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Generalized regularized long wave equation with white noise dispersion

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Abstract In this article, we address the generalized BBM equation with white noise dispersion which reads

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0,$$

in the Stratonovich formulation, where $W(t)$ is a standard real valued Brownian motion. We first investigate the well-posedness of the initial value problem for this equation. We then prove theoretically and numerically that for a deterministic initial data, the expectation of the L_x^∞ norm of the solutions decays to zero at $O(t^{-\frac{1}{6}})$ as t approaches to $+\infty$, by assuming that $p > 8$ and that the initial data is small in $L_x^1 \cap H_x^4$. This decay rate matches the one for solutions of the linear equation with white noise dispersion.

Keywords Water wave · White noise dispersion · Decay rate

1 Introduction

The generalized Benjamin–Bona–Mahony (gBBM) equation, which is also called generalized regularized long wave equation,

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$$u_t - \frac{1}{6}u_{txx} + u_x + u^p u_x = 0, \tag{1.1}$$

has been widely studied theoretically and numerically in the mathematical literature. Examples of such investigations include study of the initial value problem as in [4] and study of the decay rate of solutions when the initial data is small [2].

These analysis and numerical simulations are based on the assumption that the outside force is zero or deterministic, so the solution is deterministic. But in reality, the bottom geography, such as the sand beach, and the air pressure on the surface, such as the wind effect, are stochastic in nature. It is therefore desirable to study the stochastic water wave equations. The exploration of stochasticity for dispersive equations initiated first by considering forcing terms with additive white noise and then moved on to forcing terms with multiplicative white noise; see [5,10,14–16,19,24,26] and the references therein. But as shown in [7], the bottom topography and the pressure variation not only affect the forcing term, but also affect the linear terms of the water wave equations.

Recently, for a particular case of nonlinear Schrödinger equations, stochasticity was introduced in the dispersion through the linear part of the equation, which oscillates following the variations of a Brownian motion. In [11,18] the authors studied nonlinear Schrödinger equation (NLS) with white noise modulation and they showed that it describes the homogenization of the deterministic NLS equations with time dependent dispersion satisfying some ergodicity properties. Such investigation furthers the study where the dispersion is driven by a deterministic oscillating function [3]. More general modulated dispersion associated with the initial value problem for NLS and Korteweg–de Vries equation was investigated recently in [8,9]. In this article, we are following this direction and introducing the stochasticity through the linear part of the generalized BBM equation.

The article is organized as follows. In Sect. 2, we introduce the mathematical framework and we state the main results of this article. In Sect. 3, we prove the global well posedness for the initial value problem associated to the generalized BBM Eq. (2.4) for any value $p \geq 1$. Here we emphasize that due to the particular structure of the Stratonovich product, the H_x^1 norm of the solution is conserved with respect to time. In Sect. 4, we address the question of the decay rate of the solutions. We shall show that, for the linear problem (2.2), the expectation of the L_x^∞ norm of the solutions is decreasing to zero, but at a slower rate than the solutions of the corresponding deterministic equation. We then prove a similar result for the nonlinear problem by assuming that $p > 8$ and that the initial data is small in $L_x^1 \cap H_x^4$; here the initial data is assumed to be deterministic. We shall finish this work with numerical simulations that demonstrate the theoretical results obtained on decay rates are sharp.

We complete this introduction with some notations. In general, the function $f(s)$ is a random function in $\mathcal{F}_s$. The number s is assumed to satisfy $s < t$. The standard assumptions on $W(t)$ are that $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, where $\mathcal{N}(0, t-s)$ is a normally distributed random numbers with mean 0 and variance $t - s$, and that $W(t) - W(s)$ is independent of the past $\mathcal{F}_s$. We denote by H_x^m the standard Hilbert space on the x variable. The notation L_x^p will be used for standard Lebesgue spaces. The Banach space $L_x^1 \cap H_x^m$ will be endowed with the sum of the L_x^1 and H_x^m norms.

The notation $\widehat{u}$ denotes the Fourier transform of u in space. We set A for the operator $(1-\Delta)^{-1}\partial_x$ which maps H_x^m into H_x^{m+1} for any m. For the sake of simplicity, the random variable ω and the space variable x may be omitted throughout this article. The constant c is a numerical positive constant that may vary from one line to another and we set $< x >= \sqrt{1+x^2}$.

2 Preliminaries and main results

2.1 White noise modulated dispersion

In this article, we plan to address the generalized BBM equation with white noise dispersion. Let $W(t)$ be a standard real valued Brownian motion. Associated with this Brownian motion, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Now we consider the linear BBM equation, that reads in the deterministic setting as

$$u_t - u_{xxt} + u_x = 0, \tag{2.1}$$

with a white noise dispersion. For simplicity of notation, we set the constant to be 1 for the moment. For later use, it is convenient to observe that (2.1) reads also as the ODE in $H^1(\mathbb{R})$

$$u_t + Au = 0,$$

where $A = (I - \partial_x^2)^{-1}\partial_x$ is the bounded skew symmetric operator whose symbol is $i\xi(1+\xi^2)^{-1}$.

Consider now u_s being a random $\mathcal{F}_s$ variable which takes values in some Hilbert space such as H_x^1. We seek for $s < t$ and $x \in \mathbb{R}$ a process $u(t,x)$ that is a solution of the Stochastic Differential Equation (SDE) in $H^1(\mathbb{R})$

$$\begin{aligned} du + AudW - \frac{1}{2}A^2udt &= 0, \\ u(s) &= u_s. \end{aligned} \tag{2.2}$$

This SDE (written in its Ito formulation) reads in its formally equivalent Stratonovich formulation (a short hand version) as

$$\begin{aligned} du + Au \circ dW &= 0, \\ u(s) &= u_s. \end{aligned}$$

It is well known (see [23]) that a solution of (2.2) is defined through its Fourier transform in space as follows

$$\widehat{u}(t,\xi) = e^{-\frac{i\xi}{1+\xi^2}(W(t)-W(s))}\widehat{u_s}(\xi). \tag{2.3}$$

In comparison, the solution of usual BBM equation with deterministic dispersion (2.1) reads

$$\widehat{u}(t,\xi) = e^{-\frac{i\xi}{1+\xi^2}(t-s)}\widehat{u}_s(\xi).$$

It is worth to point out that the Brownian motion acts both on the drift term and on the dispersive term. This can be seen from the following. The BBM equation is equivalent to the KdV equation in the long wavelength regime, namely when ξ is sufficiently small. In this regime

$$\frac{i\xi}{1+\xi^2}(W(t)-W(s)) \simeq (i\xi - i\xi^3)(W(t)-W(s)).$$

Therefore, formally, the variation of the Brownian motion acts also on the KdV term $i\xi^3$ which is the dispersion term.

The nonlinear equation we will address reads

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0, \tag{2.4}$$

in Stratonovich's formulation. Once again this is a short-hand notation for an equation in Ito's formulation that reads

$$du + AudW - \frac{1}{2}A^2 udt + A\left(\frac{u^{p+1}}{p+1}\right)dt = 0. \tag{2.5}$$

2.2 Statements of the main results

We first prove the well posedness of the initial value problem that is valid for any $p \geq 1$.

Theorem 2.1 *Let u_0 be $\mathcal{F}_0$ measurable and in $L^2(\Omega; H^1_x)$. Then there exists a unique solution $u(t)$ of the Eq. (2.5), adapted to the filtration $\mathcal{F}_t$, with paths almost surely (a.s.) in $C(0,+\infty; H^1_x)$. Moreover a.s. the H^1_x norm of the solution is conserved, that is $||u(t)||_{H^1_x} = ||u_0||_{H^1_x}$ for all t.*

The classical deterministic result (see [2]) asserts that if the initial data u_0 is small enough in $L^1_x \cap H^4_x$ and p is large enough, then the solution decays in L^∞_x as $O(t^{-\frac{1}{3}})$ when $t \to \infty$, which is the same rate as for the solutions to the corresponding linear problem. We expect an analogous result for the equation with stochastic dispersion management, but with a decay rate which is half of the deterministic case. We consider the set of random solutions starting from a deterministic initial data, and we prove that in expectation the random solutions converge as fast as the solutions of the linear random equation.

Theorem 2.2 *Fix $p > 8$. Consider $u_0 \in L^1_x \cap H^4_x$ such that $\|u_0\|_{L^1_x \cap H^4_x} < \varepsilon_0$ where ε_0 is small enough. Then the solution u defined in Theorem 2.1 satisfies for any $t > 0$*

$$\mathbb{E}(||u(t)||_{L^\infty_x}) \leq C(\varepsilon_0) < t >^{-\frac{1}{6}}. \tag{2.6}$$

3 The initial value problem

Denote the solution of Eq. (2.2) by $u(t) = S(t,s)u(s)$ given by (2.3). Then almost surely in ω the linear operator $S(t,s)$ defines an isometry in any Sobolev space H_x^m. Our goal is to seek a mild solution of the Eq. (2.4) with initial condition $u(x,0) = u_0$, namely a solution of the following Duhamel form

$$u(t) = S(t,0)u_0 - \frac{1}{p+1}\int_0^t S(t,s)A(u^{p+1}(s))ds, \tag{3.1}$$

where $A = (1-\Delta)^{-1}\partial_x$.

To prove Theorem 2.1, we follow a classical strategy for stochastic PDEs (see [11,18,23]). We truncate the nonlinearity to have a globally Lipschitz mapping acting on the Banach algebra $H^1(\mathbb{R})$, and then we pass to the limit.

3.1 Solving a truncated equation

To begin with, we recall that for any $m \geq 1$, there exists a constant $c_{m,p} > 0$, that depends on m and p, such that for any function f in H_x^m

$$||f^{p+1}||_{H_x^m} \leq c_{m,p}||f||_{L_x^\infty}^p ||f||_{H_x^m}. \tag{3.2}$$

A general proof of (3.2) for Lebesgue spaces appeared in [21]. In addition, due to the Sobolev embedding $H_x^1 \subset L_x^\infty$, it is straightforward to check that the map $u \to Au^{p+1}$ is a locally Lipschitz mapping in H_x^1.

We now introduce a smooth monotonous decreasing cutoff function θ that satisfies $\theta(s) = 1$ for $|s| \leq 1$ and $\theta = 0$ for $|s| \geq 2$ and we investigate the solution of the equation

$$u_R(t) = S(t,0)u_0 - \frac{1}{p+1}\int_0^t S(t,s)\theta_R(s)A\left(u_R^{p+1}(s)\right)ds, \tag{3.3}$$

where $\theta_R(s) = \theta(\frac{||u_R(s)||_{H_x^1}}{R})$.

Proposition 3.1 *Let u_0 be $\mathcal{F}_0$ measurable and in $L^2(\Omega; H_x^1)$. Then there exists a unique solution u_R of the Eq. (3.3) in $L^2(\Omega; C([0,+\infty); H_x^1))$, $u_R(s)$ being adapted to the filtration $\mathcal{F}_s$.*

Proof We omit the subscript R to write $u_R = u$ for the sake of simplicity. Now, fixing R_0 such that $R_0 \geq 2||u_0||_{L^1(\Omega;H_x^1)}$, we plan to use the fixed point argument for the Eq. (3.3) in the ball B_0 of radius R_0 in the Banach space $X_T = \{u \in L^2(\Omega; C([0,T]; H_x^1)); u(s)$ adapted to $\mathcal{F}_s\}$ where T will be specified later.

Let $\mathcal{T}_1$ be the right hand side of (3.3) and let u, v be in B_0. Since for $0 < t < T$,

$$\begin{aligned}||\mathcal{T}_1(u)(t)||_{H_x^1} \leq& ||u_0||_{H_x^1} + c\int_0^T \theta_R(s)||u(s)||_{H_x^1}^{p+1}ds\\ \leq& ||u_0||_{H_x^1} + cT(2R)^{p+1},\end{aligned}$$

the expectation satisfies

$$\mathbb{E}\left(||\mathcal{T}_1(u)||^2_{L^\infty(0,T;H_x^1)}\right) \leq \frac{R_0^2}{2} + cT^2(2R)^{2p+2} \leq R_0^2.$$

For T small enough, which may depend on R_0 and on R, $\mathcal{T}_1$ maps B_0 into B_0.

Now consider the difference between the two solutions u and v. Here we write $\theta_u(s) = \theta(\frac{||u(s)||_{H_x^1}}{R})$. We proceed as in [11] and obtain

$$||\mathcal{T}_1(u)(t) - \mathcal{T}_1(v)(t)||_{H_x^1} \leq c\int_0^T ||\theta_u(s)u(s)^{p+1} - \theta_v(s)v(s)^{p+1}||_{H_x^1}ds. \quad (3.4)$$

We seek an upper bound for $||\theta_u(s)u(s)^{p+1} - \theta_v(s)v(s)^{p+1}||_{H_x^1}$. Assume that for a given s and a given ω, $||v(s)||_{H_x^1} \leq ||u(s)||_{H_x^1}$. This is equivalent to $\theta_u(s) \leq \theta_v(s)$. By the triangle inequality

$$\begin{aligned}&||\left(\theta_u(s)u(s)^{p+1} - \theta_v(s)v(s)^{p+1}\right)||_{H_x^1}\\ &\leq \theta_u(s)||u(s)^{p+1} - v(s)^{p+1}||_{H_x^1} + |\theta_u(s) - \theta_v(s)|||v(s)^{p+1}||_{H_x^1}.\end{aligned}$$

Firstly

$$\theta_u(s)||u(s)^{p+1} - v(s)^{p+1}||_{H_x^1} \leq c(2R)^p||u(s) - v(s)||_{H_x^1}.$$

Since $||v(s)||_{H_x^1} > 2R$ implies $|\theta_u(s) - \theta_v(s)| = 0$, we assume $||v(s)||_{H_x^1} \leq 2R$. In this case we have

$$|\theta_u(s) - \theta_v(s)| \leq cR^{-1}||u(s) - v(s)||_{H_x^1},$$

and therefore

$$||v(s)^{p+1}||_{H_x^1}|\theta_u(s) - \theta_v(s)| \leq cR^p||u(s) - v(s)||_{H_x^1}.$$

The reverse case $\theta_u(s) \geq \theta_v(s)$ is done similarly. Then in any case and for $s < t$

$$\left|\left|\left(\theta_u(s)u(s)^{p+1} - \theta_v(s)v(s)^{p+1}\right)\right|\right|_{H_x^1} \leq cR^p||u(s) - v(s)||_{H_x^1}.$$

Coming back to (3.4) and considering the expectation, we find

$$\mathbb{E}(||\mathcal{T}_1(u) - \mathcal{T}_1(v)||^2_{L^\infty(0,T;H_x^1)}) \leq cR^{2p}T^2\mathbb{E}(||u - v||^2_{L^\infty(0,T;H_x^1)}).$$

When T is small enough depending on R and R_0, $\mathcal{T}_1$ is a contraction mapping and there exists a unique fixed point in X_T. □

4 Passing to the limit $R \to +\infty$

We first state and prove

Proposition 4.1 *The mild solution given by Proposition 3.1 is the classical solution of the truncated SDE*

$$du_R + Au_R - \frac{1}{2}A^2 u_R dt + \theta_R A\left(\frac{u^{p+1}}{p+1}\right)dt = 0. \tag{4.1}$$

Proof Proceeding as Theorem 8.1.8 in [22] (or as in [20,23]), we can prove that the Eq. (4.1) has a unique solution in $L^2(\Omega; C[0,T](H^1(\mathbb{R})))$ constructed by the fixed point theorem on the mapping

$$v \mapsto -\int_0^t Au(s)dW_s + \int_0^t A\left(Au - \theta_R \frac{u^{p+1}}{p+1}\right)ds.$$

Moreover the mild solution reads $u_R(t) = S(t,0)Y(t)$, where $Y(t) = u_0 - \frac{1}{p+1}\int_0^t S(0,-s)\theta_R(s) f(u_R(s))ds$. Applying Ito's chain rule formula we establish that u_R is also the solution of (4.1). □

We now prove that the H^1_x norm of this solution does not depend on t.

Lemma 4.2 *Almost surely in ω, $||u_R(t)||_{H^1(\mathbb{R})} = ||u_0||_{H^1(\mathbb{R})}$.*

Proof We drop the subscript R in the proof. We then introduce the process $X = ||u||^2_{H^1_x}$. Formally, the Fourier transform of Eq. (4.1) reads

$$d\widehat{u} = \mu(\widehat{u})dt + \sigma(\widehat{u})dW, \tag{4.2}$$

with

$$\mu(\widehat{u}) = -(1+\xi^2)^{-1}\left(\frac{\xi^2}{2(1+\xi^2)}\hat{u} + \theta_R(t) i\xi \frac{\widehat{u^{p+1}}}{p+1}\right),$$

and

$$\sigma(\widehat{u}) = -\frac{i\xi}{1+\xi^2}\widehat{u}.$$

By Ito's chain rule formula

$$dX = 2\mathrm{Re}\left(\int (1+\xi^2)\overline{\widehat{u}}\sigma(\widehat{u})d\xi\right)dW$$
$$+ 2\mathrm{Re}\left(\int (1+\xi^2)\overline{\widehat{u}}\mu(\widehat{u})d\xi\right)dt + \int (1+\xi^2)|\sigma(\widehat{u})|^2 d\xi dt.$$

The first term vanishes because $\mathrm{Re}\left(\int |\widehat{u}|^2 i\xi d\xi\right) = 0$. The second term expands as

$$-\mathrm{Re}\left(\int \frac{\xi^2}{1+\xi^2}|\hat{u}(\xi)|^2 d\xi\right) - 2\theta_R(t)\mathrm{Re}\left(\int i\xi\overline{\widehat{u}}\frac{\widehat{u^{p+1}}}{p+1}d\xi\right). \tag{4.3}$$

We observe that the first term in the right hand side of (4.3) cancels with $\int(1+\xi^2)|\sigma(\widehat{u})|^2 d\xi$. The remaining term can be computed using Plancherel formula as

$$\theta_R(t)\int u_x u^{p+1} dx = 0.$$

Finally $dX = 0$, and $||u_R(t)||^2_{H^1_x} = ||u_0||^2_{H^1_x}$ holds a.s. In fact, this formal computation can be proved rigorously, as in the proof of Theorem 4.1 in [11], mollifying u using a suitable truncation function of $\widehat{u}$ in the Fourier space and by a limiting argument. □

To complete the proof of Theorem 2.1, it remains to construct a solution of the original equation without truncation.

We proceed as follows. Consider an increasing sequence R_n that diverges towards $+\infty$. Let u_n be the sequence of solutions constructed by the fixed point argument on the truncated Eq. (3.3) at level $R = R_n$. Actually u_n is solution to

$$u_n(t) = S(t,0)u_0 - \frac{1}{p+1}\theta\left(\frac{||u_0||_{H^1_x}}{R_n}\right)\int_0^t S(t,s)A(u_n^{p+1}(s))ds, \tag{4.4}$$

since the H^1 norm is conserved. Pick now $\omega \in \Omega$ such that $||u_0||_{H^1_x} < +\infty$. This assertion is possible on a subset of Ω of probability 1. Taking n is large enough, depending on ω, so that $R_n \geq ||u_0||_{H^1_x}$. Then we have $\theta(\frac{||u_0||_{H^1_x}}{R_n}) = 1$. Therefore $u_{R_n} = u_{R_{n+m}}$ for any $m \geq 0$. The limit $u = \lim_{n\to+\infty} u_{R_n}$, that converges a.s., is solution of the original equation. Actually, we can prove that u_n converges towards u in $L^2([0,T]\times\Omega; H^1_x)$. For the uniqueness, consider two solutions u, v starting from u_0. For a given ω such that $R > ||u_0||_{H^1}$, the paths $u(t), v(t)$ starting from u_0 are solutions to the mild Eq. (3.3) at level R. Since we have uniqueness for this equation for solutions in $C(0,T;H^1_x)$, then $u = v$ a.s.

5 Decay of solutions

5.1 Linear BBM equation with white noise dispersion

Consider a solution of the linear BBM Eq. (2.2).
We have:

Lemma 5.1 *Assume $u(s)$ be $\mathcal{F}_s$ measurable and in $L_x^1 \cap H_x^4$ a.s., then*

$$||S(t,s)u(s)||_{L_x^\infty} \le c \frac{||u(s)||_{L_x^1\cap H_x^4}}{< W(t) - W(s) >^{\frac{1}{3}}}. \tag{5.1}$$

Proof Due to the stationary phase lemma in [2], we have

$$||S(t,s)u(s)||_{L_x^\infty} \le c \frac{||u(s)||_{L_x^1\cap H_x^4}}{|W(t) - W(s)|^{\frac{1}{3}}}. \tag{5.2}$$

This provides the estimate (5.1) for $|W(t) - W(s)|$ larger than 1. We also have

$$||S(t,s)u(s)||_{L_x^\infty} \le c||S(t,s)u(s)||_{H_x^1} \le c||u(s)||_{H_x^1}.$$

which gives the estimate for $|W(t) - W(s)|$ smaller than 1 and completes the proof. □

Using estimate (5.2), the following theorem can be obtained.

Theorem 5.2 *Assume $u(s)$ be $\mathcal{F}_s$ measurable and in $L^1(\Omega; L_x^1 \cap H_x^4)$, then for $t > s$*

$$\mathbb{E}(||S(t,s)u(s)||_{L_x^\infty}) \le c \frac{\mathbb{E}(||u(s)||_{L_x^1\cap H_x^4})}{|t-s|^{\frac{1}{6}}}. \tag{5.3}$$

Proof Since $W(t) - W(s)$ is independent of $\mathcal{F}_s$,

$$\mathbb{E}\left(\frac{||u(s)||_{L_x^1\cap H_x^4}}{|W(t) - W(s)|^{\frac{1}{3}}}\right) = \mathbb{E}(|W(t) - W(s)|^{-\frac{1}{3}})\mathbb{E}(||u(s)||_{L_x^1\cap H_x^4}).$$

Now because $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ which has probability density function $\frac{1}{\sqrt{2\pi(t-s)}} \exp(-\frac{x^2}{2(t-s)})$,

$$\mathbb{E}\left(|W(t) - W(s)|^{-\frac{1}{3}}\right) = c \int \exp\left(-\frac{x^2}{2(t-s)}\right) \frac{dx}{|t-s|^{\frac{1}{2}}|x|^{\frac{1}{3}}} \le c\frac{1}{|t-s|^{\frac{1}{6}}}.$$

Using (5.2), we obtain the desired result. □

Remark 5.3 It is worth to point out that the average decay rate of the solution is $\frac{1}{6}$ which is half of the rate for the BBM equation with deterministic dispersion. This may be explained because $\mathbb{E}(|W(t) - W(s)|^2) = t - s$. Then the oscillations of solutions propagate in average twice as slow as in the deterministic case.

5.2 Decay for the nonlinear problem

We now handle the proof of Theorem 2.2. The proof is divided into several lemmata. Throughout this section, u_0 is a deterministic function. We first prove that if the initial data is in H^3_x, then the solution remains in H^3_x.

Lemma 5.4 *Consider $u(t)$ the solution of the equation given by Theorem 2.1. If u_0 belongs to H^3_x and satisfies $||u_0||_{H^3_x} \leq \varepsilon_0$, there exists a constant $c > 0$ that does not depend on t such that a.s.*

$$||u(t)||_{H^3_x} \leq c\varepsilon_0 \left(1 + \int_0^t ||u(s)||^p_{L^\infty_x} ds\right)^2. \tag{5.4}$$

Proof Since

$$u(t) = S(t,0)u_0 - \frac{1}{p+1}\int_0^t S(t,s)A(u^{p+1}(s))ds,$$

and A maps H^2_x into H^3_x, we have

$$||u(t)||_{H^3_x} \leq ||u_0||_{H^3_x} + c\int_0^t ||u^{p+1}(s)||_{H^2_x} ds. \tag{5.5}$$

Gathering (3.2) with (5.5), we find

$$||u(t)||_{H^3_x} \leq ||u_0||_{H^3_x} + c\int_0^t ||u(s)||_{H^2_x}||u(s)||^p_{L^\infty_x} ds.$$

Using interpolation and the conservation of the H^1_x norm, we obtain

$$||u(t)||_{H^3_x} \leq c\varepsilon_0 + c\sqrt{\varepsilon_0}\int_0^t ||u(s)||^{\frac{1}{2}}_{H^3_x}||u(s)||^p_{L^\infty_x} ds. \tag{5.6}$$

Let $\varphi(t)$ be the right hand side of (5.6), we can write

$$\dot{\varphi}(t) \leq c\sqrt{\varepsilon_0}||u(t)||^p_{L^\infty_x}\sqrt{\varphi(t)}.$$

Integrating in time completes the proof of the lemma. □

Remark 5.5 A similar result holds for any H^m_x norm with $m \geq 1$, namely

$$||u(t)||_{H^m_x} \leq c||u_0||_{H^m_x}\left(1 + \int_0^t ||u(s)||^p_{L^\infty_x} ds\right)^{m-1}.$$

A consequence of these inequalities is that any solution that has enough decay in L^∞_x is uniformly bounded in any H^m_x.

We now prove a key inequality (5.7) which provides a link between the modulus of the Wiener process $< W(t) >$ and $||u(t)||_{L^\infty_x}$.

Lemma 5.6 *Consider $u(t)$ the solution of the equation given by Theorem 2.1. Assume $u_0 \in L^1_x \cap H^4_x$ and $||u_0||_{L^1_x \cap H^4_x} \leq \varepsilon_0$. Then a.s we have*

$$||u(t)||_{L^\infty_x} \leq \frac{\varepsilon_0}{< W(t) >^{\frac{1}{3}}} \left(1 + (\varepsilon_0 \int_0^t < W(s) >^{\frac{1}{3}} ||u(s)||^{p-1}_{L^\infty_x} ds)^3 \right). \tag{5.7}$$

Proof Using the linear estimate (5.1) we obtain that for a solution of (2.4)

$$||u(t)||_{L^\infty_x} \leq c \frac{||u_0||_{L^1_x \cap H^4_x}}{< W(t) >^{\frac{1}{3}}} + c \int_0^t \frac{||Au^{p+1}||_{L^1_x \cap H^4_x}}{< W(t) - W(s) >^{\frac{1}{3}}} ds.$$

The right hand side can be bounded by observing

$$||Au^{p+1}||_{L^1_x} \leq c||u_x u^p||_{L^1_x} \leq c||u||^2_{H^1_x} ||u||^{p-1}_{L^\infty_x}.$$

Using (3.2) and the Sobolev embedding $H^1_x \subset L^\infty_x$,

$$||Au^{p+1}||_{H^4_x} \leq c||u||^p_{L^\infty_x} ||u||_{H^3_x} \leq c||u_0||_{H^1_x} ||u||^{p-1}_{L^\infty_x} ||u||_{H^3_x}.$$

These inequalities, combined with the H^3 bound (5.4), give

$$||u(t)||_{L^\infty_x} \leq c \frac{\varepsilon_0}{< W(t) >^{\frac{1}{3}}} + c\varepsilon_0^2 \int_0^t \frac{||u(s)||^{p-1}_{L^\infty_x} (1 + \int_0^s ||u||^p_{L^\infty_x} ds)^2}{< W(t) - W(s) >^{\frac{1}{3}}} ds. \tag{5.8}$$

From the triangular inequality, we have

$$< W(t) > \leq \sqrt{2} < W(t) - W(s) >< W(s) > .$$

Therefore (5.8) implies

$$||u(t)||_{L^\infty_x} \leq$$
$$c \frac{\varepsilon_0}{< W(t) >^{\frac{1}{3}}} \left(1 + c\varepsilon_0 \int_0^t < W(s) >^{\frac{1}{3}} ||u(s)||^{p-1}_{L^\infty_x} (1 + \int_0^s ||u||^p_{L^\infty_x} ds)^2) ds \right).$$

Set

$$I(t) = 1 + \varepsilon_0 \int_0^t < W(s) >^{\frac{1}{3}} ||u||^{p-1}_{L^\infty_x} ds,$$

we observe that due to $||u||_{L^\infty_x} \leq c||u||_{H^1_x} \leq c\varepsilon_0 \leq c$,

$$1 + \int_0^t ||u||^p_{L^\infty_x} ds \leq cI(t).$$

Putting together these inequalities, it is found

$$||u(t)||_{L^\infty_x} \leq c\frac{\varepsilon_0}{< W(t) >^{\frac{1}{3}}}\left(1 + c\int_0^t I'(s)I^2(s)ds\right).$$

This completes the proof of the Lemma by mere computations. □

We now gather the previous results to derive an upper bound for the solutions. We now state a preliminary result. Introduce the auxiliary function

$$a(t) = \left(\frac{< t >^{\frac{1}{2}}}{< W(t) >}\right)^{\frac{p-2}{3p-3}}, \tag{5.9}$$

with the corresponding maximal function $a^*(t) = \sup_{s \leq t} a(s)$.

Lemma 5.7 *There exists $c_p > 0$ that does not depend on t such that*

$$\mathbb{E}\left(a^*(t)^{\frac{5}{2}}\right) \leq c_p. \tag{5.10}$$

Proof By a convexity argument, for $q > 0$, the function $< W(t) >^{-q}$ is a submartingale. Then we can apply Doob's maximal L^p inequality [25] that leads to, for any given $t > 0$,

$$\mathbb{E}(a^*(t)^{\frac{5}{2}}) \leq c_p \mathbb{E}\left(\left(\frac{< t >^{\frac{1}{2}}}{< W(t) >}\right)^{\frac{5p-10}{6p-6}}\right).$$

For $t \leq 1$ we have that $\frac{<t>^{\frac{1}{2}}}{<W(t)>} \leq c$. For $t \geq 1$, since $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, proceeding as (5.4) we have that for $q < 1$

$$\mathbb{E}(< W(t) >^{-q}) \leq c\int_{\mathbb{R}} \frac{\exp(-\frac{x^2}{2t})}{|x|^{-q}\sqrt{t}}dx \leq c\frac{1}{t^{\frac{q}{2}}}.$$

Then for $t \geq 1$, since $\frac{5p-10}{6p-6} < 1$

$$\mathbb{E}\left(\left(\frac{< t >^{\frac{1}{2}}}{< W(t) >}\right)^{\frac{5p-10}{6p-6}}\right) \leq c.$$

□

From the previous results, we now derive an upper bound for the solutions. Let us set

$$f : \mathbb{R}^+ \to \mathbb{R}^+, z \mapsto \frac{z}{1 + \varepsilon_0^{\frac{15}{2}} z^{3p-3}}.$$

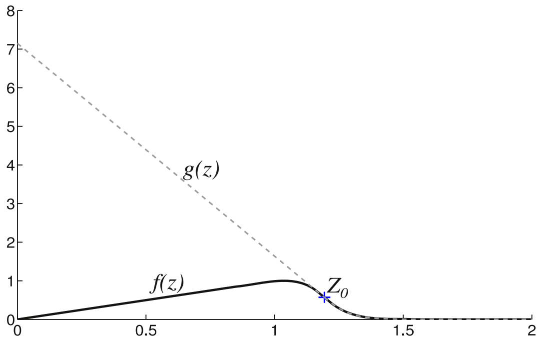

Fig. 1 Graphical representation of the function f (*dashed line*) and the convex upper bound g (*dotted line*)

We observe that $f(z)$ is a convex function for $z \geq Z_0 = ((\frac{3p-4}{3p-2})\varepsilon_0^{\frac{15}{2}})^{-\frac{1}{3p-3}}$. We consider a decreasing convex function g as follows (see Fig. 1)

$$g(z) = \begin{cases} f(Z_0) + f'(Z_0)(z - Z_0), & \text{for } z \leq Z_0, \\ f(z), & \text{for } z \geq Z_0. \end{cases}$$

Let us define the auxiliary function

$$y(t) =< t >^{\frac{p-2}{6(p-1)}} < W(t) >^{\frac{1}{3(p-1)}} ||u(t)||_{L^\infty_x} \tag{5.11}$$

and the corresponding maximal function $y^*(t) = \sup_{s\leq t} y(s)$. We now state

Proposition 5.8 *Assume ε_0 is small enough. Then there exists $c > 0$ that depends on p such that for any $t \in \mathbb{R}$,*

$$\mathbb{E}(y^*(t)^{\frac{5}{4}}) \leq c\varepsilon_0^{\frac{5}{4}}. \tag{5.12}$$

Proof The proof is divided into two steps. We first prove

Lemma 5.9 *Assume ε_0 is small enough. Then, there exists $c > 0$ that depends on p such that for any $t \in \mathbb{R}$,*

$$\mathbb{E}(y^*(t)^{\frac{5}{2}}) \leq c\varepsilon_0^{-\frac{5}{2p-2}}. \tag{5.13}$$

Proof We infer from (5.7), with a as in (5.9),

$$y(t) \leq c\varepsilon_0 a(t) \left(1 + \left(\varepsilon_0 \int_0^t \frac{y(s)^{p-1}}{< s >^{\frac{p-2}{6}}} ds\right)^3\right).$$

Using the assumption that $p > 8$, which ensures $< s >^{-\frac{p-2}{6}}$ is integrable, we then have

$$y^*(t) \leq c\varepsilon_0 a^*(t) \left(1 + \varepsilon_0^3 y^*(t)^{3p-3}\right). \tag{5.14}$$

This implies, with another constant c,

$$(y^*(t))^{\frac{5}{2}} \leq c\varepsilon_0^{\frac{5}{2}}(a^*(t))^{\frac{5}{2}}\left(1+\varepsilon_0^{\frac{15}{2}}(y^*(t)^{\frac{5}{2}})^{3p-3}\right).$$

The inequality (5.14) is equivalent to

$$f(y^*(t)^{\frac{5}{2}}) \leq c\varepsilon_0^{\frac{5}{2}}(a^*(t))^{\frac{5}{2}}.$$

We now compute $\mathbb{E}(y^*(t)^{\frac{5}{2}}) = G(t) + B(t)$, where

$$G(t) = \mathbb{E}(y^*(t)^{\frac{5}{2}} | y^*(t)^{\frac{5}{2}} \leq Z_0) = \int_{\{y^*(t)^{\frac{5}{2}} \leq Z_0\}} y^*(t)^{\frac{5}{2}} \mathrm{d}\mathbb{P}(\omega) \leq Z_0,$$

and

$$B(t) = \mathbb{E}(y^*(t)^{\frac{5}{2}} | y^*(t)^{\frac{5}{2}} \geq Z_0).$$

The Jensen inequality (see [6]) gives

$$g(B(t)) \leq \mathbb{E}(g(y^*(t)^{\frac{5}{2}}) | y^*(t)^{\frac{5}{2}} \geq Z_0) \leq \mathbb{E}(f(y^*(t)^{\frac{5}{2}}) \leq c\varepsilon_0^{\frac{5}{2}}\mathbb{E}(a^*(t)^{\frac{5}{2}}),$$

and thanks to Lemma 5.7

$$g(B(t)) \leq cc_p\varepsilon_0^{\frac{5}{2}}. \tag{5.15}$$

We now choose ε_0 small enough so that $g(B(t)) \leq g(Z_0)$; actually, we have to pick ε_0 such that

$$cc_p\varepsilon_0^{\frac{5}{2}} \leq g(Z_0) = \frac{\left(\frac{3p-4}{3p-2}\right)^{-\frac{1}{3p-3}}}{1+\frac{3p-4}{3p-2}}\varepsilon_0^{-\frac{5}{2(p-1)}}.$$

Since g is a decreasing function, then $B(t) \geq Z_0$. Therefore (5.15) implies

$$f(B(t)) = \frac{B(t)}{1+\varepsilon_0^{\frac{15}{2}}B(t)^{3p-3}} \leq c\varepsilon_0^{\frac{5}{2}}.$$

Finally,

$$\mathbb{E}\left(y^*(t)^{\frac{5}{2}}\right) = G(t) + B(t) \leq Z_0 + c\varepsilon_0^{\frac{5}{2}}\left(1+\varepsilon_0^{\frac{15}{2}}\mathbb{E}\left(y^*(t)^{\frac{5}{2}}\right)^{3p-3}\right).$$

The function $X \mapsto c\varepsilon_0^{10} X^{3p-3} - X + Z_0 + c\varepsilon_0^{\frac{5}{2}}$ is negative for $X = 2Z_0$ (for ε_0 small enough). We observe that $t \mapsto \mathbb{E}(y^*(t)^{\frac{5}{2}})$ is continuous and that $\mathbb{E}(y^*(0)^{\frac{5}{2}}) \leq c\varepsilon_0^{\frac{5}{2}} \ll Z_0$. Then the function $\mathbb{E}(y^*(s)^{\frac{5}{2}})$ remains trapped in $[0, 2Z_0]$. □

We now complete the proof of Proposition 5.8. We define

$$\tau_\omega = \inf\{t > 0;\, y^*(t) > 2c\varepsilon_0 a^*(t)\}. \tag{5.16}$$

Either $\tau_\omega = +\infty$ or $y^*(\tau_\omega) = 2c\varepsilon_0 a^*(\tau_\omega)$. In this second case, from (5.14), $y^*(\tau_\omega) \leq 2c\varepsilon_0^4 a^*(\tau_\omega) y^*(\tau_\omega)^{3p-3}$, which leads to $1 \leq c\varepsilon_0^{3p} a^*(\tau_\omega)^{3p-3}$, and then to $1 \leq c\varepsilon_0^{\frac{5p}{2p-2}} a^*(\tau_\omega)^{\frac{5}{2}}$. Therefore, for any $T > 0$,

$$\mathbb{P}(\tau_\omega \leq T) \leq \int_{\{\tau_\omega \leq T\}} c\varepsilon_0^{\frac{5p}{2p-2}} a^*(T)^{\frac{5}{2}} \mathrm{d}\mathbb{P}(\omega),$$

thanks to Lemma 5.7

$$\mathbb{P}(\tau_\omega \leq T) \leq c c_p \varepsilon_0^{\frac{5p}{2p-2}}. \tag{5.17}$$

We now write

$$\mathbb{E}(y^*(t)^{\frac{5}{4}}) = \mathbb{E}(y^*(t)^{\frac{5}{4}} | \tau_\omega > t) + \mathbb{E}(y^*(t)^{\frac{5}{4}} | \tau_\omega \leq t). \tag{5.18}$$

The first term of the right hand side of (5.18) is bounded by above by $c\varepsilon_0^{\frac{5}{4}} \mathbb{E}(a^*(t)^{\frac{5}{4}}) \leq \tilde{c}\varepsilon_0^{\frac{5}{4}}$. We bound the second term by Cauchy–Schwarz inequality as

$$\mathbb{E}(y^*(t)^{\frac{5}{4}} | \tau_\omega \leq t) \leq \mathbb{E}(y^*(t)^{\frac{5}{2}})^{\frac{1}{2}} \mathbb{P}(\tau_\omega \leq t)^{\frac{1}{2}}.$$

Using Lemma 5.9 and estimate (5.17) concludes the proof of the proposition. □

We now complete the proof of Theorem 2.2.

Using the Hölder inequality in Ω and gives

$$\mathbb{E}(||u(t)||_{L^\infty_x}) \leq \mathbb{E}(y^{\frac{5}{4}}(t))^{\frac{4}{5}} \mathbb{E}((< W(t) >^{-\frac{1}{3}} < t >^{-\frac{p-2}{6}})^{\frac{5}{p-1}})^{\frac{1}{5}}. \tag{5.19}$$

The result follows promptly, due to Proposition 5.8 and to

$$\mathbb{E}\left(\left(< W(t) >^{-\frac{1}{3}} < t >^{-\frac{p-2}{6}}\right)^{\frac{5}{p-1}}\right)^{\frac{1}{5}} \leq c < t >^{-\frac{1}{6}}.$$

6 Numerical simulations

6.1 Space and time discretizations

Spectral methods are suitable to the discretization of equations such as (2.4) because the derivatives and the nonlinearity appearing in (2.4) can be easily treated.

For the numerical computations, we consider (2.4) on a bounded domain $[-K, K]$, with K being a large fixed value. The discrete Fourier basis consisting of trigonometric polynomial functions

$$\mathrm{e}^{\mathrm{i}\frac{k\pi}{K}x}, \quad \mathrm{i}^2 = -1\,, \quad -\frac{N_x}{2} \le k \le \frac{N_x}{2} - 1,$$

where $N_x \in 2\mathbb{N}^*$ represents the number of modes, any periodic and locally integrable scalar function u can be approximated by u_K, where u_K in the considered discrete basis is as follows,

$$u_K(x) = \sum_{k=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \widehat{u}\left(\frac{k\pi}{K}\right) \mathrm{e}^{\mathrm{i}\frac{k\pi}{K}x}\,,$$

with

$$\widehat{u}\left(\frac{k\pi}{K}\right) = \frac{1}{2K} \int_{-K}^{K} u(x)\mathrm{e}^{-\mathrm{i}\frac{k\pi}{K}x}\, dx\,.$$

The problem now becomes a system of N_x ordinary differential equations with N_x unknowns: for each ξ, find $\widehat{u}(\xi, t)$, such that

$$\begin{cases} (1+\xi^2)\widehat{u}_t(\xi,t) + \mathrm{i}\xi\widehat{u}(\xi,t) \circ dW_t + \mathrm{i}\xi\,\widehat{\frac{u^{p+1}}{p+1}}(\xi,t) \;=\; 0\,, \quad \forall\, t > 0, \\ \widehat{u}(\xi,0) \;=\; \widehat{u}_0(\xi)\,, \end{cases} \tag{6.1}$$

where

$$\xi = \frac{k\pi}{K}\,, \; -\frac{N_x}{2} \le k \le \frac{N_x}{2} - 1.$$

For the time discretization, the definition of the Stratonovich integral has to be used. This integral $\int_0^T u(t) \circ dW_t$ is defined as the limit in probability of the sum

$$\sum_{n=1}^{N} \frac{u(t_{n+1}) + u(t_n)}{2} (W(t_{n+1}) - W(t_n)).$$

Thus, for consistent numerical computations on long time intervals, the Cranck–Nicolson method is used.

Let $\Delta t > 0$ be the stepsize of the time discretization, and set $t_n = n\Delta t$. For $n \in \mathbb{N}$, we denote by $\widehat{u}_n$ an approximation of $\widehat{u}(\xi, t_n)$ and by u_n an approximation of $u(x, t_n)$, where $x \in [-K, K]$. The numerical scheme reads

$$(1+\xi^2)(\widehat{u}_{n+1} - \widehat{u}_n) + i\xi \frac{\widehat{u}_{n+1} + \widehat{u}_n}{2} \Delta W_n + \frac{\mathrm{i}\xi\Delta t}{2} \left(\frac{\widehat{u_{n+1}^{p+1}}}{p+1} + \frac{\widehat{u_n^{p+1}}}{p+1} \right) = 0$$

where $\Delta W_n = W(t_{n+1}) - W(t_n) = \sqrt{dt}\,\mathrm{Randn}(1)$ and $W(0) = 0$ with Randn(1) being the normally distributed random numbers.

For linear case, the evolution of u_{n+1} is straightforward. For the nonlinear case, the u_{n+1} is computed iteratively. Let $M \geq 1$ be a maximal number of iterations of the Picard iterations, the algorithm for solving (6.1) is as follows:

- Set u_0 and compute $\widehat{u}_0$.
- For $n = 0, 1, \ldots,$ compute:
  - For $m = 0, 1, \ldots, M-1$, $\widehat{u}_{n+1,0} = \widehat{u}_n$, and

$$\widehat{u}_{n+1,m+1} := \frac{\widehat{u}_n \left(2(1+\xi^2) - \mathrm{i}\xi\Delta W_n\right) - \mathrm{i}\xi\Delta t \left(\frac{\widehat{u_{n+1,m}^{p+1}}}{p+1} + \frac{\widehat{u_n^{p+1}}}{p+1} \right)}{2(1+\xi^2) + \mathrm{i}\xi\Delta W_n}. \tag{6.2}$$

The iterations are stopped if one of the two following cases happen:
- when $\|u_{n+1,m+1} - u_{n+1,m}\|_{l^2} / \|u_{n+1,0}\|_{l^2} \leq \tau$, with $\tau > 0$ being a fixed tolerance. We then set $u_{n+1} := u_{n+1,m+1}$;
- or when $m = M - 1$. Here, we set $u_{n+1} := u_{n+1,m}$.

Remark 6.1 The Stratonovich integral is used since Itô's formulation (2.5) is not suitable for the discretization of these stochastic partial differential equations. Indeed, Itô's integral is explicitly defined as

$$\sum_{n=1}^{N} u(t_n)(W(t_{n+1}) - W(t_n)),$$

and an explicit discretization of the equations implies a loss of conservation laws.

6.2 The linear case

The numerical scheme is first tested for the stochastic linear equation

$$du - du_{xx} + u_x \circ dW = 0. \tag{6.3}$$

In this case, the scheme is explicit and no iteration is required. Simulations are performed with $K = 100$, $N_x = 2^{12}$, $\Delta t = 0.1$. To obtain a precise computation of the

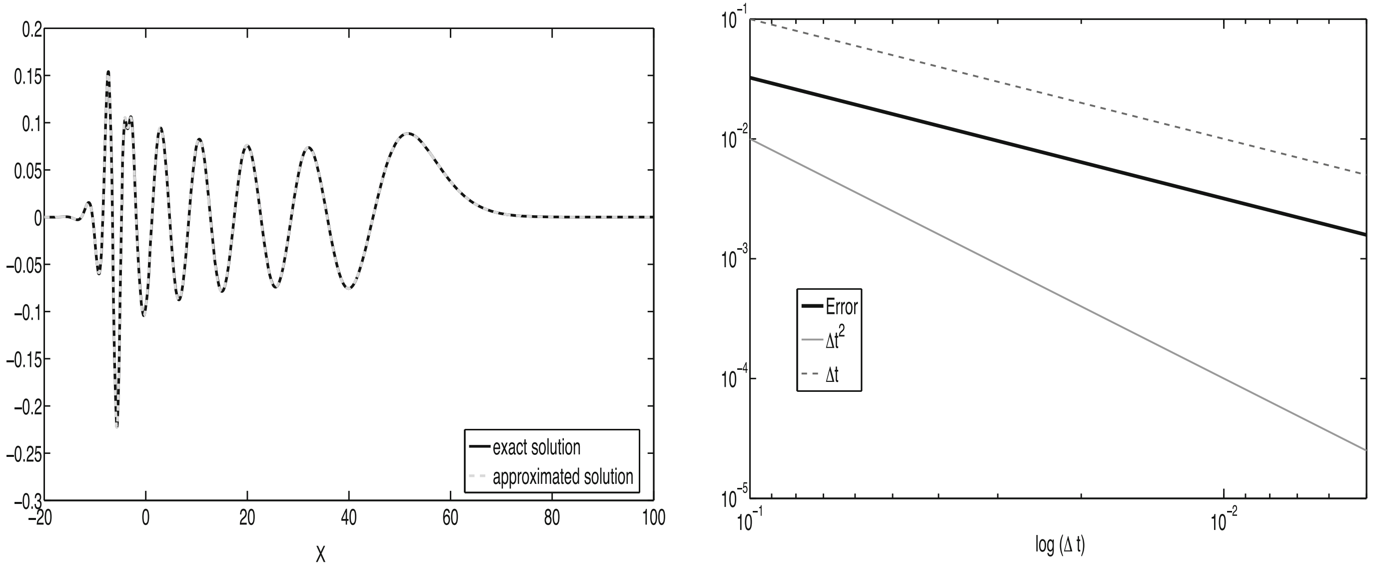

Fig. 2 On the *left*, exact and approximate solutions at $t = 1000$ obtained for one Brownian motion. At *right*, order of convergence of the Crank–Nicolson scheme with respect to Δt

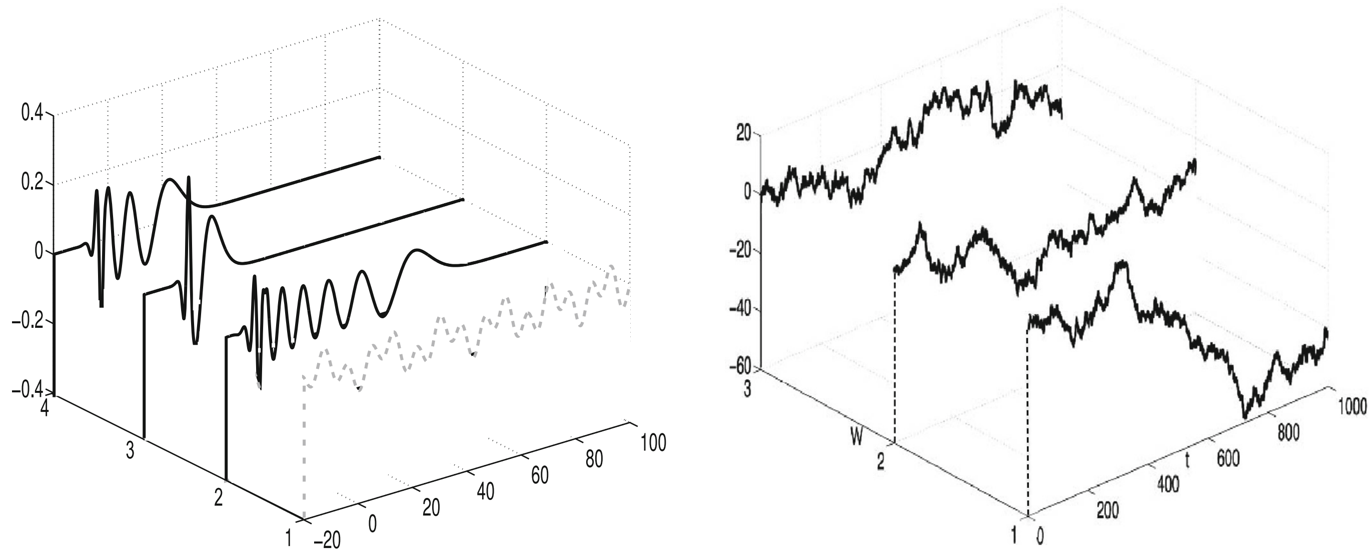

Fig. 3 On the *left*, solutions of the linear equation obtained with $\alpha = 0.5$ for three distinct stochastic processes at time $t = 1000$. The *dashed line* represents the solution of the deterministic linear equation. On the *right*, the corresponding Brownian motions with respect to time

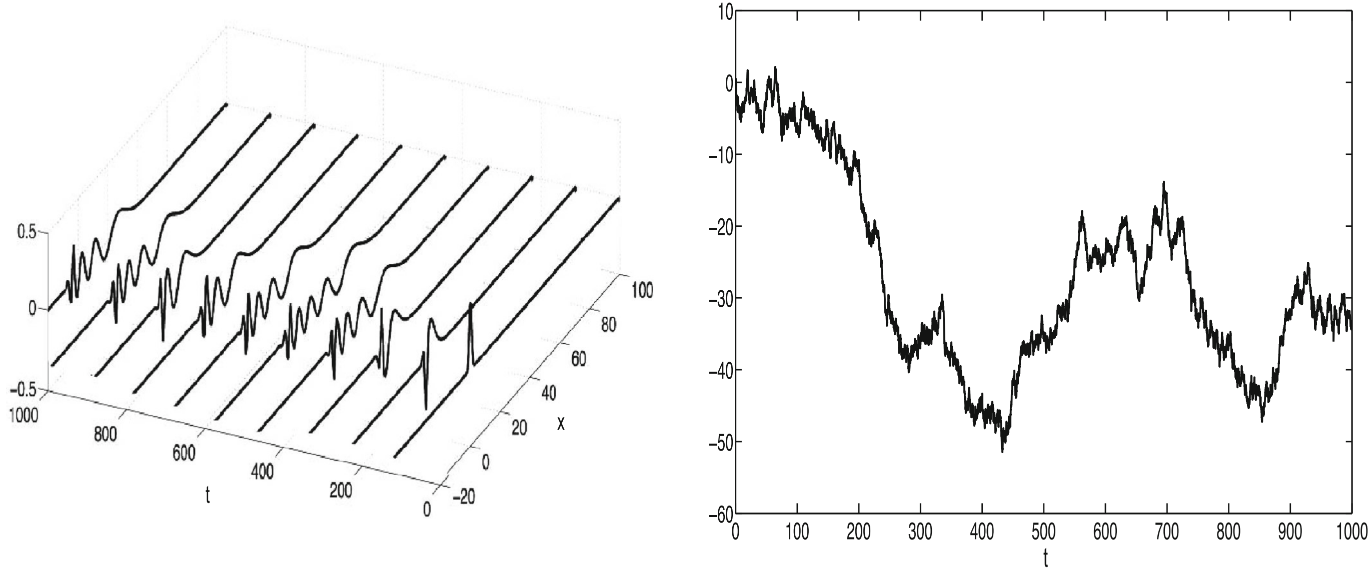

Fig. 4 On the *left*, evolution of the solution of the linear equation obtained for $\alpha = 0.5$ with respect to time. On the *right*, the corresponding Brownian motion with respect to time

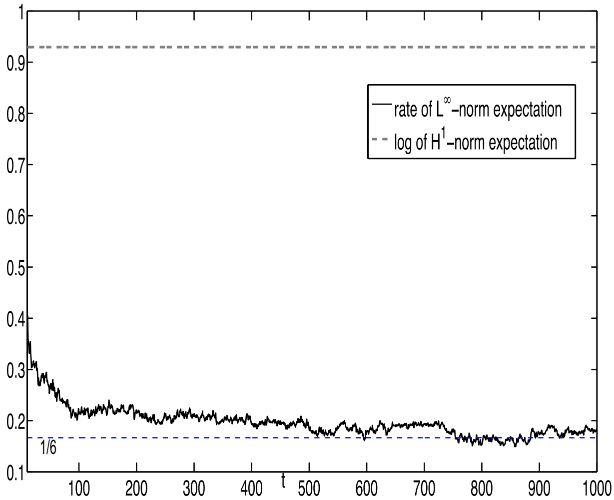

Fig. 5 Decay rate of the L^∞-norm expectation (*dashed line*) and logarithm of the H^1-norm (*dotted line*) with respect to time

expectation with respect to the Brownian motion, a Monte-Carlo method is performed with 95% confidence interval using the standard estimator

$$\sigma_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}\Big(||u^{\omega_i}||_{L^\infty} - \mathbb{E}(||u^N||_{L^\infty})\Big)^2.$$

Simulations start from the initial Gaussian datum:

$$u_0(x) = \alpha e^{-x^2}$$

with $\alpha = 0.5$. In the left figure of Fig. 2, the approximate solution, computed with Crank–Nicolson scheme as stated before and the exact solution

$$\hat{u}_{ex}(\xi, t) = e^{-\frac{i\xi}{1+\xi^2}(W_t - W_0)}\hat{u}_0(\xi).$$

at $t = 1000$ are plotted. The two solutions are almost on top of each other. The right figure of Fig. 2 shows the L^∞ error of the solutions at $t = 1000$ with respect to Δt (Δt is computed uniformly from .005 to .1). Contrary to the deterministic case, the Crank–Nicolson scheme applied to stochastic differential equations is of order 1. To demonstrate the scheme is indeed order 1, we also plotted two additional curves in the figure, one is Δt and another is Δt^2.

Figure 3 shows the solutions of the linear equation for three distinct Brownian motions (W_t) at $t = 1000$. We note that for any (W_t), the solution disperses, but the dispersion tails are relatively large. The corresponding solution of the deterministic linear equation is also shown in the left figure. In comparison, the wave fronts in the stochastic case moves slower than the deterministic case. For the three solutions

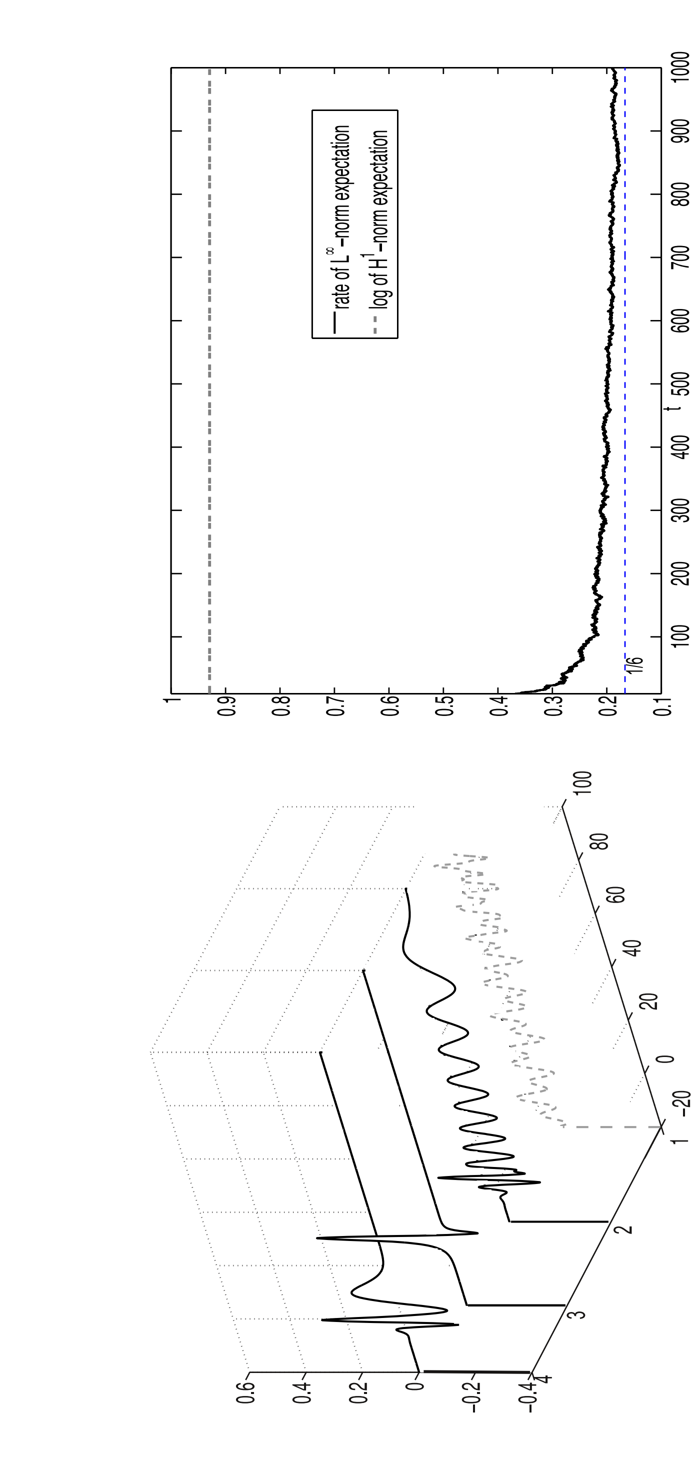

Fig. 6 On the *left*, solutions of the nonlinear equation obtained with $\alpha = 0.5$ for three distinct stochastic processes at time $t = 1000$. The *dashed line* represents the solution of the deterministic nonlinear equation. On the *right*, decay rate of the L^{∞}-norm expectation (*dashed line*) and logarithm of the H^1-norm (*dotted line*) with respect to time

presented in Fig. 3, the decay rates of the L^∞ norm of the solutions are computed and they varies between 0.22 and 0.18 at $t = 1000$.

In Fig. 4, the time evolution of the solution is plotted for one Brownian motion. It is observed that the dispersion of the solution is highly dependent on the stochastic motion. Each turn in the Brownian motion induces or prevents the dispersion. We see in Fig. 4 that in the early stages, the decrease of the Wiener process benefits the dispersion. The growth of the process reduces it. The same phenomena can be observed in also in Fig. 3.

To check that the decay rate for the expectation of the solution is $\mathcal{O}(t^{1/6})$ we compute in log–log scale, i.e. we plot in Fig. 5 the ratio $\frac{\log\left(\frac{\mathbb{E}(||u(t+\Delta t)||_{L_x^\infty})}{\mathbb{E}(||u(t)||_{L_x^\infty})}\right)}{\log\left(\frac{t+\Delta t}{t}\right)}$. It approaches approximately $\frac{1}{6} \approx 0.167$ as t increases, which demonstrates that our result in Theorem 5.3 is sharp. The expectation of the H^1 norm of the solution is also plotted and it is well preserved. In fact, the H^1 norm of the solution is well preserved for all Brownian motions by the numerical scheme. Recall that, on average, the solution of the stochastic equation (Theorem 2.2) moves twice as slow as the one of the deterministic problem [1,2] as illustrated by Figs. 3 and 6.

6.3 Decay of small solutions of the nonlinear problem

We now evaluate the decay rate of the L^∞ norm expectation to the nonlinear problem

$$du - du_{xx} + u_x \circ dW + u^p u_x dt = 0.$$

The behavior of small solutions of the nonlinear problem is very close to that of the linear problem solutions. Although each stochastic solution disperses differently, the decay rate of the L^∞ norm expectation, obtained in Fig. 6, goes to $\frac{1}{6}$ as t increases. Here again, the H^1 norm is well preserved for all Brownian motions by the numerical scheme.

Remark 6.2 Similar results are obtained for other values of α, in particular for large α which implies large amplitude initial data where the theorem does not cover. It seems that the stochastic nature of the dispersion makes easier for the solutions to decrease. One may wonder if we can prove the decay rate estimate for any solution of the equation, for instance removing the smallness assumption on the initial data, by probability arguments as in [17]. This raises technical difficulties and will be addressed in a forthcoming work. The decay result also implies that there is no solitary wave in expectation. Nevertheless, it seems possible to build stochastic modulations that allow the solution to remain close to the deterministic soliton $\phi(x - ct)$ [12,13].

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