\boldsymbol{w}} = \{(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{u}})_n\}$ be given by $\begin{bmatrix}\tilde{\eta}_n \\ \tilde{u}_n\end{bmatrix} = \frac{1}{1+n^2}\begin{bmatrix}1 \\ 1\end{bmatrix}$. Clearly $\tilde{\boldsymbol{w}} \neq 0$ and $\tilde{\boldsymbol{w}} \in K$. Again, suppose to the contrary that there exist a $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in \partial K_R$ and a $\lambda \geqslant 0$ such that for all n,

$$\begin{bmatrix}\eta_n \\ u_n\end{bmatrix} = \frac{1}{\omega^2\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)^2 - 1}\begin{bmatrix}\omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\boldsymbol{\eta} \times \boldsymbol{u})_n + \frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \\ \frac{\omega}{2}\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\boldsymbol{u} \times \boldsymbol{u})_n + (\boldsymbol{\eta} \times \boldsymbol{u})_n\end{bmatrix} + \lambda\tilde{\boldsymbol{w}}. \tag{3.17}$$

In particular,

$$\begin{aligned}\eta_0 &= \frac{1}{\omega^2-1}\left(\omega\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2\right)+\lambda,\\ u_0 &= \frac{1}{\omega^2-1}\left(\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{\omega}{2}\|\boldsymbol{u}\|_2^2\right)+\lambda.\end{aligned}\tag{3.18}$$

Since $\boldsymbol{w} = \boldsymbol{0} \notin \partial K_R$, and that $\eta_0 = 0$ if and only if $u_0 = 0$, $\eta_0 \neq 0$ and $u_0 \neq 0$. Consequently, we obtain from (3.18) the bounds $0 < \eta_0 \leqslant \omega^2 - 1$, $0 < u_0 \leqslant \dfrac{\omega^2-1}{\omega}$, $0 \leqslant \lambda \leqslant \dfrac{\omega^2-1}{\omega}$ and

$$\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2 \leqslant \frac{(\omega^2-1)^2}{\omega}.\tag{3.19}$$

One can see from (3.17) and from the fact that τ_n is decreasing in $|n|$ that

$$\begin{aligned}\eta_n &\leqslant \|T_n^{-1}\|_{l^\infty}\left(\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2\right)+\frac{\lambda}{1+n^2},\\ u_n &\leqslant \|T_n^{-1}\|_{l^\infty}\left(\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2\right)+\frac{\lambda}{1+n^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}R^2 = \sum_{n=-\infty}^{\infty}\eta_n^2 + u_n^2 &\leqslant 2\sum_{n=-\infty}^{\infty}\left\{\|T_n^{-1}\|_{l^\infty}\left(\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2\right)+\frac{\lambda}{1+n^2}\right\}^2\\ &\leqslant 4\sum_{n=-\infty}^{\infty}\|T_n^{-1}\|_{l^\infty}^2\left(\sum_{k=-\infty}^{\infty}\eta_k u_k + \frac{1}{2}\|\boldsymbol{u}\|_2^2\right)^2 + 4\sum_{n=-\infty}^{\infty}\frac{\lambda^2}{(1+n^2)^2}.\end{aligned}$$

Hence, from (3.11), (3.19) and using the fact that $\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)^2} \leqslant 3$, the conclusion

$$R \leqslant \frac{2(\omega^2-1)}{\omega}(\gamma(\omega^2-1)+\sqrt{3})\tag{3.20}$$

is drawn, which contradicts the assumption on R. □

Theorem 3.8. *Let r and R be as above. Then the fixed point index of $\mathcal{A}$ on $K_r^R = \{\boldsymbol{w} \in K : r < \|\boldsymbol{w}\|_X < R\}$ is* $\mathrm{i}(K, \mathcal{A}, K_r^R) = -1$.

Proof. This follows immediately from theorem 2.4 and propositions 3.6 and 3.7. □

An immediate consequence of theorem 3.8 is that there must be at least one fixed point of $\mathcal{A}$ in K_r^R. However, the analysis is not yet complete since the constant periodic solution $(\boldsymbol{p}^*, \boldsymbol{q}^*) \in K_r^R$ could be the only fixed point in K_r^R. This case is excluded through the following. Let $\epsilon = \epsilon(l) > 0$ be an arbitrarily fixed, sufficiently small number whose value will be determined later. Let

$$K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*) = \{\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in K : \|(\boldsymbol{\eta}, \boldsymbol{u}) - (\boldsymbol{p}^*, \boldsymbol{q}^*)\|_X < \epsilon\}$$

and

$$\partial K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*) = \{\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in K : \|(\boldsymbol{\eta}, \boldsymbol{u}) - (\boldsymbol{p}^*, \boldsymbol{q}^*)\|_X = \epsilon\}.$$

Lemma 3.9. *If* $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ *is the only fixed point of* $\mathcal{A}$ *in* K_r^R, *then when the half-period* $l > 0$ *is chosen large enough,* $\mathrm{i}(K, \mathcal{A}, K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*)) = 0$.

Proof. The ϵ will be chosen small enough so $\bar{K}_\epsilon$ is in K_r^R. Therefore, the lemma is proved, owing to lemma 2.3, if one can show that $(I - \mathcal{A})\partial K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*)$ omits the ray $\{\lambda\tilde{\boldsymbol{w}} : \lambda \geqslant 0\}$, where $\tilde{\boldsymbol{w}} \in K$ is defined as in proposition 3.7.

Now suppose that there are a $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in \partial K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*)$ and a $\lambda \geqslant 0$ such that $\boldsymbol{w} - \mathcal{A}\boldsymbol{w} = \lambda\tilde{\boldsymbol{w}}$. Then for all $n \in \mathbb{Z}$,

$$\begin{bmatrix} \eta_n \\ u_n \end{bmatrix} = \frac{1}{\omega^2\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)^2 - 1} \begin{bmatrix} \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\boldsymbol{\eta} \times \boldsymbol{u})_n + \frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \\ \frac{\omega}{2}\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\boldsymbol{u} \times \boldsymbol{u})_n + (\boldsymbol{\eta} \times \boldsymbol{u})_n \end{bmatrix} + \lambda\tilde{\boldsymbol{w}}. \tag{3.21}$$

In particular, for $n = 1$,

$$\begin{aligned} \eta_1 &\geqslant A(\eta_0 u_1 + \eta_1 u_0) + B u_0 u_1 + \lambda/2, \\ u_1 &\geqslant A u_0 u_1 + B(\eta_0 u_1 + \eta_1 u_0) + \lambda/2, \end{aligned} \tag{3.22}$$

where

$$A = \frac{\omega\left(1 + \frac{1}{6}\left(\frac{\pi}{l}\right)^2\right)}{\omega^2\left(1 + \frac{1}{6}\left(\frac{\pi}{l}\right)^2\right)^2 - 1} \qquad \text{and} \qquad B = \frac{1}{\omega^2\left(1 + \frac{1}{6}\left(\frac{\pi}{l}\right)^2\right)^2 - 1}.$$

Thus

$$\eta_1 + u_1 \geqslant u_0(A + B)(\eta_1 + u_1) + (A + B)\eta_0 u_1 + \lambda. \tag{3.23}$$

Since $\boldsymbol{w} \in \partial K_\epsilon(\boldsymbol{p}^*, \boldsymbol{q}^*)$, it can be written as $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) = (\boldsymbol{p}^*, \boldsymbol{q}^*) + \epsilon(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{u}})$, where $\|(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{u}})\|_X = 1$. Note that for $n \geqslant 1$

$$\begin{cases} \tilde{\eta}_n = \eta_n/\epsilon \geqslant 0, \\ \tilde{u}_n = u_n/\epsilon \geqslant 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{\eta}_n \geqslant \tilde{\eta}_{n+1}, \\ \tilde{u}_n \geqslant \tilde{u}_{n+1}. \end{cases}$$

In terms of the new variables $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{u}})$, (3.23) can be written as

$$\epsilon(\tilde{\eta}_1 + \tilde{u}_1) \geqslant \frac{\epsilon}{\omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1}(q_0 + \epsilon\tilde{u}_0)(\tilde{\eta}_1 + \tilde{u}_1) + \frac{\epsilon\tilde{u}_1(p_0 + \epsilon\tilde{\eta}_0)}{\omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1} + \lambda. \tag{3.24}$$

The half-period l can be chosen large enough so that

$$q_0 = \frac{3\omega - \sqrt{\omega^2 + 8}}{2} > \omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1. \tag{3.25}$$

The explicit condition for l is

$$l^2 > L_0^2 \equiv \frac{\pi^2\omega(\omega + 2 + \sqrt{\omega^2 + 8})}{12(\omega - 1)}. \tag{3.26}$$

Notice that $p_0 \geqslant q_0$, so

$$p_0 > \omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1. \tag{3.27}$$

The number ϵ can then be chosen small enough to satisfy

$$q_0 + \epsilon\tilde{u}_0 > \omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1 \qquad \text{and} \qquad p_0 + \epsilon\tilde{\eta}_0 > \omega + \frac{\omega}{6}\left(\frac{\pi}{l}\right)^2 - 1.$$

It follows immediately from (3.24) that $\lambda = 0$ and $\tilde{\eta}_1 = \tilde{u}_1 = 0$, which imply that $\tilde{\eta}_n = \tilde{u}_n = 0$ for all $n \neq 0$. Using (3.21) for $n = 0$ yields that $\tilde{\eta}_0 = \tilde{u}_0 = 0$. This contradicts the assumption that $\boldsymbol{w} = (\boldsymbol{\eta}, \boldsymbol{u}) \in \partial K_\epsilon(\boldsymbol{p}, \boldsymbol{q})$. □

Theorem 3.10. *For $w > 1$, if the half-period l is chosen large enough as in (3.26), then the operator $\mathcal{A}$ has a non-trivial fixed point $\bar{\boldsymbol{w}} = (\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{u}})$ in the cone segment K_r^R. Moreover,*

(i) $\bar{\eta}_n, \bar{u}_n > 0$ for every $n \in \mathbb{Z}$ and
(ii) for any $\sigma \geqslant 0$, the sequences $\{|n|^\sigma \bar{\eta}_n\}$ and $\{|n|^\sigma \bar{u}_n\}$ are in l_1. Therefore, the non-trivial fixed point solution is infinitely smooth.

Proof. The existence of a non-trivial fixed point is the consequence of theorem 3.8 and lemma 3.9. It is left to establish (i) and (ii).

Let $n = N$ be the smallest non-negative integer such that either $\bar{\eta}_N$ or $\bar{u}_N$ is zero. Notice that $N > 1$ since the solution is non-trivial and $\eta_1 = 0$ if and only if $u_1 = 0$, which will lead to the trivial solution. Recall that as a fixed point of $\mathcal{A}$, $\bar{\boldsymbol{w}} = (\bar{\boldsymbol{\eta}}, \bar{\boldsymbol{u}}) \in l_2 \times l_2 = X$ is given by

$$\begin{bmatrix} \bar{\eta}_n \\ \bar{u}_n \end{bmatrix} = \frac{1}{\omega^2\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)^2 - 1} \begin{bmatrix} \omega\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n + \frac{1}{2}(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n \\ \frac{\omega}{2}\left(1 + \frac{1}{6}\left(\frac{n\pi}{l}\right)^2\right)(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n + (\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n \end{bmatrix}. \tag{3.28}$$

If $\bar{\eta}_N = 0$, it follows from (3.28) and from both $\bar{\eta}_k$ and $\bar{u}_k > 0$ for every $k \in [-N+1, N-1]$ that

$$\begin{aligned} 0 &= \omega\left(1 + \frac{1}{6}\left(\frac{N\pi}{l}\right)^2\right)(\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_N + \frac{1}{2}(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_N \\ &\geqslant \omega\left(1 + \frac{1}{6}\left(\frac{N\pi}{l}\right)^2\right)\bar{\eta}_{N-1}\bar{u}_1 + \frac{1}{2}\bar{u}_{N-1}\bar{u}_1 > 0 \end{aligned}$$

which is certainly not true. The case $\bar{u}_N = 0$ can be handled similarly. Thus $\bar{\eta}_n, \bar{u}_n > 0$ for every $n \in \mathbb{Z}$ and (i) is proved.

Since $(\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n \leqslant \|\bar{\boldsymbol{\eta}}\|_2\|\bar{\boldsymbol{u}}\|_2 < \infty$ and $(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n \leqslant \|\bar{\boldsymbol{u}}\|_2^2 < \infty$, it follows that for $|n| \geqslant 1$, there exists a constant C_0 independent of n satisfying

$$\bar{\eta}_n \leqslant C\|T_n^{-1}\|_{l^\infty} \leqslant \frac{C_0}{n^2} \qquad \text{and} \qquad \bar{u}_n \leqslant C\|T_n^{-1}\|_{l^\infty} \leqslant \frac{C_0}{n^2}. \tag{3.29}$$

Therefore, $\{\bar{\eta}_n\}$ and $\{\bar{u}_n\}$ are in l_1.

The continuous bootstrapping argument can now be utilized to show that for any $\sigma \geqslant 0$, the sequences $\{|n|^\sigma \bar{\eta}_n\}$ and $\{|n|^\sigma \bar{u}_n\}$ are in l_1. The arguments start with noting from (3.29) that for $|n| \geqslant 1$,

$$|n|\bar{\eta}_n \leqslant \frac{C_0}{|n|}, \qquad |n|\bar{u}_n \leqslant \frac{C_0}{|n|}$$

and

$$\sum_{n=-\infty}^{\infty} (1 + |n|)^2\bar{\eta}_n^2 \leqslant C, \qquad \sum_{n=-\infty}^{\infty} (1 + |n|)^2\bar{u}_n^2 \leqslant C. \tag{3.30}$$

Therefore

$$(\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n = \sum_{k=-\infty}^{\infty} \bar{\eta}_{n-k}\bar{u}_k = \sum_{k=-\infty}^{\infty} \frac{(1+|n-k|)\bar{\eta}_{n-k}(1+|k|)\bar{u}_k}{(1+|n-k|)(1+|k|)}$$

$$\leqslant \frac{1}{1+|n|} \sum_{k=-\infty}^{\infty} (1+|n-k|)\bar{\eta}_{n-k}(1+|k|)\bar{u}_k \leqslant \frac{C}{1+|n|}.$$

Similarly,

$$(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n = \sum_{k=-\infty}^{\infty} \bar{u}_{n-k}\bar{u}_k \leqslant \frac{C}{1+|n|}.$$

It now follows that for $|n| \geqslant 1$,

$$\bar{\eta}_n \leqslant \|T_n^{-1}\|_{l^\infty} \left((\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n + \frac{1}{2}(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n \right) \leqslant \frac{C_1}{|n|^3},$$

$$\bar{u}_n \leqslant \|T_n^{-1}\|_{l^\infty} \left((\bar{\boldsymbol{\eta}} \times \bar{\boldsymbol{u}})_n + \frac{1}{2}(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{u}})_n \right) \leqslant \frac{C_1}{|n|^3}.$$

Therefore the sequences $\{|n|\bar{\eta}_n\}$ and $\{|n|\bar{u}_n\}$ are in l_1.

Similarly, one can show that for any $\sigma \geqslant 2$ and for $|n| \geqslant 1$,

$$\begin{aligned} &|n|^\sigma \bar{\eta}_n \leqslant \frac{C}{|n|}, && |n|^\sigma \bar{u}_n \leqslant \frac{C}{|n|}, \\ &\sum_{n=-\infty}^{\infty} (1+n^2)^\sigma \bar{\eta}_n^2 \leqslant C, && \sum_{n=-\infty}^{\infty} (1+n^2)^\sigma \bar{u}_n^2 \leqslant C \end{aligned} \tag{3.31}$$

and hence

$$\bar{\eta}_n \leqslant \frac{C_\sigma}{|n|^{\sigma+2}} \qquad \text{and} \qquad \bar{u}_n \leqslant \frac{C_\sigma}{|n|^{\sigma+2}} \tag{3.32}$$

which leads to the conclusion that the sequences $\{|n|^\sigma \bar{\eta}_n\}$ and $\{|n|^\sigma \bar{u}_n\}$ are in l_1. □

The results are now summarized in the following theorem.

Theorem 3.11. *For any phase speed $|\omega| > 1$ and for any $l > L_0$ where L_0 is defined in (3.26), there exists an infinitely smooth non-trivial cnoidal wave solution with period $2l$ in the form of*

$$\eta(x,t) = \eta(x-\omega t) = \sum_{n=-\infty}^{\infty} \eta_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)},$$

$$u(x,t) = u(x-\omega t) = \sum_{n=-\infty}^{\infty} u_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)}.$$

Moreover, the norm of the solution satisfies

$$(2l)^{\frac{1}{2}} r_0 < ||(\eta, u)||_{L^2\times L^2} < (2l)^{\frac{1}{2}} R_0,$$

where r_0 and R_0 are functions of ω and l defined in (3.14) and (3.16).

3.2. The general case

Attention is now turned to the general four-parameter family of Boussinesq systems (1.1) and (1.2). Upon substituting the form of the solution (1.4) into (1.1) and carrying out similar calculations, the following infinite system is obtained:

$$D_n \begin{bmatrix} \eta_n \\ u_n \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\eta} \times \boldsymbol{u})_n \\ \frac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \end{bmatrix}, \tag{3.33}$$

where

$$D_n = \begin{bmatrix} \omega\left(1 + b\left(\frac{n\pi}{l}\right)^2\right) & \left(a\left(\frac{n\pi}{l}\right)^2 - 1\right) \\ \left(c\left(\frac{n\pi}{l}\right)^2 - 1\right) & \omega\left(1 + d\left(\frac{n\pi}{l}\right)^2\right) \end{bmatrix}.$$

To use the same argument as for the coupled BBM-system, the matrix

$$D_n^{-1} = \frac{1}{\det(D_n)} \begin{bmatrix} \omega\left(1 + d\left(\frac{n\pi}{l}\right)^2\right) & \left(1 - a\left(\frac{n\pi}{l}\right)^2\right) \\ \left(1 - c\left(\frac{n\pi}{l}\right)^2\right) & \omega\left(1 + b\left(\frac{n\pi}{l}\right)^2\right) \end{bmatrix},$$

where

$$\det(D_n) = \omega^2 - 1 + \left(\frac{n\pi}{l}\right)^2 (\omega^2(b+d) + (a+c)) + \left(\frac{n\pi}{l}\right)^4 (\omega^2 bd - ac)$$

is required to have positive and even entries for all n and each entry is to be square-summable and decreasing with respect to $|n|$. Therefore ω has *two* admissible regions, namely $|\omega|^2 > \max\{1, \frac{ac}{bd}\}$ and $|\omega|^2 < \min\{1, -\frac{a+c}{b+d}, \frac{ac}{bd}\}$. When a or c is zero, which includes the case of the coupled BBM-system, the two regions degenerate to one which is $|\omega| > 1$.

An operator $\mathcal{B}$ is now defined as follows: for any $\boldsymbol{w} = \{(\boldsymbol{\eta}, \boldsymbol{u})_n\} \in X$, $\mathcal{B}\boldsymbol{w} = \{(\mathcal{B}\boldsymbol{w})_n\}$, where

$$(\mathcal{B}\boldsymbol{w})_n = D_n^{-1} \begin{bmatrix} (\boldsymbol{\eta} \times \boldsymbol{u})_n \\ \dfrac{1}{2}(\boldsymbol{u} \times \boldsymbol{u})_n \end{bmatrix}.$$

System (3.33) can be written in the form $\boldsymbol{w} = \mathcal{B}\boldsymbol{w}$ and the fixed points of operator $\mathcal{B}$ are solutions of (3.33). Hence, one arrives at the following conclusions.

Theorem 3.12. *For a Boussinesq system with $b, d > 0$, $a, c \leqslant 0$ which satisfy the consistency condition (1.2) and for any phase speed $|\omega|^2 > \max\left\{1, \dfrac{ac}{bd}\right\}$ and for $l > L_0$, where L_0 depends on ω and the dispersive constants a, b, c, d, there exists an infinitely smooth non-trivial cnoidal wave solution with period $2l$ in the form of*

$$\eta(x, t) = \eta(x - \omega t) = \sum_{n=-\infty}^{\infty} \eta_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x - \omega t)},$$

$$u(x, t) = u(x - \omega t) = \sum_{n=-\infty}^{\infty} u_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x - \omega t)}.$$

By introducing the new variables $(\bar{\eta}, \bar{u}) = (-\eta, -u)$ and applying the same arguments for small phase velocity, namely for $|\omega|^2 < \min\{1, -(a+c)/(b+d), ac/(bd)\}$, the following existence theorem is a straightforward consequence.

Theorem 3.13. *For a Boussinesq system with $b, d > 0$, $a, c \leqslant 0$ which satisfy the consistency condition (1.2) and for any phase speed $|\omega|^2 < \min\left\{1, -\frac{a+c}{b+d}, \frac{ac}{bd}\right\}$ and for $l > L_0$, where L_0 depends on ω and the dispersive constants a, b, c, d, there exists an infinitely smooth non-trivial cnoidal wave solution with period $2l$ in the form of*

$$\eta(x,t) = \eta(x-\omega t) = \sum_{n=-\infty}^{\infty} \eta_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)},$$

$$u(x,t) = u(x-\omega t) = \sum_{n=-\infty}^{\infty} u_n \mathrm{e}^{\mathrm{i}(n\pi/l)(x-\omega t)}.$$

4. Explicit cnoidal wave solutions

In this section, the explicit cnoidal wave solutions of the coupled BBM-system are found by using the Jacobi elliptic function series on (3.9). The solutions in this form also exist for other systems in (1.1) and details will appear elsewhere. For completeness, the definition of the Jacobi elliptic functions is recalled here.

Let

$$v = \int_0^{\phi} \frac{1}{\sqrt{1-m^2\sin^2 t}}\,\mathrm{d}t \qquad \text{for } 0 \leqslant m \leqslant 1,$$

which is denoted as $v = F(\phi, m)$. Then $\phi = F^{-1}(v, m)$. The two basic Jacobi elliptic functions $\mathrm{cn}(v, m)$ and $\mathrm{sn}(v, m)$ are defined as

$$\mathrm{sn}(v,m) = \sin(\phi) = \sin(F^{-1}(v,m)) \qquad \text{and} \qquad \mathrm{cn}(v,m) = \cos(\phi) = \cos(F^{-1}(v,m)),$$

where m is known as the elliptic modulus. It is easy to see that $\mathrm{cn}(v, m)$ and $\mathrm{sn}(v, m)$ are periodic functions in v with period

$$4\int_0^{\pi/2} \frac{1}{\sqrt{1-m^2\sin^2 t}}\,\mathrm{d}t.$$

Moreover, these functions are generalizations of the trigonometric and hyperbolic functions which satisfy

$$\mathrm{sn}(v,0) = \sin(v), \qquad \mathrm{cn}(v,0) = \cos(v),$$

$$\mathrm{cn}(v,1) = \mathrm{sech}(v), \qquad \mathrm{sn}(v,1) = \tanh(v).$$

To search for explicit solutions of (3.3), a function u in the form of the Jacobi elliptic function series satisfying (3.9) is sought. The deviation η can then be evaluated from the second equation in (3.3).

Let

$$u(\xi) = \sum_{j=0}^{M} a_j \mathrm{cn}^j(\lambda\xi, m)$$

and substitute it into (3.9). Using the basic facts, such as

$$\begin{aligned}&\mathrm{cn}'(v,m) = -\mathrm{sn}(v,m)\sqrt{1-m+m\,\mathrm{cn}^2(v,m)} \qquad \text{and}\\ &\mathrm{cn}^2(v,m) + \mathrm{sn}(v,m)^2 = 1,\end{aligned} \tag{4.1}$$

an equation in terms of cn is obtained. By balancing the terms in the highest order, it yields $M = 2$. Assuming, for simplicity, that u is an even solution, it then takes the form

$$u(\xi) = a_0 + a_2\mathrm{cn}^2(\lambda\xi, m). \tag{4.2}$$

The corresponding half-period l of u can therefore be computed using an elliptic integral

$$l = \frac{1}{\lambda}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-m^2\sin^2 t}}\,\mathrm{d}t, \tag{4.3}$$

by noting that in (4.2) there are no odd power terms of cn, so the fundamental period is half of that for $\mathrm{cn}(\lambda\xi, m)$.

Substituting (4.2) into (3.9) and also using

$$(\mathrm{cn}^2)''(v,m) = 2 - 2m + 4(2m^2-1)\mathrm{cn}^2(v,m) - 6m^2\mathrm{cn}^4(v,m),$$

one obtains an equation in the form of

$$c_0 + c_2\mathrm{cn}(\lambda\xi,m)^2 + c_4\mathrm{cn}(\lambda\xi,m)^4 + c_6\mathrm{cn}(\lambda\xi,m)^6 = 0,$$

where c_i are functions of $a_0, a_2, \omega, \lambda$ and m. By requiring all the coefficients $c_i = 0, i = 0, 2, 4$ and 6, one arrives at a nonlinear algebraic system in $a_0, a_2, \omega, \lambda$ and m. With the help of the software Maple, a set of non-trivial solutions is found where

$$a_0 = \frac{1}{9}\omega(9 - 20m^2\lambda^2 + 10\lambda^2), \qquad a_2 = \frac{10}{3}\omega\lambda^2 m^2,$$

with λ being a root of

$$F(m^2,\lambda^2) \equiv 160(m^2-2)(2m^2-1)(m^2+1)\lambda^6 + 1188(m^4-m^2+1)\lambda^4 - 729 = 0 \tag{4.4}$$

and ω^2 satisfying

$$135(m^2-2)(2m^2-1)(m^2+1)\omega^2 = -G_1(m^2,\lambda^2),$$

where

$$\begin{aligned} G_1(m^2,\lambda^2) &= 270(m^2-2)(2m^2-1)(m^2+1) + 3267\lambda^2(m^4-m^2+1)^2 \\ &\quad + 440\lambda^4(m^2-2)(2m^2-1)(m^2+1)(m^4-m^2+1). \end{aligned} \tag{4.5}$$

Therefore, for a fixed $m \in [0, 1]$ and $m^2 \neq \frac{1}{2}$, the cnoidal wave solutions exist if there are real positive solutions λ^2 to $F(m^2, \lambda^2) = 0$ which satisfy

$$\omega^2(m^2,\lambda^2) = -\frac{G_1(m^2,\lambda^2)}{135(m^2-2)(2m^2-1)(m^2+1)} \geqslant 0. \tag{4.6}$$

It is worth noting that without loss of generality, only the positive solutions of λ and ω need to be considered because λ and $-\lambda$ offer the same solution and ω and $-\omega$ offer a pair of solutions, one is right propagating and the other is left propagating.

In figure 1, the curves where $F(m^2, \lambda^2) = 0$ are plotted and the regions are separated by the signs of $\omega^2(m^2, \lambda^2)$ which is calculated from (4.6). It is shown that for each fixed $m^2 > \frac{1}{2}$, there are two positive λ^2 satisfying $F(m^2, \lambda^2) = 0$. By denoting the two solutions of $F = 0$ as $\lambda_1^2(m^2)$ and $\lambda_2^2(m^2)$ with $|\lambda_1| \leqslant |\lambda_2|$, it is clear that λ_1^2 is in the ω^2-positive region. This is also the case for λ_2^2 with $0.501 \leqslant m^2 \leqslant 1$. Figure 1(*b*) zooms in a region near the curve $\lambda_2^2(m^2)$ to show this fact. For m^2 in $(0.5, 0.501)$, the loss of significant digits occurs in the computation. Specifically, when $m^2 = 0.501$, $\lambda_2^2 \approx 1.24 \times 10^3$ and $\omega^2 \approx 8.82 \times 10^{-7}$. The situation is more severe as $m^2 \to \frac{1}{2}^+$ with the resulting $\lambda_2^2 \to \infty$ and ω^2 very close to 0. The sign of ω^2, which dictates the existence of the second branch of cnoidal wave solutions, is hard to identify. It is worth noting that we are not particularly interested in this region since as $\lambda_2^2 \to \infty$ and

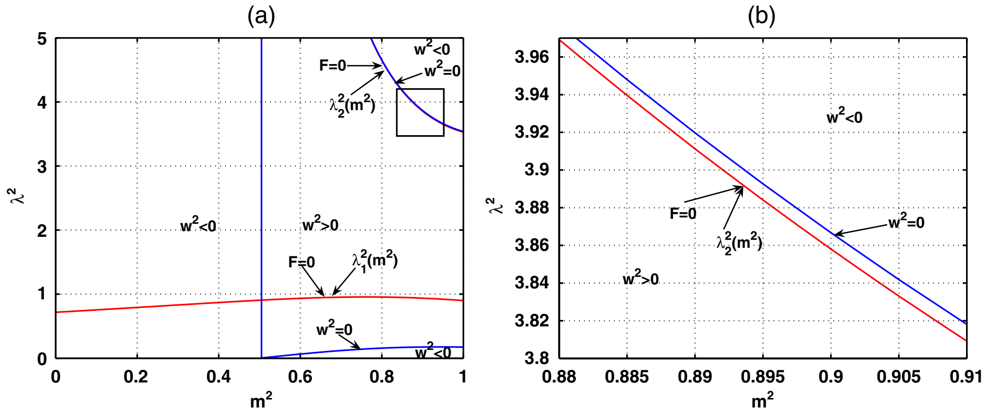

Figure 1. (*a*) The location $F(m^2, \lambda^2) = 0$ and the region where $\omega^2(m^2, \lambda^2) \geqslant 0$ are indicated; (*b*) same as (*a*) with a zoom in the boxed region.

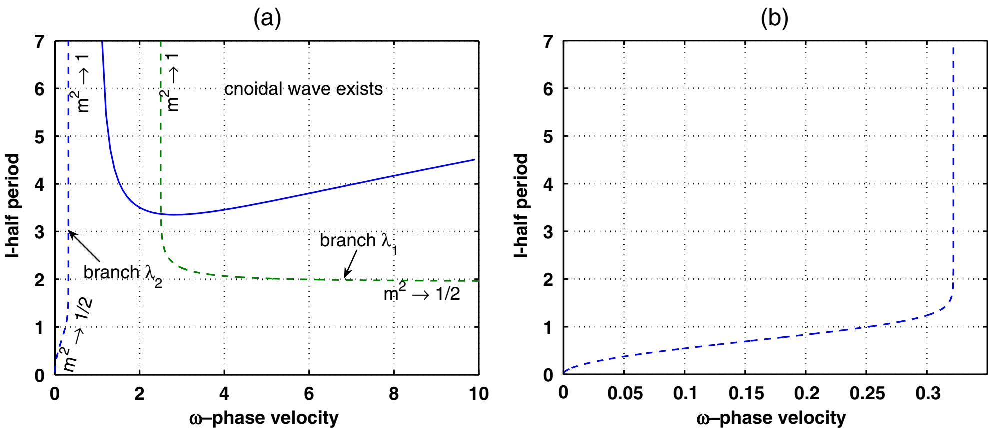

Figure 2. (*a*) Location where the cnoidal wave solutions exist: at any point above the solid line (existence theory) and at any point on dashed lines (explicit formula); (*b*) same as (*a*) with a zoom near the origin.

$m^2 \to \frac{1}{2}^+$, the corresponding half-period l, according to (4.3), is approaching zero. Therefore, the wave length is small which is outside the validity region of the Boussinesq systems.

The corresponding half-period l and the phase velocity ω for the solutions corresponding to branches λ_1 and λ_2 are plotted in figure 2(*a*). The branch corresponding to λ_1 has phase velocity from $\frac{5}{2}$ to ∞, as m^2 varies from 1 to $\frac{1}{2}$, with the corresponding half-period from $+\infty$ to around 1.95. As $m^2 \to \frac{1}{2}^+$ which corresponds to the right 'endpoint', $\lambda_1^2 \to \sqrt{\frac{9}{11}}$ and $\omega^2 \to \infty$. When $m^2 = 1$ which corresponds to the left 'endpoint', one recovers the known solitary wave solution

$$
\begin{aligned}
u(\xi) &= \pm\frac{15}{2}\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}\xi\right),\\
\eta(\xi) &= \frac{15}{4}\left(2\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}\xi\right) - 3\mathrm{sech}^4\left(\frac{3}{\sqrt{10}}\xi\right)\right),
\end{aligned}
\tag{4.7}
$$

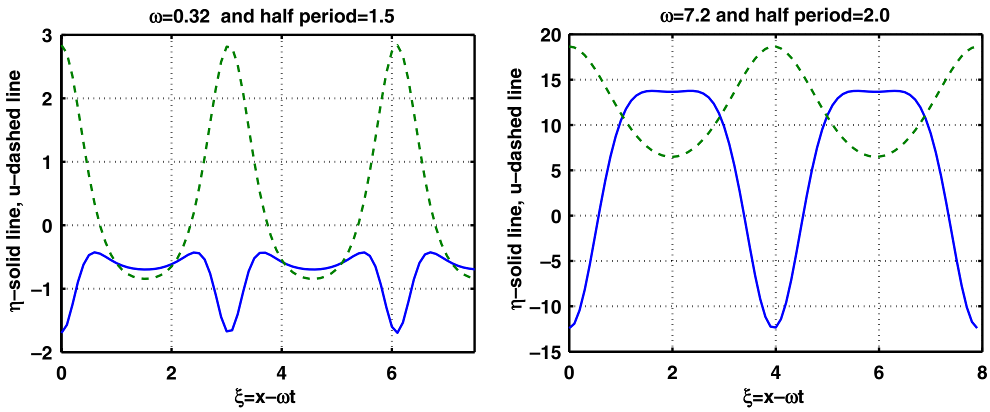

Figure 3. Examples of solutions.

with $\omega = \frac{5}{2}$, found in [3, 8]. This calculation provides an explicit and concrete proof that the solitary wave (4.7) is a limit of cnoidal wave solutions as l approaches infinity, a fact which may not be easy to prove otherwise. It is expected that this is also the case for other branches of cnoidal wave solutions. The branch corresponding to λ_2 are cnoidal wave solutions with the phase velocity between 0 and $c_0 = (-\frac{49}{8} + \frac{33}{40}\sqrt{57})^{\frac{1}{2}} \approx 0.3219$. It is worth noting that these solutions are not in the parameter range where the index theory in section 3 is applicable. As $m^2 \to \frac{1}{2}^+$, the phase velocity and half-period appear to approach zero and the detail around that point is shown in figure 2(*b*). At $m^2 = 1$, this is again a solitary wave solution found in [8].

Figure 2(*a*) also shows the region, which is above the solid curve, where for any point in the region, namely for a specified half-fundamental-period l and a phase speed ω, there is a cnoidal wave solution corresponding to that point (the result of section 3). The dashed lines indicate the parameters where the explicit solutions exist. The solutions corresponding to the part of branch λ_1 located below the solid curve and the ones corresponding to branch λ_2 are not covered by theorem 3.11 in section 3 (which is non-constructive). Therefore it demonstrates that the condition (3.26) for existence of the cnoidal wave solution is sufficient, but not necessary. In figure 3, two pairs of explicit solutions, one with $\omega = 0.32$ and another with $\omega = 7.2$, are plotted. The solid lines are for $\eta(x,t) = \eta(\xi)$ and the dashed lines are for $u(x,t) = u(\xi)$. It is noted that these solutions are not in the regime covered by theorem 3.11 according to figure 2(*a*).

The above results are summarized in the following theorem.

Theorem 4.1. *For any fixed m with* $\frac{1}{2} < m^2 \leqslant 1$, *there are two positive roots, denoted by* λ_1^2 *and* λ_2^2 *with* $|\lambda_1| \leqslant |\lambda_2|$, *of* $F(m^2, \lambda^2) = 0$, *where F is defined in (4.4). Denoting* $\xi = x - \omega t$, *the corresponding cnoidal wave solutions in the form of*

$$u(x,t) = u(\xi) = a_0 + a_2 \mathrm{cn}^2(\lambda\xi, m) \tag{4.8}$$

exist for $\lambda = \lambda_1(m)$ *with* $\frac{1}{2} < m^2 \leqslant 1$ *and for* $\lambda = \lambda_2(m)$ *with* $0.501 \leqslant m^2 \leqslant 1$. *The phase speed* ω *is defined by (4.5) and (4.6) and*

$$a_0 = \frac{1}{9}\omega(9 + 10m^2\lambda^2 + 10\lambda^2), \qquad a_2 = -\frac{10}{3}\omega\lambda^2 m^2.$$

Moreover, the solution $\eta(x,t)$ *has the form*

$$\eta(x,t) = \omega u - \frac{1}{2}u^2 - \frac{\omega}{6}\lambda^2 u'_{\xi\xi} = b_0 + b_2 cn^2(\lambda\xi, m) + b_4 cn^4(\lambda\xi, m),$$

where

$$b_0 = \frac{1}{2}\omega^2 - \frac{10}{81}\omega^2\lambda^4(11m^4 - 11m^2 + 5), \qquad b_2 = \frac{40}{27}\omega^2\lambda^4 m^2(2m^2 - 1),$$

$$b_4 = -\frac{20}{9}\omega^2\lambda^4 m^4.$$

In summary, for the coupled BBM-system, we proved the existence of cnoidal waves in a 2D region on the l–ω plane, where l is the half-period and ω the phase speed, by a topological method and find explicit cnoidal wave solutions for $l = l(\omega)$ which is determined by (4.3)–(4.6). The region and the function $l = l(\omega)$ are shown in figure 2(a). For Boussinesq systems with $b, d > 0$ and $a, c \leqslant 0$ which satisfy (1.2), we proved, using a topological method, the existence of cnoidal waves in two regions, which degenerate to one when a or c is 0, on the $l - \omega$ plane.

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