Solitary-Wave Solutions to Boussinesq Systems with Large Surface Tension

Min Chen[1], Nghiem V. Nguyen[1,2] and Shu-Ming Sun[3]
[1]Department of Mathematics, Purdue University,
West Lafayette, IN 47907, USA

[2]Department of Mathematics and Statistics, Utah State University,
Logan, UT 84322, USA

[3]Department of Mathematics, Virginia Tech,
VA 24061, USA

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Abstract

Considered herein are certain Boussinesq systems with the presence of large surface tension. The existence and stability of solitary waves are established by using techniques introduced earlier by Buffoni [7] and Lions [9, 10].

1 Introduction

The four-parameter family of Boussinesq systems

$$\begin{cases} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{cases} \tag{1.1}$$

is introduced in [4] (generalized to include the surface tension in [8]) to describe the motion of small-amplitude long waves on the surface of an ideal fluid under the force of gravity. All the variables are scaled with length scale h_0 and time scale $\sqrt{h_0/g}$, where g is the gravitational constant and h_0 (scaled to 1) the undisturbed average water depth. The quantity $\eta(x,t)$ is the deviation of free surface with respect to the undisturbed state, and $\eta(x,t)+1$ corresponds to the total depth of the liquid at (x,t), while $u(x,t)$ is the dimensionless horizontal velocity field at height θ, where $0 \le \theta \le 1$. From the derivation of (1.1), the parameters a, b, c, d are not independently specified but satisfy the consistency condition

$$a + b + c + d = \frac{1}{3} - \tau\,, \tag{1.2}$$

where τ is the surface tension coefficient. If a_0 connotes a typical wave amplitude and λ a typical wavelength, the condition of "small amplitude and long wavelength" just mentioned amounts to

$$\alpha = \frac{a_0}{h_0} << 1, \quad \beta = \frac{h_0^2}{\lambda^2} << 1, \quad \frac{\alpha}{\beta} = \frac{a_0\lambda^2}{h_0^3} \approx 1. \tag{1.3}$$

Systems (1.1) are first-order approximations in α and β to Euler's equations, justified rigorously by Bona, Colin and Lannes in [6]. We refer the readers to the papers [4] and [5] for further discussion about the derivation and well-posedness of these systems.

These systems are free of the presumption of unidirectionality that is the hallmark of KdV-type equations. One therefore expects that these Boussinesq systems will have more intrinsic interest than the one-way models on account of their considerably wider range of potential applicability. Because dissipation is ignored in the derivation of (1.1) and the overlying Euler equations are Hamiltonian, it is expected that some of the systems in (1.1) will likewise possess a Hamiltonian form. One finds indeed that whenever $b = d$, the functional

$$\mathcal{H}(\eta, u) = \frac{1}{2}\int_{-\infty}^{\infty}\Big(-c\eta_x^2 - au_x^2 + \eta^2 + (1+\eta)u^2\Big)dx \tag{1.4}$$

serves as a Hamiltonian and the systems have the following conserved quantities

$$\int_{-\infty}^{\infty} u(x,t)dx, \quad \int_{-\infty}^{\infty} \eta(x,t)dx, \quad \mathcal{I}(\eta,u) = \int_{-\infty}^{\infty}\Big(\eta u + b\eta_x u_x\Big)dx \tag{1.5}$$

along with $\mathcal{H}(\eta, u)$ (see Remark 4.1 in [5]).

In this manuscript, the existence and stability of solitary waves of the system (1.1) with

$$b = d > 0, \quad a, c < 0, \quad ac > b^2 \tag{1.6}$$

are studied. It is noted that condition (1.6) implies $a + b + c + d < 0$ and therefore $\tau > \frac{1}{3}$, which corresponds to systems with large surface tension. The special properties of this class of systems include established global well-posedness and previously stated conserved quantities which enable the use of the technique of constrained global minimization. For general system (1.1) with a, b, c, d satisfying $b = d > 0$ and $a, c < 0$, the existence of solitary-wave solutions can be proved, which includes the case with zero surface tension. However, the stability of the solitary-wave solutions cannot be obtained for this general case.

Consideration is given to the initial-value problem. In the context of (1.1) and (1.6), one imagines being provided with an initial wave profile, say at $t = 0$,

$$(\eta(x,0), u(x,0)) = (\phi(x), \psi(x)), \tag{1.7}$$

for $x \in \mathbb{R}$ which is very close to a traveling solitary wave solution $(\eta(x,t), u(x,t)) = (\eta(x-\omega t), u(x-\omega t))$ of system (1.1) with ω being a fixed positive constant. One then inquires into the subsequent evolution under (1.1). This presumes that the initial-value problem (1.1) is a well-posed problem so that a unique solution $(\eta(x,t), u(x,t))$ departs from $(\phi(x), \psi(x))$. A summary of what is needed regarding the well-posedness issue will be provided in the next section.

The stability established in this manuscript is often regarded as "set stability", that is, the set of certain constrained minimizers is stable. For $r > 0$, denote

$$B_r = \{\eta(x) \in H^1(\mathbb{R}) \bigm| \|\eta\|_{H^1(\mathbb{R})} \le r\} \quad \text{and} \quad \mathcal{B}_r = \{\eta(x) \in H^2(\mathbb{R}) \bigm| \|\eta\|_{H^2(\mathbb{R})} \le r\}. \tag{1.8}$$

For $\mu > 0$, define a real number $\mathcal{H}_{r,\mu}$ to be

$$\mathcal{H}_{r,\mu} = \inf_{(\eta,u)} \{\mathcal{H}(\eta,u) \bigm| (\eta,u) \in \mathcal{B}_r \times H^1(\mathbb{R}) \text{ and } \mathcal{I}(\eta,u) = 2\mu\}.$$

The *set of minimizers* for $\mathcal{H}_{r,\mu}$ is defined to be

$$D(r,\mu) = \left\{(\eta,u) \in \mathcal{B}_r \times H^1(\mathbb{R}) \bigm| \mathcal{H}(\eta,u) = \mathcal{H}_{r,\mu} \text{ and } \mathcal{I}(\eta,u) = 2\mu\right\}, \tag{1.9}$$

while a *minimizing sequence* for $\mathcal{H}_{r,\mu}$ is any sequence $\{(\eta_n, u_n)\} \in \mathcal{B}_r \times H^1(\mathbb{R})$ satisfying

$$\mathcal{I}(\eta_n, u_n) = 2\mu, \quad \text{for all } n \text{ and } \quad \lim_{n\to\infty} \mathcal{H}(\eta_n, u_n) = \mathcal{H}_{r,\mu}.$$

For $(f,g) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, denote

$$dist((f,g), D(r,\mu)) = \inf_{(\eta,u)\in D(r,\mu)} \{\|(f,g) - (\eta,u)\|_{H^1\times H^1}\}.$$

The precise statement of the result is as follows:

Theorem 1.1. *(Existence) There exist an $r_0 > 0$ and a $\mu_0 > 0$ (which depends on r_0), such that for $r \in (0, r_0)$ and $\mu \in (0, \mu_0)$, the set of minimizers $D(r,\mu)$ is non-empty and $dist\big((\eta_n, u_n), D(r,\mu)\big) \to 0$ for every minimizing sequence $\{(\eta_n, u_n)\} \in \mathcal{B}_r \times H^1(\mathbb{R})$.*

Theorem 1.1 shows the set of minimizers is in the space $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. By a bootstrapping argument, it can be shown that the minimizers are in $H^{m+1}(\mathbb{R}) \times H^m(\mathbb{R})$ for any $m > 1$, which implies that the minimizers decay to zero at infinity.

For the large surface tension case, the Euler equations have solitary-wave solutions whose first-order approximations are the solitary-wave solutions of the KdV equation. However, the relationship between the minimizers obtained in this paper and the KdV solitary-wave solutions or the solitary-wave solutions of Euler equations is not well understood. Although it is most likely that the minimizers here are approximations to the solitary-wave solutions of Euler equations, and KdV solitary-wave solutions are approximations to the solutions in the set of minimizers, there are no rigorous proofs for that.

The stability result for the set of minimizers is the consequence of the above theorem (see Theorem 5.2) which reads

Theorem 1.2. *(Stability) There exist an $r_0 > 0$ and a $\mu_0 > 0$ (which depends on r_0) such that if $r \in (0, r_0)$ and $\mu \in (0, \mu_0)$, the set $D(r, \mu)$ is stable. That is, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if*

$$(\phi, \psi) \in \mathcal{B}_r \times H^2(\mathbb{R}), \quad dist\big((\phi, \psi), D(r, \mu)\big) < \delta,$$

and if $\big(\eta(x,t), u(x,t)\big)$ is a solution of (1.1)-(1.2)-(1.6) with initial data (ϕ, ψ) and if $\eta(x,t) \in \mathcal{B}_r$ for $t \geq 0$, then

$$dist\big((\eta(\cdot,t), u(\cdot,t)), D(r, \mu)\big) < \epsilon \quad \text{for all} \quad t \in [0, \infty).$$

Remark 1.3. *Here, we note that the initial data are in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and the stability result is in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. A stronger result would be the stability of the solitary wave $(f(x-ct), g(x-ct))$, i.e, for a fixed $\epsilon > 0$, there is a corresponding $\delta > 0$ such that*

$$\|(\phi, \psi) - (f, g)\|_{H^1} \leq \delta \quad \text{implies}$$

$$\inf_{y \in \mathbb{R}} \|(\eta(\cdot + y, t), u(\cdot + y, t)) - (f, g)\|_{H^1} \leq \epsilon, \tag{1.10}$$

for all $t > 0$. However, being unable to obtain the uniform bounds of solutions in $H^2(\mathbb{R})$ with respect to t as well as to determine whether the set $D(r, \mu)$ consists only of a singleton prevents us from obtaining the more desirable form of stability (1.10). Nevertheless, such stability result is fairly common and sometimes the best possible for complicated problems (see [11] and [7]). To the best of our knowledge, our stability result is the first one for any Boussinesq systems.

Remark 1.4. *Generally speaking, a Boussinesq system is much simpler to analyze than the Euler equations and the solutions of the Boussinesq system for the parameters in some regions exist globally. However, the Boussinesq system is derived from the Euler equations under the condition of small amplitude and long-wave length and some features of the Boussinesq system are very different from those of the Euler equations.*

For example, in the momentum functional $\mathcal{I}$ of (1.5), the order of the derivative is one, while the order in the Euler equation is one half (see [7]). The higher order derivative causes a big problem for our case, when the estimates of the minimizers are obtained and the convergence of the minimizing sequences is proved.

The manuscript is organized as follows. Section 2 provides a brief summary of relevant known results for the Boussinesq systems. In Section 3, some necessary estimates for functionals are given. Section 4 gives the existence proof of minimizers of $\mathcal{H}(\eta, u)$ with $\mathcal{I}(\eta, u) = 2\mu$. In Section 5, the stability of the set of minimizers is obtained. Section 6 is an appendix which provides proofs that are left unproved in the previous sections.

The standard notations are used. For $1 \leq p < \infty$, L^p is the usual Banach space of measurable functions on $\mathbb{R}$ with norm given by $\|f\|_{L^p} = (\int_{-\infty}^{\infty} |f|^p \, dx)^{1/p}$. The space L^∞ consists of the measurable, essentially bounded functions f on $\mathbb{R}$ with norm $|f|_\infty = \text{ess sup}_{x\in\mathbb{R}}|f(x)|$. For $s \in \mathbb{R}$, the L^2-based Sobolev space $H^s = H^s(\mathbb{R})$ (see [1]) is the set of all tempered distributions f on $\mathbb{R}$ whose Fourier transforms $\widehat{f}$ are measurable functions on $\mathbb{R}$ satisfying

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + |k|^2 + \cdots + |k|^{2s}) |\widehat{f}(k)|^2 \, dk < \infty.$$

2 Review of the Boussinesq Systems

As expected, prior to a discussion of stability as formulated above in terms of perturbations of the initial data, there should be a theory for the initial-value problem itself. Local existence and continuous dependence on initial data have been studied in [5] for numerous cases of (1.1). In order to extend the local result to a global one, some kind of control on the norms is needed in the energy estimates. Whenever $b = d$, the systems (1.1) admit the conservation laws (1.4) and (1.5) which allow one to obtain the control needed. Moreover, in this case, the systems (1.1) with (1.6) can be written as

$$\partial_t \begin{bmatrix} \eta \\ u \end{bmatrix} = J \text{ grad } \mathcal{H}(\eta, u),$$

where the operator J is defined as

$$J = \begin{bmatrix} 0 & (I - b\partial_x^2)^{-1}\partial_x \\ (I - b\partial_x^2)^{-1}\partial_x & 0 \end{bmatrix},$$

and grad $\mathcal{H}$ stands for the gradient or Euler derivative, computed with respect to the $L^2 \times L^2$-inner product, of the functional $\mathcal{H}$. Because the operator J is skew-adjoint, $\mathcal{H}$ can be seen as a Hamiltonian for the systems.

Because none of the conserved quantities is composed only of positive terms, they do not on their own provide the *a priori* information one needs to conclude the global existence of solutions to the initial-value problem. However, a time-dependent relationship can be coupled with the invariance of the Hamiltonian to give suitable information leading to a global existence theory. The global existence needed in this manuscript has been established in [5] as follows.

Theorem 2.1. *Let $s \geq 1$ and suppose $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ is such that*

$$\inf_{x\in\mathbb{R}}\{1+\phi(x)\} > 0 \quad \textit{and} \quad |\mathcal{H}(\phi,\psi)| < |c|^{1/2}.$$

Then the solution (η, u) of (1.1)-(1.7)-(1.6) exists and is in $C(\mathbb{R}_+; H^s(\mathbb{R}))\times C(\mathbb{R}_+; H^s(\mathbb{R}))$. Moreover, the H^1-norm of both η and u is uniformly bounded in t and $1 + \eta(x,t) \geq 1 - \frac{|\mathcal{H}(\phi,\psi)|}{\sqrt{|c|}} = \alpha > 0$.

Remark 2.2. *The constants r_0 and μ_0 in Theorems 1.1-1.2 will be chosen such that*

$$r_0 < 1 \quad \textit{and} \quad \mu_0 < \frac{1}{2}|c|^{\frac{1}{2}}$$

which gives the conditions for the global existence of the solution

$$\inf_{x\in\mathbb{R}}\{1+\phi(x)\} > 0 \quad \textit{and} \quad |\mathcal{H}(\phi,\psi)| < |c|^{1/2}$$

using the facts that

$$\|\phi(x)\|_{L^\infty} \leq \|\phi(x)\|_{H^1} \leq r_0 < 1$$

and $\mathcal{H}(\phi,\psi) \leq 2\mu \leq 2\mu_0$ (see Lemma 3.8).

3 Estimates of Functionals

By a solitary wave solution we shall mean a solution (η, u) of (1.1) of the form

$$\eta(x,t) = \eta(x-\omega t) \text{ and } u(x,t) = u(x-\omega t) \tag{3.1}$$

where $\omega > 0$ denotes the speed of the wave. In what follows, we require that η, $u \in H^1(\mathbb{R})$, $\|\eta\|_{H^1} \leq 1$ and restrict ourselves to the case (1.6).

Let $\xi = x - \omega t$ and substitute the form of the solution (3.1) into (1.1), integrate once and evaluate the constants of the integrations using the fact that $\eta, u \in H^1(\mathbb{R})$, one sees that (η, u) must satisfy

$$\begin{aligned} c\eta_{\xi\xi} + \eta - \omega u + b\omega u_{\xi\xi} + \frac{1}{2}u^2 = 0, \\ au_{\xi\xi} + u - \omega\eta + b\omega\eta_{\xi\xi} + \eta u = 0. \end{aligned} \tag{3.2}$$

It is worth to mention that traveling wave solutions are critical points of minimization problem on $\mathcal{H}(\eta, u)$ with constraint $\mathcal{I}(\eta, u) = 2\mu$ and the Lagrange multiplier is the phase speed of the waves.

To prove existence and stability of a traveling wave, we use the idea introduced by Buffoni [7] which makes use of two conserved quantities associated with the system, namely $\mathcal{I}(\eta, u)$ and $\mathcal{H}(\eta, u)$. We first fix η and minimize $\mathcal{H}(\eta, u)$ with respect to u using the constraint $\mathcal{I}(\eta, u) = 2\mu$ for some $\mu > 0$. Substituting the minimizer u_η into $\mathcal{H}(\eta, u)$, the problem becomes finding the minimizer for $\mathcal{H}(\eta) = \mathcal{H}(\eta, u_\eta)$ without constraints. The last step will be to show that the original minimization problem is equivalent to this two-step approach. We will use x as the independent variable when there is no confusion.

We now carry out the steps in details. First, fix an $\eta \in B_r$ (later, we use $\mathcal{B}_r \subset B_r$) where $r < 1$ according to Remark 2.2 and minimize $\mathcal{H}(\eta, u)$ with respect to u under the condition $\mathcal{I}(\eta, u) = 2\mu$ for some $0 < \mu < |c|^{1/2}/2$. Denote the minimum by $u_\eta(x)$ and let λ_η be the corresponding Lagrange multiplier; $u_\eta(x)$ and λ_η satisfy

$$a(u_\eta)_{xx} + u_\eta + \eta u_\eta = \lambda_\eta(\eta - b\eta_{xx}). \tag{3.3}$$

The function u_η and λ_η can be expressed explicitly in terms of μ and η. From (3.3), it follows that

$$u_\eta = \lambda_\eta G^{-1}(\eta)\big(\eta - b\eta_{xx}\big), \tag{3.4}$$

where

$$G(\eta) = a\partial_{xx} + (1 + \eta). \tag{3.5}$$

Substituting (3.4) into $\mathcal{I}(\eta, u)$ and using the constraint $I(\eta, u_\eta) = 2\mu$, one obtains

$$2\mu = \lambda_\eta \int_{-\infty}^{\infty} \big(\eta G^{-1}(\eta)\big(\eta - b\eta_{xx}\big) + b\eta_x[G^{-1}(\eta)\big(\eta - b\eta_{xx}\big)]_x\big)\ dx$$

and

$$\lambda_\eta = \frac{\mu}{\mathcal{L}_1(\eta)} \tag{3.6}$$

with

$$\mathcal{L}_1(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} (\eta - b\eta_{xx}\big)G^{-1}(\eta)\big(\eta - b\eta_{xx}\big)dx, \tag{3.7}$$

after integration by parts once. Substituting (3.4) and (3.6) into $\mathcal{H}$, we have by using (3.3) and (3.6) that

$$\mathcal{H}(\eta, u_\eta) = \frac{1}{2}\int_{-\infty}^{\infty} \big(-c\eta_x^2 + \eta^2\big)dx + \frac{1}{2}\int_{-\infty}^{\infty} (\eta - b\eta_{xx}\big)\lambda_\eta u_\eta dx\,. \tag{3.8}$$

The second step is to find the minimizers of $\mathcal{H}(\eta)$ for fixed $0 < \mu < |c|^{1/2}/2$ and $\eta \in B_r$ where $r < 1$, with no constraints, where

$$\mathcal{H}(\eta) := \mathcal{H}(\eta, u_\eta) = \mathcal{L}_0(\eta) + \frac{\mu^2}{\mathcal{L}_1(\eta)}, \tag{3.9}$$

with $\mathcal{L}_1(\eta)$ defined in (3.7), $\int_{-\infty}^{\infty}(\eta - b\eta_{xx})u_\eta \, dx = 2\mu$ and

$$\mathcal{L}_0(\eta) = \frac{1}{2}\int_{-\infty}^{\infty}(-c\eta_x^2 + \eta^2)dx. \tag{3.10}$$

The concentration compactness theory developed in [10, 9, 2, 3] will be the center piece of the arguments. We start by studying $\mathcal{L}_1(\eta)$ which requires the investigation of the operator $G(\eta)$.

Lemma 3.1. *Let $G(\eta) = a\partial_{xx} + (1+\eta)$ with $a < 0$, $\eta \in B_r$ where $r < 1$. Then $G^{-1}(\eta)$ maps $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$ and*

(i) *(boundedness) $\|G^{-1}(0)f\|_{L^2} \le \|f\|_{L^2}$, $\|G^{-1}(\eta)f\|_{H^1} \le C\|f\|_{L^2}$, where C depends on r and a, for any $f \in L^2(\mathbb{R})$;*

(ii) *(symmetry) $\int_{-\infty}^{\infty} fG(\eta)g dx = \int_{-\infty}^{\infty} gG(\eta)f dx$ for any f and g in $H^1(\mathbb{R})$, and $\int_{-\infty}^{\infty} fG^{-1}(\eta)g \, dx = \int_{-\infty}^{\infty} gG^{-1}(\eta)f \, dx$, for any f and g in $L^2(\mathbb{R})$;*

(iii) *(positivity) $\int_{-\infty}^{\infty} fG(\eta)f \, dx \ge 0$, for any f in $H^1(\mathbb{R})$, and $\int_{-\infty}^{\infty} fG^{-1}(\eta)f \, dx \ge 0$, for any f in $L^2(\mathbb{R})$;*

(iv) *$\int_{-\infty}^{\infty} fG^{-1}(0)f \, dx = \int_{-\infty}^{\infty} \frac{|\widehat{f}|^2}{1 - a|k|^2} dk \le \|f\|_{L^2}^2$, for any f in $L^2(\mathbb{R})$,*

where the derivatives of f may be considered in a generalized sense.

Proof. In the following, we let $h = G^{-1}(\eta)f$, which gives $f = ah_{xx} + (1+\eta)h$. The first part of (i) can be obtained by transforming it into the Fourier space. For the second part, from

$$\begin{aligned}\|h\|_{H^1}^2 &= \int_{-\infty}^{\infty}\big(h_x^2 + h^2\big)dx \le \frac{1}{\min(\alpha, -a)}\int_{-\infty}^{\infty}\big(-ah_x^2 + (1+\eta)h^2\big)dx \\ &\le \frac{1}{\min(\alpha, -a)}\int_{-\infty}^{\infty}(ah_{xx} + (1+\eta)h)h dx \le \frac{1}{\min(\alpha, -a)}\|f\|_{L^2}\|h\|_{H^1}\end{aligned}$$

with α defined in Theorem 2.1, one obtains (i).

The first part in (ii) is obtained by integration in parts and the second part is obtained by noticing

$$\int_{-\infty}^{\infty} fG^{-1}(\eta)gdx = \int_{-\infty}^{\infty}(ah_{xx} + (1+\eta)h)G^{-1}(\eta)gdx = \int_{-\infty}^{\infty} G(\eta)h(G^{-1}(\eta)g)dx$$
$$= \int_{-\infty}^{\infty} hG(\eta)(G^{-1}(\eta)g)dx = \int_{-\infty}^{\infty} hgdx = \int_{-\infty}^{\infty} gG^{-1}(\eta)fdx.$$

The statement in (iii) follows from the fact that

$$\int_{-\infty}^{\infty} fG^{-1}(\eta)f\ dx = \int_{-\infty}^{\infty}\Big(-ah_x^2 + (1+\eta)h^2\Big)dx \geq 0.$$

The statement in (iv) comes from $G(0) = a\partial_{xx} + 1$ and $f = ah_{xx} + h$, then

$$\int_{-\infty}^{\infty} fG^{-1}(0)fdx = \int_{-\infty}^{\infty} fhdx = \int_{-\infty}^{\infty}(-ah_x^2 + h^2)dx$$
$$= \int_{-\infty}^{\infty}(-a|k|^2 + 1)|\widehat{h}|^2 dk = \int_{-\infty}^{\infty}\frac{|\widehat{f}|^2}{1 - a|k|^2}dk.$$

□

We now study in detail the structure of $\mathcal{L}_1(\eta)$ for η small. Letting $\eta \in B_r$ and

$$w = G^{-1}(\eta)(\eta - b\eta_{xx}), \tag{3.11}$$

one sees that

$$\mathcal{L}_1(\eta) = \frac{1}{2}\int_{-\infty}^{\infty} wG(\eta)wdx, \tag{3.12}$$

which is positive from Lemma 3.1 (iii). We split $\mathcal{L}_1(\eta)$ into three parts consisting of quadratic, cubic and higher-order terms in η respectively by starting from splitting w into three parts. From (3.11),

$$(a\partial_{xx} + (1+\eta))w = (\eta - b\eta_{xx}),$$

$$G(0)w = \eta - b\eta_{xx} - \eta w,$$

and therefore

$$w = G^{-1}(0)(\eta - b\eta_{xx}) - G^{-1}(0)(\eta w). \tag{3.13}$$

With $G^{-1}(0)$ being a linear bounded operator, (3.13) can be used to give an expansion of w in terms of η. Specifically, letting $w_1(\eta) = G^{-1}(0)(\eta - b\eta_{xx})$ and using the equation (3.13) twice, one has

$$\begin{aligned} w =& w_1 - G^{-1}(0)\Big(\eta\big(w_1 - G^{-1}(0)(\eta w)\big)\Big) \\ =& w_1 - G^{-1}(0)(\eta w_1) + G^{-1}(0)\Big(\eta G^{-1}(0)(\eta w)\Big) \\ =& w_1(\eta) + w_2(\eta) + w_3(\eta) \end{aligned}$$

where $w_2 := -G^{-1}(0)(\eta w_1)$ and $w_3 := G^{-1}(0)\Big(\eta G^{-1}(0)(\eta w)\Big)$. The following Lemma gives a relation between the terms w_1, w_2 and w_3.

Lemma 3.2. *If*

$$w = G^{-1}(\eta)(\eta - b\eta_{xx}) = w_1 + w_2 + w_3, \tag{3.14}$$

one has

$$G(0)w_1 = \eta - b\eta_{xx}, \quad G(0)w_2 = -\eta w_1, \quad G(0)w_3 = -\eta(w_2 + w_3). \tag{3.15}$$

Proof. The first two equalities in (3.15) are the definitions. For the third equality, applying $G(0)$ on (3.13), $G(0)w + \eta w = G(0)w_1$, it is deduced after using the definition of w_2 that

$$G(0)w_3 = -\eta w_1 - \eta w_2 - \eta w_3 - G(0)w_2 = -\eta(w_2 + w_3).$$

□

Substituting (3.14) into (3.12), noticing that $G(\eta) = G(0) + \eta$, and grouping them in terms of the order in η, one finds

$$\begin{aligned} \mathcal{L}_1(\eta) =& \frac{1}{2}\int_{-\infty}^{\infty} (w_1 + w_2 + w_3)(G(0) + \eta)(w_1 + w_2 + w_3)dx \\ =& \mathcal{L}_{1,0}(\eta) + \mathcal{L}_{1,1}(\eta) + \mathcal{L}_{1,2}(\eta)\,, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} \mathcal{L}_{1,0}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty} w_1 G(0) w_1 dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(bk^2+1)^2}{(1-ak^2)}|\widehat{\eta}|^2 dk \geq 0, \\ \mathcal{L}_{1,1}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty} -\eta w_1^2 dx\,, \\ \mathcal{L}_{1,2}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty} (w_2 G(0) w_2 + w_2 G(0) w_3) dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (w_2 G(0) w_2 - \eta w_1 w_3) dx, \end{aligned} \tag{3.17}$$

with the use of self-adjointness of $G(0)$ and last two equalities of (3.15) to simplify the terms in $\mathcal{L}_{1,2}$.

Lemma 3.3. *For any $\eta \in H^1(\mathbb{R})$, there exist positive constants C_1, C_2, C_3 and C_4 such that,*

(i) $\|w_1\|_{L^2}^2 \le \displaystyle\int_{-\infty}^{\infty} \frac{(1+bk^2)^2}{(1-ak^2)^2}|\widehat{\eta}|^2 dk \le C_1\|\eta\|_{L^2}^2, \quad \|w_1\|_{H^1} \le C_2\|\eta\|_{H^1}$;

(ii) $\|w_2\|_{L^2} \le \|\eta\|_{H^1}\|w_1\|_{L^2}, \quad \|w_2\|_{H^1} \le C_3\|\eta\|_{H^1}^2$;

(iii) *if* $\eta \in B_{\frac{1}{2}}, \quad \|w_3\|_{L^2} \le 2\|\eta\|_{H^1}\|w_2\|_{L^2}$;

(iv) *if* $\eta \in B_r$ *with* $r < r_0$ *and* $r_0 \le \frac{1}{2}\min\{1, -a\}, \quad \|w_3\|_{H^1} \le C_4\|\eta\|_{H^1}^3$;

(v) *if* $\eta \in B_{r_0}$ *with* r_0 *chosen in (iv), then* $\|w_1(\eta)\|_{H^1}, \|w_2(\eta)\|_{H^1}$, *and* $\|w_3(\eta)\|_{H^1}$ *are bounded by a constant.*

Proof. From the expression $w_1(\eta) = G^{-1}(0)(\eta - b\,\eta_{xx})$, one obtains the first part of (i) with the use of Fourier transform. Notice now that

$$\min\{1,-a\}\|w_1\|_{H^1}^2 \le \int_{-\infty}^{\infty}(-a(w_1)_x^2 + w_1^2)dx = \int_{-\infty}^{\infty} w_1(\eta)(\eta - b\eta_{xx})dx$$
$$\le \|w_1(\eta)\|_{L^2}\|\eta\|_{L^2} + b\|(w_1)_x\|_{L^2}\|\eta_x\|_{L^2} \le \max\{1,b\}\|w_1\|_{H^1}\|\eta\|_{H^1}.$$

From this, the second part of (i) is implied with $C_2 = \dfrac{\max\{1,b\}}{\min\{1,-a\}}$.

Similarly, as $w_2(\eta) = -G^{-1}(0)(\eta w_1)$, Lemma 3.1 (i) implies

$$\|w_2\|_{L^2} \le \|\eta w_1\|_{L^2} \le \|\eta\|_{H^1}\|w_1\|_{L^2}$$

and with a use of (i), one derives that

$$\min\{1,-a\}\|w_2\|_{H^1}^2 \le \int_{-\infty}^{\infty}(-a(w_2)_x^2 + w_2^2)dx = \left|\int_{-\infty}^{\infty} w_2(\eta)\eta w_1(\eta)dx\right|$$
$$\le \|w_2(\eta)\|_{L^2}\|\eta w_1(\eta)\|_{L^2} \le \|w_2(\eta)\|_{L^2}\|\eta\|_{H^1}\|w_1(\eta)\|_{L^2} \le C_2\|w_2\|_{H^1}\|\eta\|_{H^1}^2$$

which gives (ii), with $C_3 = \dfrac{C_2}{\min\{1,-a\}}$.

From Lemma 3.2, namely $w_3 = -G^{-1}(0)[\eta(w_2 + w_3)]$, one therefore has

$$\|w_3\|_{L^2} \le \|\eta(w_2 + w_3)\|_{L^2} \le \|\eta\|_{H^1}\|w_2\|_{L^2} + \|\eta\|_{H^1}\|w_3\|_{L^2}.$$

So

$$\|w_3\|_{L^2} \le \frac{\|\eta\|_{H^1}}{1-\|\eta\|_{H^1}}\|w_2\|_{L^2} \le 2\|\eta\|_{H^1}\|w_2\|_{L^2}. \tag{3.18}$$

Similarly,

$$\begin{aligned}\min\{1,-a\}\|w_3\|_{H^1}^2 &\le \int_{-\infty}^{\infty}(-a(w_3)_x^2 + w_3^2)\,dx = \left|\int_{-\infty}^{+\infty} w_3 G(0) w_3\,dx\right| \\ &= \left|\int_{-\infty}^{\infty}\eta w_2(\eta)w_3(\eta)dx + \int_{-\infty}^{\infty}\eta w_3^2(\eta)dx\right| \le \|\eta\|_{H^1}\|w_2(\eta)\|_{L^2}\|w_3(\eta)\|_{L^2} \\ &+ \|\eta\|_{H^1}\|w_3(\eta)\|_{L^2}^2 \le C_3\|w_3(\eta)\|_{L^2}\|\eta\|_{H^1}^3 + \|w_3(\eta)\|_{L^2}^2\|\eta\|_{H^1}.\end{aligned}$$

from which one obtains that

$$(\min\{1,-a\} - \|\eta\|_{H^1})\|w_3(\eta)\|_{H^1}^2 \le C_3\|w_3(\eta)\|_{H^1}\|\eta\|_{H^1}^3.$$

Choosing $r_0 \le \frac{1}{2}\min\{1,-a\}$, one has

$$\|w_3\|_{H^1} \le C_4\|\eta\|_{H^1}^3$$

with $C_4 = \frac{2C_3}{\min\{1,-a\}}$. Since $r_0 < 1$ and $\eta \in B_{r_0}$,

$$\max\{\|w_1(\eta)\|_{H^1}, \|w_2(\eta)\|_{H^1}, \|w_3(\eta)\|_{H^1}\} \le r_0 \max\{C_2, C_3, C_4\}$$

. □

Lemma 3.4. *For $\eta \in H^1(\mathbb{R})$, let*

$$\mathcal{H}_0(\eta) = \mathcal{L}_0(\eta) + \frac{\mu^2}{\mathcal{L}_{1,0}(\eta)}.$$

Then

(i) $\mathcal{H}_0(\eta) \ge 2\mu > 0$, *and*

(ii) *there exists a sequence $\{\eta_\alpha\}_{\alpha\in(0,1)}$ in $H^1(\mathbb{R})$ with $\lim_{\alpha\to 0}\|\eta_\alpha\|_{H^1}^2 = 2\mu$ such that*

$$\lim_{\alpha\to 0}\mathcal{H}_0(\eta_\alpha) = 2\mu.$$

Proof. Notice first that $ac \ge b^2$ from (1.6) implies $-a-c \ge 2b$ since $(a+c)^2 \ge 4ac \ge 4b^2$. Therefore for any k, $(-ck^2+1)(-ak^2+1) \ge (bk^2+1)^2$, and it follows from (3.10) and (3.17) that

$$\frac{\mathcal{L}_0(\eta)}{\mathcal{L}_{1,0}(\eta)} = \frac{\displaystyle\int_{-\infty}^{\infty}(-ck^2+1)|\widehat{\eta}|^2dk}{\displaystyle\int_{-\infty}^{\infty}\frac{(bk^2+1)^2}{-ak^2+1}|\widehat{\eta}|^2dk} \ge 1.$$

Consequently, one has by using Cauchy-Schwartz inequality,

$$\mathcal{H}_0(\eta) = \mathcal{L}_0(\eta) + \frac{\mu^2}{\mathcal{L}_{1,0}(\eta)} \geq 2\sqrt{\frac{\mu^2\mathcal{L}_0(\eta)}{\mathcal{L}_{1,0}(\eta)}} \geq 2\mu > 0.$$

For (ii), let f be a function in $H^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f^2(x)dx = 1$. Let $\eta_\alpha = Af(\alpha x)$, where $A^2 = 2\mu\alpha$ (this α is not related to the small parameter defined in (1.3)) and we study $\lim_{\alpha\to 0}\mathcal{H}_0(\eta_\alpha)$. It is observed that

$$\mathcal{L}_0(\eta_\alpha) = \frac{A^2}{2}\int_{-\infty}^{\infty}\Big[-c\alpha^2\big(f'(\alpha x)\big)^2 + f^2(\alpha x)\Big]dx = \mu - \mu\int_{-\infty}^{\infty} c\alpha^2k^2|\widehat{f}|^2dk \longrightarrow \mu$$

as $\alpha \to 0$, which also yields that $\lim_{\alpha\to 0}\|\eta_\alpha\|^2_{H^1} = 2\mu$ by replacing $-c = 1$. Now for the term $\mathcal{L}_{1,0}(\eta_\alpha)$, by changing variable $y = \alpha x$,

$$\int_{-\infty}^{\infty}\eta_\alpha(x)G^{-1}(0)\eta_\alpha(x)dx = 2\mu\int_{-\infty}^{\infty} f(y)(a\alpha^2\partial_{yy}+1)^{-1}f(y)dy = 2\mu\int_{-\infty}^{\infty}\frac{|\widehat{f}|^2}{1-a\alpha^2k^2}dk$$

$$= 2\mu\int_{-\infty}^{\infty}|\widehat{f}|^2dk + 2\mu\int_{-\infty}^{\infty}\frac{a\alpha^2k^2|\widehat{f}|^2}{1-a\alpha^2k^2}dk = 2\mu + 2\mu\int_{-\infty}^{\infty}\frac{a\alpha^2k^2|\widehat{f}|^2}{1-a\alpha^2k^2}dk.$$

Using the same technique, it shows

$$\begin{aligned}\mathcal{L}_{1,0}(\eta_\alpha) &= \frac{1}{2}\int_{-\infty}^{\infty} w_1G(0)w_1dx = \frac{1}{2}\int_{-\infty}^{\infty}\Big(\eta_\alpha - b(\eta_\alpha)_{xx}\Big)G^{-1}(0)\Big(\eta_\alpha - b(\eta_\alpha)_{xx}\Big)dx\\ &= \mu\int_{-\infty}^{\infty}|\widehat{f}|^2\frac{(1+bk^2\alpha^2)^2}{1-ak^2\alpha^2}dk = \mu + \int_{-\infty}^{\infty}|\widehat{f}|^2\frac{(1+bk^2\alpha^2)^2-1+ak^2\alpha^2}{1-ak^2\alpha^2}dk \to \mu\end{aligned}$$

as $\alpha \to 0$. Consequently,

$$\lim_{\alpha\to 0}\mathcal{H}_0(\eta_\alpha) = 2\mu.$$

□

To establish the lower and upper bounds for $\mathcal{L}_{1,0}, \mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ in Lemma 3.6, the following inequalities are needed.

Lemma 3.5. *Let $\eta \in H^1(\mathbb{R})$ with $\|\eta\|_{H^1} \leq 1/2$. Then*

$$|\mathcal{L}_{1,1}(\eta)| + |\mathcal{L}_{1,2}(\eta)| \leq \|\eta\|_{H^1}(1 + \|\eta\|_{H^1} + 2\|\eta\|^2_{H^1})\mathcal{L}_{1,0}(\eta).$$

Proof. From Lemma 3.3 (i) and the first equation in (3.17), we have

$$\|w_1\|_{L^2}^2 \le 2\mathcal{L}_{1,0}(\eta). \tag{3.19}$$

Therefore

$$|\mathcal{L}_{1,1}(\eta)| = \left|\frac{1}{2}\int_{-\infty}^{\infty} \eta w_1^2 dx\right| \le \frac{1}{2}\|\eta\|_{H^1}\|w_1\|_{L^2}^2 \le \|\eta\|_{H^1}\mathcal{L}_{1,0}(\eta).$$

The first term in $\mathcal{L}_{1,2}(\eta)$ (defined in (3.17)) can be bounded by using Lemma 3.1 (iv) and (3.19), namely

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} w_2 G(0) w_2 dx &= \frac{1}{2}\int_{-\infty}^{\infty} (\eta w_1) G^{-1}(0)(\eta w_1) dx \le \frac{1}{2}\|\eta w_1\|_{L^2}^2 \\ &\le \frac{1}{2}\|\eta\|_{H^1}^2\|w_1\|_{L^2}^2 \le \|\eta\|_{H^1}^2\mathcal{L}_{1,0}(\eta)\,.\end{aligned}$$

From Lemma 3.3 (ii), (iii), the second term is bounded by

$$\begin{aligned}\frac{1}{2}\left|\int_{-\infty}^{\infty} \eta w_1 w_3 dx\right| &\le \frac{1}{2}\|\eta\|_{H^1}\|w_1\|_{L^2}\|w_3\|_{L^2} \\ &\le \|\eta\|_{H^1}^2\|w_1\|_{L^2}\|w_2\|_{L^2} \le \|\eta\|_{H^1}^3\|w_1\|_{L^2}^2 \le 2\|\eta\|_{H^1}^3\mathcal{L}_{1,0}(\eta).\end{aligned}$$

The lemma is obtained by combining above inequalities. □

The upper and lower bounds on $\mathcal{H}(\eta)$ are given in the next lemma. It is worth to note that the upper bound is for information only and is not used in the rest of the paper.

Lemma 3.6. *There exists an r_0 with $0 < r_0 \le 1/2$ such that for $0 < \|\eta\|_{H^1} < r_0$, one has*

$$\frac{1}{2}\min\{-c,1\}\|\eta\|_{H^1}^2 \le \mathcal{H}(\eta) \le \frac{1}{2}\max\{-c,1\}\|\eta\|_{H^1}^2 + 4\mu^2\frac{\max\{1,-a\}}{(\min\{1,b\})^2\|\eta\|_{H^1}^2}. \tag{3.20}$$

Proof. The left inequality is true because (3.9) and $\mathcal{L}_1(\eta) \ge 0$ from Lemma 3.1 (iii). Using Lemma 3.5, it is possible to find $0 < r_0 \le 1/2$ such that when $0 < \|\eta\|_{H^1} < r_0$, one has

$$|\mathcal{L}_{1,1}(\eta)| + |\mathcal{L}_{1,2}(\eta)| \le \frac{\mathcal{L}_{1,0}(\eta)}{2}. \tag{3.21}$$

Consequently, from (3.17),

$$\mathcal{L}_1(\eta) \ge \mathcal{L}_{1,0}(\eta) - |\mathcal{L}_{1,1}(\eta)| - |\mathcal{L}_{1,2}(\eta)| \ge \frac{1}{2}\mathcal{L}_{1,0}(\eta) \ge \frac{1}{4}\frac{(\min\{1,b\})^2}{\max\{1,-a\}}\|\eta\|_{H^1}^2 > 0 \tag{3.22}$$

which leads to the advertised bounds. □

The following Lemma is used in Lemma 3.8 which establishes an upper bound on the minimization of $\mathcal{H}(\eta)$ in terms of μ.

Lemma 3.7. *For any $f \in H^1(\mathbb{R})$ and for any $0 < \mu < 1$*

$$\left|2\mu^2\mathcal{L}_{1,1}(f(\mu^2 x)) + \int_{-\infty}^{\infty} f^3(x)dx\right| \leq C\mu^2\|f\|_{H^1}^3$$

where $C = \dfrac{2|a+b|}{\sqrt{-a}} + \dfrac{(a+b)^2}{-a}$.

Proof. From the definition of $\mathcal{L}_{1,1}(\eta)$ and a change of variable $y = \mu^2 x$, one finds

$$\begin{aligned}2\mu^2\mathcal{L}_{1,1}(f(\mu^2 x)) &= -\int_{-\infty}^{\infty} f\left\{(1+a\mu^4\partial_{xx})^{-1}(1-b\mu^4\partial_{xx})f\right\}^2 dx = -\int_{-\infty}^{\infty} f^3(x)dx \\ &+ 2(a+b)\mu^4\int_{-\infty}^{\infty} f^2(1+a\mu^4\partial_{xx})^{-1}f_{xx}dx - (a+b)^2\mu^8\int_{-\infty}^{\infty} f[(1+a\mu^4\partial_{xx})^{-1}f_{xx}]^2dx\end{aligned}$$

by noticing

$$(1+a\mu^4\partial_{xx})^{-1}(1-b\mu^4\partial_{xx})f = f - (a+b)\mu^4(1+a\mu^4\partial_{xx})^{-1}f_{xx}.$$

Therefore

$$\begin{aligned}&\left|2\mu^2\mathcal{L}_{1,1}(f(\mu^2 x)) + \int_{-\infty}^{\infty} f^3(x)dx\right| \\ &\leq 2|a+b|\mu^4\|f\|_{H^1}^2\|(1+a\mu^4\partial_{xx})^{-1}f_{xx}\|_{L^2} + (a+b)^2\mu^8\|f\|_{H^1}\|(1+a\mu^4\partial_{xx})^{-1}f_{xx}\|_{L^2}^2.\end{aligned} \tag{3.23}$$

Since

$$\left\|\sqrt{-a}\mu^2\Big((a\mu^4\partial_{xx}+1)^{-1}f_x\Big)\right\|_{L^2}^2 = \int_{-\infty}^{\infty}\frac{-a\mu^4k^2}{(1-a\mu^4k^2)^2}|\widehat{f}(k)|^2dk \leq \|f\|_{L^2}^2,$$

one has

$$\|\sqrt{-a}\mu^2(a\mu^4\partial_{xx}+1)^{-1}f_{xx}\|_{L^2}^2 \leq \|f_x\|_{L^2}^2 \leq \|f\|_{H^1}^2\,,$$

which yields the conclusion of the lemma by applying it in (3.23). □

The next lemma gives an upper bound for the minimization of $\mathcal{H}(\eta)$.

Lemma 3.8. *There exists a μ_0, where $2\mu_0 + 2\mu_0^5 \leq \frac{1}{4}$ and for any $\mu < \mu_0$, there exists a function $g \in H^1(\mathbb{R})$ such that $\|g\|_{H^1}^2 = 2\mu + 2\mu^5$, and $\mathcal{H}(g) < 2\mu - C_0\mu^{5/2}$, where the constant C_0 is positive and independent of μ.*

Proof. Let f be a function in $H^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} f_x^2 dx = 1$, $\int_{-\infty}^{\infty} f^3(x)dx < 0$, which gives $\|f\|_{H^1} = \sqrt{2}$. Let $g(x) = Af(\mu^2 x)$ where $A^2 = 2\mu^3$. Then

$$\|g\|_{H^1}^2 = 2\int_{-\infty}^{\infty} (\mu^5 f_x^2 + \mu f^2)dx = 2\mu + 2\mu^5 = O(\mu).$$

The requirement on μ_0 guarantees that $g \in B_{\frac{1}{2}}$ and $\mu_0 < 1$. Thus, Lemma 3.3(iii) and Lemma 3.7 are valid.

We now start the computation of $\mathcal{H}(g)$ term by term.

$$\begin{aligned}
\mathcal{L}_0(g) &= \mu - \mu^5 \int_{-\infty}^{\infty} c f_x^2(x)dx = \mu - c\mu^5, \\
\mathcal{L}_{1,0}(g) &= \frac{1}{2}\int_{-\infty}^{\infty} (g - bg_{xx})G^{-1}(0)(g - bg_{xx})dx \\
&= \frac{1}{2}\frac{A^2}{\mu^2}\int_{-\infty}^{\infty} (f - b\mu^4 f_{yy})(a\mu^4\partial_{yy} + 1)^{-1}(f - b\mu^4 f_{yy})dy \\
&= \mu + \mu\int_{-\infty}^{\infty} \frac{(1 + b\mu^4k^2)^2 - 1 + a\mu^4k^2}{1 - a\mu^4k^2}|\widehat{f}|^2 dk,
\end{aligned}$$

which leads to

$$|\mathcal{L}_{1,0}(g) - \mu| \leq C\mu^5 \leq C\mu^3.$$

The next term in $\mathcal{L}_1(g)$ reads

$$\mathcal{L}_{1,1}(g) = A^3\mathcal{L}_{1,1}(f(\mu^2 x)) = 2^{\frac{3}{2}}\mu^{\frac{9}{2}}\mathcal{L}_{1,1}(f(\mu^2 x)).$$

Using Lemma 3.7,

$$|\mathcal{L}_{1,1}(g) + \sqrt{2}\mu^{\frac{5}{2}}\int_{-\infty}^{\infty} f^3(x)dx| \leq \sqrt{2}\mu^{\frac{5}{2}}\left|2\mu^2\mathcal{L}_{1,1}(f(\mu^2 x)) + \int_{-\infty}^{\infty} f^3(x)dx\right| \leq C\mu^{\frac{9}{2}} \leq C\mu^3.$$

For $\mathcal{L}_{1,2}$, one has, by using Lemma 3.3 (i), (ii) and (iii), $\|w_3(g)\|_{L^2} \leq C\|g\|_{H^1}^3$. Hence, by Lemma 3.3(i)

$$\begin{aligned}
|\mathcal{L}_{1,2}(g)| &= \frac{1}{2}\Big|\int_{-\infty}^{\infty} (w_2G(0)w_2 - gw_1w_3)dx\Big| = \frac{1}{2}\Big|\int_{-\infty}^{\infty} \Big((gw_1)G^{-1}(0)(gw_1) - gw_1w_3\Big)dx\Big| \\
&\leq \frac{1}{2}\Big|\int_{-\infty}^{\infty} g^2w_1^2 dx\Big| + \frac{1}{2}\Big|\int_{-\infty}^{\infty} gw_1w_3 dx\Big| \\
&\leq \frac{1}{2}\|g\|_{L^4}^2\|w_1\|_{L^4}^2 + \frac{1}{2}\|g_x\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|w_1\|_{L^2}\|w_3\|_{L^2} \\
&\leq C(\|g\|_{L^4}^2\|w_1\|_{H^1}\|w_1\|_{L^2} + \|g_x\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|g\|_{H^1}^4) \\
&\leq C(\|g\|_{L^4}^2\|g\|_{H^1}^2 + \|g_x\|_{L^2}^{\frac{1}{2}}\|g\|_{H^1}^{\frac{9}{2}}).
\end{aligned}$$

From the construction of $g(x)$, one has $\|g\|_{H^1} = O(\sqrt{\mu})$, $\|g_x\|_{L^2} = O(\mu^{\frac{5}{2}})$, and $\|g\|_{L^4} = o(\mu)$, it is concluded $\mathcal{L}_{1,2}(g) \leq C\mu^3$. With those calculations at hand, it is deduced that

$$\mathcal{L}_1(g) = \mu\left\{1 - \sqrt{2}\mu^{3/2}\int_{-\infty}^{\infty} f^3(x)dx + O(\mu^2)\right\},$$

and therefore

$$\frac{\mu^2}{\mathcal{L}_1(g)} = \mu\left\{1 + \sqrt{2}\mu^{3/2}\int_{-\infty}^{\infty} f^3(x)dx\right\} + O(\mu^3)$$

because $\mathcal{L}_1(g)$ is continuous with respect to μ. Consequently, one obtains

$$\mathcal{H}(g) = 2\mu + \sqrt{2}\mu^{5/2}\int_{-\infty}^{\infty} f^3(x)dx + O(\mu^3).$$

Hence, there is a μ_0 with $0 < \mu_0 \leq 1/2$ such that for $0 < \mu < \mu_0$, one has

$$\|g\|_{H^1}^2 = 2\mu(1 + \mu^4), \tag{3.24}$$

and

$$\mathcal{H}(g) < 2\mu - C_0\mu^{5/2}$$

where the positive constant C_0 depends on f (and μ_0) but not on μ. □

4 Convergence of Minimizing Sequences

We now investigate the minimization problem for any r with $0 < r < r_0 \leq 1/2$, and any μ with $\mu < \mu_0 \leq 1/2$ and $\mu < r^2/4$ such that Lemmas 3.6 and 3.8 hold and the function g is in B_r. Define a real number

$$\widetilde{c}(\mu) = \inf_{\{\eta \in B_r, \|\eta\|_{H^1} \neq 0\}} \left\{\mathcal{H}(\eta) = \mathcal{L}_0(\eta) + \frac{\mu^2}{\mathcal{L}_1(\eta)}\right\}$$

and let

$$C(\mu) = \{\eta \in B_r \setminus \{0\} | \mathcal{H}(\eta) = \widetilde{c}(\mu)\}$$

be the set of minimizers of $\mathcal{H}$. A sequence $\{\eta_n\} \in B_r \setminus \{0\}$ is called a minimizing sequence if $\lim_{n\to\infty} \mathcal{H}(\eta_n) = \widetilde{c}(\mu)$. It follows from Lemma 3.8 that $\widetilde{c}(\mu) < 2\mu - C_0\mu^{5/2}$. From now on it is always assumed that Lemmas 3.1-3.8 hold. We now prove the following two lemmas which will lead to the proof the fact that $\widetilde{c}(\mu)$ is strictly subadditive.

Lemma 4.1. *For $r \in (0, r_0)$ and $\mu \in (0, \mu_0)$, let $\{\eta_n\} \in B_r \setminus \{0\}$ be a minimizing sequence. Then there exist positive constants C_i, $i = 5, 6, 7$ (may depend on μ) and a positive integer N such that for all $n \geq N$,*

(i) $\mathcal{L}_{1,0}(\eta_n) \geq C_5 > 0$,

(ii) $\|\eta_n\|_{H^1}^2 \leq C_6\mu$,

(iii) $\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n) \geq C_7 > 0$.

Proof. For (i), taking a subsequence $\{\eta_{n_k}\}$ if necessary, suppose to the contrary that $\mathcal{L}_{1,0}(\eta_{n_k}) \to 0$. This implies that $\|\eta_{n_k}\|_{H^1} \to 0$ from (3.22). Using (3.21), one has

$$|\mathcal{L}_1(\eta)| \leq |\mathcal{L}_{1,0}(\eta)| + |\mathcal{L}_{1,1}(\eta)| + |\mathcal{L}_{1,2}(\eta)| \leq \frac{3}{2}|\mathcal{L}_{1,0}(\eta)|.$$

Therefore $\mathcal{L}_1(\eta_{n_k}) \to 0$ and consequently $\mathcal{H}(\eta_{n_k}) \to \infty$, which contradicts with $\widetilde{c}(\mu) < 2\mu - C_0\mu^{5/2}$.

Since there exists an N, such that for $n > N$, $\mathcal{H}(\eta_n) < 2\mu$, Lemma 3.6 then yields (ii) with $C_6 = \dfrac{4}{\min\{1, -c\}}$.

From Lemma 3.4, one can see that

$$\frac{\mu^2}{\mathcal{L}_{1,0}(\eta_n)} - \frac{\mu^2}{\mathcal{L}_{1,0}(\eta_n) + \mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)}$$
$$= \mathcal{H}_0(\eta_n) - \mathcal{H}(\eta_n) \geq 2\mu - (2\mu - C_0\mu^{5/2}) = C_0\mu^{5/2}.$$

Therefore from

$$\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n) \geq C_0\mu^{1/2}\mathcal{L}_{1,0}(\eta_n)\Big(\mathcal{L}_{1,0}(\eta_n) + \mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)\Big)$$

and (3.21), one obtains

$$\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n) \geq \frac{C_0}{2}\mu^{1/2}\mathcal{L}_{1,0}^2(\eta_n) \geq \frac{C_0}{2}C_5^2\mu^{1/2} = C_7 > 0.$$

□

Lemma 4.2. *There exists a* $\mu_0(r) > 0$ *such that for* $\sigma > 1$ *and* $\mu > 0$ *satisfying* $\sigma\mu < \mu_0(r)$*, one has* $\widetilde{c}(\sigma\mu) < \sigma\widetilde{c}(\mu)$.

Proof. Let $\{\eta_n\}$ be a minimizing sequence for $\widetilde{c}(\mu)$, that is

$$\lim_{n\to\infty} \mathcal{H}(\eta_n) = \widetilde{c}(\mu) \text{ and } \|\eta_n\|_{H^1} \leq r.$$

From Lemma 4.1, $\|\eta_n\|_{H^1}^2 \leq C_6\mu$ for $n \geq N$. Now, to study $\widetilde{c}(\sigma\mu)$ one considers the sequence $\{\sqrt{\sigma}\eta_n\}$. Because

$$\|\sqrt{\sigma}\eta_n\|_{H^1}^2 \leq \sigma C_6\mu \leq C_6\mu_0(r),$$

by requiring $\mu_0(r) \le \frac{r^2}{4C_6}$, we have $\{\sqrt{\sigma}\eta_n\} \in B_{r/2} \subset B_r$ for $n \ge N$. Therefore, for $n \ge N$,

$$\widetilde{c}(\sigma\mu) \le \mathcal{L}_0(\sqrt{\sigma}\eta_n) + (\sigma\mu)^2/\mathcal{L}_1(\sqrt{\sigma}\eta_n).$$

Now if one can show that

$$\frac{1}{\mathcal{L}_1(\sqrt{\sigma}\eta_n)} \le \frac{1}{\sigma\mathcal{L}_1(\eta_n) + C_7(\sigma^{3/2} - \sigma)} \tag{4.1}$$

with $C_7 > 0$ defined in Lemma 4.1, then

$$\begin{aligned}
&\mathcal{L}_0(\sqrt{\sigma}\eta_n) + (\sigma\mu)^2/\mathcal{L}_1(\sqrt{\sigma}\eta_n) \\
&\le \sigma\mathcal{L}_0(\eta_n) + (\sigma\mu)^2\Big(\frac{1}{\sigma\mathcal{L}_1(\eta_n)} - \frac{C_7(\sigma^{3/2} - \sigma)}{\sigma\mathcal{L}_1(\eta_n)(\sigma\mathcal{L}_1(\eta_n) + C_7(\sigma^{3/2} - \sigma))}\Big) \\
&\le \sigma\widetilde{c}(\mu) - \sigma\mu^2\frac{C_7(\sigma^{3/2} - \sigma)}{\mathcal{L}_1(\eta_n)(\sigma\mathcal{L}_1(\eta_n) + C_7(\sigma^{3/2} - \sigma))}.
\end{aligned}$$

Since $|\mathcal{L}_1(\eta_n)| \le \frac{3}{2}|\mathcal{L}_{1,0}(\eta_n)| \le C\|\eta_n\|_{H^1}^2$ by letting $n \to \infty$ and noticing that $\mathcal{L}_1(\eta_n)$ is bounded and positive, one obtains that the limit of the last term is strictly negative and therefore arrives at

$$\widetilde{c}(\sigma\mu) < \sigma\widetilde{c}(\mu),$$

the desired strict inequality of the lemma.

We now start the proof of (4.1), or equivalently, because the denominators are positive,

$$\mathcal{L}_{1,1}(\sqrt{\sigma}\eta_n) + \mathcal{L}_{1,2}(\sqrt{\sigma}\eta_n) \ge \sigma(\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)) + C_7(\sigma^{3/2} - \sigma)\,.$$

From the forms of $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$, one has by using (3.17) and Lemma 4.1 (iii),

$$\begin{aligned}
&\mathcal{L}_{1,1}(\sqrt{\sigma}\eta_n) + \mathcal{L}_{1,2}(\sqrt{\sigma}\eta_n) - \sigma(\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)) = (\sigma^{3/2} - \sigma)\big(\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)\big) \\
&+ \mathcal{L}_{1,2}(\sqrt{\sigma}\eta_n) - \sigma^{3/2}\mathcal{L}_{1,2}(\eta_n) \ge C_7(\sigma^{3/2} - \sigma) + \mathcal{L}_{1,2}(\sqrt{\sigma}\eta_n) - \sigma^{3/2}\mathcal{L}_{1,2}(\eta_n).
\end{aligned}$$

So the only thing remaining to show is that $\mathcal{L}_{1,2}(\sqrt{\sigma}\eta_n) - \sigma^{3/2}\mathcal{L}_{1,2}(\eta_n) \ge 0$ for $n \ge N$.

Denote $\widetilde{\eta}_n = \sqrt{\sigma}\eta_n$, $w_1(\widetilde{\eta}_n) = \widetilde{w}_1 = \sqrt{\sigma}w_1$, $w_2(\widetilde{\eta}_n) = \widetilde{w}_2 = \sigma w_2$ and $w_3(\widetilde{\eta}_n) = \widetilde{w}_3$. Using (3.17) and Lemma 3.2, one can see that

$$\begin{aligned}
&\mathcal{L}_{1,2}(\widetilde{\eta}_n) - \sigma^{3/2}\mathcal{L}_{1,2}(\eta_n) \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty}\Big(\sigma^2 w_2 G(0) w_2 + \sigma w_2 G(0)\widetilde{w}_3 - \sigma^{\frac{3}{2}}\big(w_2 G(0) w_2 + w_2 G(0) w_3\big)\Big)\,dx \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty}\Big((\sigma^2 - \sigma^{\frac{3}{2}}) w_2 G(0) w_2 - \sigma w_2 \widetilde{\eta}_n(\widetilde{w}_2 + \widetilde{w}_3) + \sigma^{\frac{3}{2}} w_2 \eta_n (w_2 + w_3)\Big)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{\infty}\big[(\sigma^2-\sigma^{3/2})w_2G(0)w_2-(\sigma^{5/2}-\sigma^{3/2})\eta_n w_2^2-\sigma^{3/2}\eta_n w_2(\widetilde{w}_3-w_3)\big]dx\\
&\geq \frac{1}{2}\Big\{(\sigma^2-\sigma^{3/2})\|w_2\|_{L^2}^2-(\sigma^{5/2}-\sigma^{3/2})\|\eta_n\|_{H^1}\|w_2\|_{L^2}^2\\
&\qquad\qquad -\sigma^{3/2}\|\eta_n\|_{H^1}\|w_2\|_{L^2}\|\widetilde{w}_3-w_3\|_{L^2}\Big\}.
\end{aligned}
$$

Using Lemma 3.1(i), Lemma 3.3 (iii) and the fact that $\widetilde{\eta}_n\in B_{\frac{r}{2}}\subset B_{\frac{1}{2}}$, one has

$$
\begin{aligned}
&\|\widetilde{w}_3-w_3\|_{L^2}=\|G^{-1}(0)\big(\widetilde{\eta}_n(\widetilde{w}_2+\widetilde{w}_3)\big)-G^{-1}(0)\big(\eta_n(w_2+w_3)\big)\|_{L^2}\\
&\quad\leq\|\widetilde{\eta}_n(\widetilde{w}_2+\widetilde{w}_3)-\eta_n(w_2+w_3)\|_{L^2}\\
&\quad\leq\|\widetilde{\eta}_n(\widetilde{w}_3-w_3)+\widetilde{\eta}_n(\widetilde{w}_2-w_2)\|_{L^2}+\|(\widetilde{\eta}_n-\eta_n)(w_2+w_3)\|_{L^2}\\
&\quad\leq\|\widetilde{\eta}_n\|_{H^1}\|\widetilde{w}_3-w_3\|_{L^2}+\|\widetilde{\eta}_n\|_{H^1}\|\widetilde{w}_2-w_2\|_{L^2}+(\sqrt{\sigma}-1)(\|w_3\|_{L^2}+\|w_2\|_{L^2})\|\eta_n\|_{H^1}\\
&\quad\leq\frac{1}{2}\|\widetilde{w}_3-w_3\|_{L^2}+(\sigma-1)\sqrt{\sigma}\|\eta_n\|_{H^1}\|w_2\|_{L^2}+2(\sqrt{\sigma}-1)\|\eta_n\|_{H^1}\|w_2\|_{L^2}.
\end{aligned}
$$

Therefore

$$
\|\widetilde{w}_3-w_3\|_{L^2}\leq 2(\sigma^{3/2}+\sqrt{\sigma}-2)\|\eta_n\|_{H^1}\|w_2\|_{L^2}\leq 4(\sigma^{3/2}-1)\|\eta_n\|_{H^1}\|w_2\|_{L^2}\,.
$$

Consequently, from Lemma 4.1, one arrives at

$$
\begin{aligned}
&\mathcal{L}_{1,2}(\widetilde{\eta}_n)-\sigma^{3/2}\mathcal{L}_{1,2}(\eta_n)\\
&\geq\frac{1}{2}\Big\{(\sigma^2-\sigma^{3/2})\|w_2\|_{L^2}^2-(\sigma^{5/2}-\sigma^{3/2})\|\eta_n\|_{H^1}\|w_2\|_{L^2}^2-4\sigma^{3/2}(\sigma^{3/2}-1)\|\eta_n\|_{H^1}^2\|w_2\|_{L^2}^2\Big\}\\
&\geq\frac{(\sigma^2-\sigma^{3/2})}{2}\|w_2\|_{L^2}^2\Big\{1-\frac{\sigma^{5/2}-\sigma^{3/2}}{\sigma^2-\sigma^{3/2}}\|\eta_n\|_{H^1}-\frac{4\sigma^{3/2}(\sigma^{3/2}-1)}{\sigma^2-\sigma^{3/2}}\|\eta_n\|_{H^1}^2\Big\}\\
&\geq\frac{(\sigma^2-\sigma^{3/2})}{2}\|w_2\|_{L^2}^2\Big\{1-2\sigma^{1/2}\|\eta_n\|_{H^1}-12\sigma\|\eta_n\|_{H^1}^2\Big\}\\
&\geq\frac{(\sigma^2-\sigma^{3/2})}{2}\|w_2\|_{L^2}^2(1-2\sqrt{C_6}\sqrt{\mu_0(r)}-12C_6\mu_0(r))\geq 0
\end{aligned}
$$

by noticing $\dfrac{\sigma^2-\sigma}{\sigma^2-\sigma^{3/2}}\leq 2$, $\dfrac{\sigma^{3/2}-1}{\sigma^{1/2}-1}\leq 3\sigma$ and by requiring $\mu_0(r)$ to satisfy

$$
1-2\sqrt{C_6}\sqrt{\mu_0(r)}-12C_6\mu_0(r)\geq 0. \tag{4.2}
$$

Therefore, by choosing $\mu_0(r)\leq\frac{r^2}{4C_6}$ satisfying (4.2), the lemma is proved. $\square$

Corollary 4.3. *With the same $\mu_0(r)$ as in Lemma 4.2, for $\mu_1, \mu_2 > 0$ with $\mu_1 + \mu_2 < \mu_0(r)$, one has*

$$\widetilde{c}(\mu_1 + \mu_2) < \widetilde{c}(\mu_1) + \widetilde{c}(\mu_2).$$

Proof. If $\mu_1 \neq \mu_2$, we assume $\mu_1 > \mu_2$ without loss of generality. Using Lemma 4.2 twice, one has

$$\begin{aligned}\widetilde{c}(\mu_1 + \mu_2) = \widetilde{c}\big(\mu_1(1 + \frac{\mu_2}{\mu_1})\big) &< (1 + \frac{\mu_2}{\mu_1})\widetilde{c}(\mu_1) = \widetilde{c}(\mu_1) + \frac{\mu_2}{\mu_1}\widetilde{c}(\mu_1) \\ &= \widetilde{c}(\mu_1) + \frac{\mu_2}{\mu_1}\widetilde{c}(\frac{\mu_1}{\mu_2}\mu_2) < \widetilde{c}(\mu_1) + \widetilde{c}(\mu_2).\end{aligned}$$

If $\mu_1 = \mu_2$, then

$$\widetilde{c}(\mu_1 + \mu_2) = \widetilde{c}(2\mu_1) < 2\widetilde{c}(\mu_1) = \widetilde{c}(\mu_1) + \widetilde{c}(\mu_2).$$

□

Let $\eta_n \in B_r$ be a minimizing sequence and consider the associated concentration function $\rho_n(x) = -c(\eta_n')^2 + \eta_n^2$. As $\|\eta_n\|_{H^1} < r$ for all n, we can extract a subsequence which we again denote as η_n, so that

$$\beta = \lim_{n\to\infty} \int_{-\infty}^{\infty} \rho_n(x)dx$$

exists. Define a sequence of nondecreasing function $M_n : [0, \infty) \to [0, \beta]$ as follows

$$M_n(s) = \sup_{y\in\mathbb{R}} \int_{y-s}^{y+s} (-c|\eta_n'|^2 + |\eta_n|^2)dx = \sup_{y\in\mathbb{R}} \int_{y-s}^{y+s} \rho_n(x)dx.$$

As $M_n(s)$ is a uniformly bounded sequence of nondecreasing function in s, one can show that it has a subsequence, which we still denote as M_n, that converges point-wisely to a nondecreasing limit function $M(s) : [0, \infty) \to [0, \beta]$. Let

$$\beta_0 = \lim_{s\to\infty} M(s) :\equiv \lim_{s\to\infty} \lim_{n\to\infty} M_n(s) = \lim_{s\to\infty} \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-s}^{y+s} \rho_n(x)dx.$$

Then $0 \leq \beta_0 \leq \beta$.

Lions' Concentration Compactness Lemma [9, 10] shows that there are three possibilities for the value of β_0:

- Case 1: (*Vanishing*) $\beta_0 = 0$. Since $M(s)$ is non-negative and non-decreasing, this is equivalent to saying $M(s) = \lim_{n\to\infty} M_n(s) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-s}^{y+s} \rho_n(x)dx = 0$ for all $s < \infty$, or

- Case 2: (*Dichotomy*) $\beta_0 \in (0, \beta)$, or
- Case 3: (*Compactness*) $\beta_0 = \beta$, which implies that there exists $\{y_n\}_{n=1} \in \mathbb{R}$ such that $\rho_n(\cdot + y_n)$ is tight, namely, for all $\epsilon > 0$, there exists $s < \infty$ such that $\displaystyle\int_{y_n-s}^{y_n+s} \rho_n(x)dx \geq \beta - \epsilon$.

Lemma 4.4. *(Vanishing cannot occur.) There exists a $\gamma > 0$ such that*

$$\lim_{n\to\infty} M_n\left(\frac{1}{2}\right) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \geq \gamma.$$

Therefore

$$\beta_0 \geq \gamma > 0.$$

Proof. Suppose that $\displaystyle\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx = 0$. Let $I_j = [j - 1/2, j + 1/2]$. On I_j, one can see that,

$$(\sup_{x\in I_j} |\eta_n(x)|)^2 \leq C \int_{I_j} (\eta_n'(s))^2 + (\eta_n(s))^2 \ ds \leq C \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \longrightarrow 0,$$

as $n \to \infty$. From the expressions of $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ in (3.17) and Lemma 3.3, it is deduced that

$$\begin{aligned}
&|\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n)| \\
&\quad = \frac{1}{2} \left| \sum_{j=-\infty}^{\infty} \int_{I_j} -\eta_n w_1^2(\eta_n) + w_2(\eta_n)G(0)w_2(\eta_n) - \eta_n w_1(\eta_n)w_3(\eta_n)dx \right| \\
&\quad \leq \frac{1}{2} \sum_{j=-\infty}^{\infty} \sup_{x\in I_j} |\eta_n| \int_{I_j} \left(\left|w_1^2(\eta_n)\right| + \left|w_1(\eta_n)w_2(\eta_n)\right| + \left|w_1(\eta_n)w_3(\eta_n)\right| \right) dx \\
&\quad \leq C \sqrt{\sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx} \\
&\qquad\qquad \times \int_{-\infty}^{\infty} \left(\left|w_1^2(\eta_n)\right| + \left|w_1(\eta_n)w_2(\eta_n)\right| + \left|w_1(\eta_n)w_3(\eta_n)\right| \right) dx \\
&\quad \leq C\|\eta_n\|_{H^1}^2 \sqrt{\sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx}.
\end{aligned}$$

Upon letting $n \to \infty$, one arrives at $\mathcal{L}_{1,1}(\eta_n) + \mathcal{L}_{1,2}(\eta_n) \to 0$, a contradiction to (iii) in Lemma 4.1. Consequently, it follows that

$$\beta_0 = \lim_{s\to\infty} M(s) \geq M(1/2) = \lim_{n\to\infty} M_n(1/2) \geq \gamma > 0.$$

$\square$

We now turn our attention to the possibility of having dichotomy, that is $0 < \beta_0 < \beta$. Assume $0 < \beta_0 < \beta$, we will construct two sequences $\rho_{1,n}, \rho_{2,n} \geq 0$ with properties stated in the lemma below, and prove this will lead to a contradiction with the strict subadditivity proved in Corollary 4.3.

Given any $\epsilon > 0$, for all sufficiently large values of s, one has

$$\beta_0 - (\epsilon/2) < M(s) \leq M(2s) \leq \beta_0. \tag{4.3}$$

Suppose for the moment that a large value of s has been chosen so that (4.3) holds. Then one can choose N large enough that

$$\beta_0 - \epsilon \leq M_n(s) \leq M_n(2s) \leq \beta_0 + \epsilon$$

for all $n \geq N$. Hence for each $n \geq N$, there exists a y_n such that

$$\int_{y_n-s}^{y_n+s} \rho_n(x)dx \geq \beta_0 - \epsilon \quad \text{and} \quad \int_{y_n-2s}^{y_n+2s} \rho_n(x)dx \leq \beta_0 + \epsilon.$$

Now choose a function $\phi \in C_0^\infty[-2,2]$ such that $\phi = 1$ on $[-1,1]$ with $0 \leq \phi \leq 1$, and let $\psi \in C^\infty(\mathbb{R})$ satisfy $\phi + \psi = 1$ on $\mathbb{R}$. For each $s \in \mathbb{R}$, let $\phi_s(x) = \phi(\frac{x}{s})$ and $\psi_s(x) = \psi(\frac{x}{s})$ and define

$$\eta_{1,n}(x) = \phi_s(x - y_n)\eta_n(x) \quad \text{and} \quad \eta_{2,n}(x) = \psi_s(x - y_n)\eta_n(x). \tag{4.4}$$

Set

$$\rho_{1,n} = -c(\eta_{1,n}')^2 + \eta_{1,n}^2 \qquad \text{and} \qquad \rho_{2,n} = -c(\eta_{2,n}')^2 + \eta_{2,n}^2.$$

Notice that both $\eta_{1,n}$ and $\eta_{2,n}$ depend on s (which has been chosen for the moment large enough so that (4.3) holds) and hence so do $\rho_{1,n}$ and $\rho_{2,n}$. One can verify the following

Lemma 4.5. *For every $\epsilon > 0$, there exist S and N large enough such that for $n \geq N$ and $s \geq S$,*

(a) $\left| \int_{-\infty}^{\infty} \rho_{1,n}(x)dx - \beta_0 \right| \leq 2\epsilon,$

(b) $\left|\int_{-\infty}^{\infty}\rho_{2,n}(x)dx-(\beta-\beta_0)\right|\leq 2\epsilon,$

(c) $\left|\int_{-\infty}^{\infty}\Big(\rho_n(x)-(\rho_{1,n}(x)+\rho_{2,n}(x))\Big)dx\right|\leq 4\epsilon\,.$

Proof. The proof follows the same way used in [3].

(a) Consider $\int_{-\infty}^{\infty}\rho_{1,n}(x)dx=\int_{-\infty}^{\infty}[-c(\eta_{1,n}')^2+\eta_{1,n}^2]dx$

$$\begin{aligned}
&=-c\int_{-\infty}^{\infty}\phi^2(\frac{x-y_n}{s})\big(\eta_n'(x)\big)^2dx+\int_{-\infty}^{\infty}\phi^2(\frac{x-y_n}{s})\eta_n^2(x)dx\\
&-c\frac{1}{s^2}\int_{-\infty}^{\infty}[\phi'(\frac{x-y_n}{s})\eta_n(x)]^2dx-c\frac{2}{s}\int_{-\infty}^{\infty}\phi'(\frac{x-y_n}{s})\phi(\frac{x-y_n}{s})\eta_n(x)\eta_n'(x)dx \qquad (4.5)\\
&\leq\int_{y_n-2s}^{y_n+2s}\rho_n(x)dx+C/s\,,
\end{aligned}$$

where the constant C is positive and independent of n and s. By choosing s large enough, one can guarantee that $C/s\leq\epsilon$. Then

$$\int_{-\infty}^{\infty}\rho_{1,n}(x)dx\leq\int_{y_n-2s}^{y_n+2s}\rho_n(x)dx+\epsilon\leq\beta_0+2\epsilon. \tag{4.6}$$

On the other hand, from (4.5) one has

$$\int_{-\infty}^{\infty}\rho_{1,n}(x)dx\geq\int_{y_n-s}^{y_n+s}\rho_n(x)dx-C/s\geq\beta_0-2\epsilon. \tag{4.7}$$

Combining (4.6) and (4.7) there obtains statement (a).

(b) Consider $\int_{-\infty}^{\infty}\rho_{2,n}(x)dx=\int_{-\infty}^{\infty}[-c(\eta_{2,n}')^2+\eta_{2,n}^2]dx$

$$\begin{aligned}
&=-c\int_{-\infty}^{\infty}\psi^2(\frac{x-y_n}{s})\big(\eta_n'(x)\big)^2dx+\int_{-\infty}^{\infty}\psi^2(\frac{x-y_n}{s})\eta_n^2(x)dx\\
&-c\frac{1}{s^2}\int_{-\infty}^{\infty}[\psi'(\frac{x-y_n}{s})\eta_n(x)]^2dx-c\frac{2}{s}\int_{-\infty}^{\infty}\psi'(\frac{x-y_n}{s})\psi(\frac{x-y_n}{s})\eta_n(x)\eta_n'(x)dx\\
&\leq\int_{-\infty}^{y_n-s}\rho_n(x)dx+\int_{y_n+s}^{\infty}\rho_n(x)dx+\tilde{C}/s
\end{aligned}$$

where the positive constant $\tilde{C}$ is independent of n and s. By choosing s large enough, one can guarantee that $\tilde{C}/s \leq \epsilon/2$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \rho_{2,n}(x)dx &\leq \int_{-\infty}^{y_n-s} \rho_n(x)dx + \int_{y_n+s}^{\infty} \rho_n(x)dx + \epsilon/2 \\ &= \int_{-\infty}^{\infty} \rho_n(x)dx - \int_{y_n-s}^{y_n+s} \rho_n(x)dx + \epsilon \leq \beta - \beta_0 + 2\epsilon.\end{aligned} \tag{4.8}$$

Again, on the other hand,

$$\int_{-\infty}^{\infty} \rho_{2,n}(x)dx \geq \int_{-\infty}^{y_n-2s} \rho_n(x)dx + \int_{y_n+2s}^{\infty} \rho_n(x)dx - \tilde{C}/s \geq \beta - \beta_0 - 2\epsilon. \tag{4.9}$$

Combining (4.8) and (4.9) there derives statement (b).

Statement (c) follows from (a) and (b) and an application of the triangular inequality.

□

Lemma 4.6. *Dichotomy cannot occur either, namely $\beta_0 \notin (0, \beta)$*

Proof. For $\mu < \mu_0(r)$, let $\{\eta_n\}$ be a minimizing sequence. Consider two sequences $\{\eta_{1,n}\}$ and $\{\eta_{2,n}\}$ defined in (4.4). Suppose dichotomy happens. Define

$$D :\equiv \mathcal{L}_1(\eta_{1,n}) + \mathcal{L}_1(\eta_{2,n}).$$

From Lemma 4.5, $\|\eta_{1,n}\|_{H^1} \geq c_0 > 0$, $\|\eta_{2,n}\|_{H^1} \geq c_0 > 0$, which implies $\mathcal{L}_1(\eta_{1,n}) \geq c_1 > 0$ and $\mathcal{L}_1(\eta_{2,n}) \geq c_1 > 0$ due to (3.22), and $D > 0$, where c_0, c_1 are independent of n. Let

$$\mu_1 = \mu\frac{\mathcal{L}_1(\eta_{1,n})}{D}, \qquad \mu_2 = \mu\frac{\mathcal{L}_1(\eta_{2,n})}{D}.$$

Using the facts that $\mu_1, \mu_2 > 0$ and $\mu_1 + \mu_2 = \mu < \mu_0(r)$, one can see that

$$\begin{aligned}\widetilde{c}(\mu_1) + \widetilde{c}(\mu_2) &\leq \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(-c(\eta_{1,n}')^2 + \eta_{1,n}^2\big)dx + \frac{\mu_1^2}{\mathcal{L}_1(\eta_{1,n})}\right\} \\ &\quad + \left\{\frac{1}{2}\int_{-\infty}^{\infty}\big(-c(\eta_{2,n}')^2 + \eta_{2,n}^2\big)dx + \frac{\mu_2^2}{\mathcal{L}_1(\eta_{2,n})}\right\} \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\big(-c(\eta_{1,n}')^2 + \eta_{1,n}^2\big)dx + \frac{\mu^2}{D^2}\mathcal{L}_1(\eta_{1,n}) \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty}\big(-c(\eta_{2,n}')^2 + \eta_{2,n}^2\big)dx + \frac{\mu^2}{D^2}\mathcal{L}_1(\eta_{2,n}).\end{aligned} \tag{4.10}$$

Taking subsequences if necessary, Lemma 4.5 implies that for all n,

(a) $\left|\int_{-\infty}^{\infty} \rho_{1,n}(x)dx - \beta_0\right| \le 2/n,$

(b) $\left|\int_{-\infty}^{\infty} \rho_{2,n}(x)dx - (\beta - \beta_0)\right| \le 2/n,$

(c) $\left|\int_{-\infty}^{\infty} \Big(\rho_n(x) - (\rho_{1,n}(x) + \rho_{2,n}(x))\Big)dx\right| \le 4/n.$

Moreover, we have the following Claim, which will be proved in Appendix 6.1.
Claim: For all $n \ge 0$, by taking subsequences if necessary, $\eta_n, \eta_{1,n}$ and $\eta_{2,n}$ satisfy

$$\left|\mathcal{L}_1(\eta_n) - \Big(\mathcal{L}_1(\eta_{1,n}) + \mathcal{L}_1(\eta_{2,n})\Big)\right| \le (C/n),$$

where C is a constant independent of n.

Then, using the fact that $\mathcal{L}_1(\eta_n) \ge C\|\eta_n\|_{H^1}^2$, there follows

$$\begin{aligned}
\widetilde{c}(\mu_1) + \widetilde{c}(\mu_2) &\le \frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\eta_n')^2 + \eta_n^2\big)dx + \frac{\mu^2}{\mathcal{L}_1(\eta_{1,n}) + \mathcal{L}_1(\eta_{2,n})} + 4/n \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\eta_n')^2 + \eta_n^2\big)dx + \frac{\mu^2}{\mathcal{L}_1(\eta_n) - C/n} + 4/n \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\eta_n')^2 + \eta_n^2\big)dx + \frac{\mu^2}{\mathcal{L}_1(\eta_n)}\left[1 + O\left(\frac{C}{n\mathcal{L}_1(\eta_n)}\right)\right] + 4/n \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\eta_n')^2 + \eta_n^2\big)dx + \frac{\mu^2}{\mathcal{L}_1(\eta_n)} + O\left(n^{-1}\right)\left(\|\eta_n\|_{H^1}^{-2} + 1\right).
\end{aligned}$$

Upon letting $n \to \infty$ and noticing that $\lim_{n\to\infty} \int_{-\infty}^{\infty} \rho_n dx = \beta$, one arrives at

$$\widetilde{c}(\mu_1) + \widetilde{c}(\mu_2) \le \widetilde{c}(\mu) = \widetilde{c}(\mu_1 + \mu_2),$$

a contradiction to the strict subadditivity condition proved in Corollary 4.3. □

Now, Lions' Concentration Compactness Principle guarantees that the minimizing sequence is compact (possibly up to translation). However, because of the special variational form $\mathcal{H}(\eta)$ in (3.9), we are unable to extract a subsequence $\{\eta_{n_k}\}$ such that $\eta_{n_k}(\cdot + y_k) \to \eta$ *strongly* in $H^1(\mathbb{R})$. Therefore, we need to consider the minimizing sequence in $H^2(\mathbb{R})$. All of the $H^1(\mathbb{R})-$bounds we established earlier are unaffected since H^2-bound certainly implies H^1-bounds.

For $r > 0$, let $\mathcal{B}_r$ be defined in (1.8) and define a real number

$$c_0(\mu) = \inf_{\{\eta\in\mathcal{B}_r, \|\eta\|_{H^2}\neq 0\}} \left\{ \mathcal{H}(\eta) = \mathcal{H}_0(\eta) = \mathcal{L}_0(\eta) + \frac{\mu^2}{\mathcal{L}_1(\eta)} \right\}$$

and let

$$C_0(\mu) = \{\eta \in \mathcal{B}_r \setminus \{0\} \mid \mathcal{H}_0(\eta) = c_0(\mu)\}$$

be the set of minimizers of $\mathcal{H}_0$. A sequence $\{\eta_n\} \in \mathcal{B}_r \setminus \{0\}$ is called a minimizing sequence if $\lim_{n\to\infty} \mathcal{H}_0(\eta_n) = c_0(\mu)$.

The following shows that Lemma 3.8 is still valid when the function g is taken to be in $H^2(\mathbb{R})$.

Lemma 4.7. *There exists a μ_0 where $2(\mu_0 + \mu_0^5 + \mu_0^9) \leq \frac{1}{4}$ and for any $\mu < \mu_0$, there exists a function $g \in H^2(\mathbb{R})$ such that $\|g\|_{H^2}^2 = 2(\mu+\mu^5+\mu^9)$, and $\mathcal{H}_0(g) < 2\mu - \tilde{C}_0\mu^{5/2}$, where the positive constant $\tilde{C}_0$ does not depend on μ.*

Proof. Let f be a function in $H^2(\mathbb{R})$ such that $\int_{-\infty}^{\infty} f^3(x)dx < 0$ and

$$\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} f_x^2(x)dx = \int_{-\infty}^{\infty} f_{xx}^2(x)dx = 1.$$

Let $g(x) = Af(\mu^2 x)$ where $A^2 = 2\mu^3$. Then

$$\|g\|_{H^2}^2 = 2\int_{-\infty}^{\infty} (\mu^9 f_{xx}^2 + \mu^5 f_x^2 + \mu f^2)dx = 2(\mu + \mu^5 + \mu^9) = O(\mu).$$

The rest of the arguments in Lemma 3.8 remains unchanged. □

Hence as before, for $r < r_0 \leq 1/2$, $\mu < \mu_0 \leq 1/2$ and $\mu < r^2/4$, we can obtain that the minimizing sequence has a subsequence that is compact. Now, we use the concentration compactness to show the minimizer is attained.

Theorem 4.8. *Let $a, c < 0$ and $b > 0$ be real numbers such that $ac > b^2$. For $r \in (0, r_0)$ and $\mu \in (0, \mu_0(r))$, let $\{\eta_n\} \subset \mathcal{B}_r \setminus \{0\}$ be a minimizing sequence. Then there exist a subsequence $\{\eta_{n_k}\}$, a sequence of points $\{y_k\}$ in $\mathbb{R}$ and $\eta \in \mathcal{B}_r \setminus \{0\}$ such that $\eta_{n_k}(\cdot + y_k) \to \eta$ strongly in $H^1(\mathbb{R})$ and $\mathcal{H}(\eta) = c_0(\mu)$.*

Proof. Let $\widetilde{\eta}_{n_k}(x)$ denote $\eta_{n_k}(x + y_k)$ for $x \in \mathbb{R}$. Lions' concentration compactness principle guarantees that the minimizing sequence is compact, that is for every $k \in \mathbf{N}$, there exists $s_k \in \mathbb{R}$ such that for all sufficiently large n

$$\int_{-s_k}^{s_k} \widetilde{\rho}_n(x)dx > \beta - \frac{1}{k}. \tag{4.11}$$

Since $\|\widetilde{\eta}_n\|_{H^2} \leq C$, hence by compact embedding of $H^2(\Omega) \subset H^1(\Omega)$ for any bounded Ω, some subsequence $\widetilde{\eta}_{n_k} \longrightarrow$ converges *strongly* in $H^1[-s_k, s_k]$ norm to a limit function $\eta \in H^1[-s_k, s_k]$ satisfying

$$\int_{-s_k}^{s_k} (-c\eta_x^2 + \eta^2)dx > \beta - \frac{1}{k}. \tag{4.12}$$

Using (4.11) and (4.12) together with the fact that $\int_{-\infty}^{\infty} \widetilde{\rho}_n(x)dx = \beta$, one can assert that some subsequence of $\{\widetilde{\eta}_n\}$ (denoted again by $\{\widetilde{\eta}_n\}$ in the following) converges in $H^1(\mathbb{R})$ norm to a nonzero function $\eta \in H^1(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} (\eta_x^2 + \eta^2)dx = \beta$.

We proceeds now to show that the minimum is indeed attained at η. In Appendix 6.2, we prove the following,

$$\begin{aligned}
&\lim_{n\to\infty} \|w_1(\tilde{\eta}_n) - w_1(\eta)\|_{H^1} = 0, \ \lim_{n\to\infty} \|w_2(\tilde{\eta}_n) - w_2(\eta)\|_{H^1} = 0, \\
&\lim_{n\to\infty} \|w_3(\tilde{\eta}_n) - w_3(\eta)\|_{L^2} = 0,
\end{aligned} \tag{4.13}$$

and

$$\lim_{n\to\infty} \|G^{-1}(\eta)(\eta - b\eta_{xx}) - G^{-1}(\tilde{\eta}_n)(\tilde{\eta}_n - b\tilde{\eta}_{nxx})\|_{H^1} = 0. \tag{4.14}$$

Notice that since η is a weak limit of $\tilde{\eta}_n$ in a Hilbert space $H^2(\mathbb{R})$, $\eta \in H^2(\mathbb{R})$. It follows from (3.21), (4.13) and (4.14) that

$$|\big\{\mathcal{L}_{1,1}(\widetilde{\eta}_{n_k}) + \mathcal{L}_{1,2}(\widetilde{\eta}_{n_k})\big\} - \big\{\mathcal{L}_{1,1}(\eta) + \mathcal{L}_{1,2}(\eta)\big\}| \leq C|\mathcal{L}_{1,0}(\widetilde{\eta}_{n_k} - \eta)|.$$

Therefore, $\mathcal{L}_{1,1}(\widetilde{\eta}_{n_k}) + \mathcal{L}_{1,2}(\widetilde{\eta}_{n_k})$ converges to $\mathcal{L}_{1,1}(\eta) + \mathcal{L}_{1,2}(\eta)$. Consequently,

$$c_0(\mu) \leq \frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\eta')^2 + \eta^2\big) + \frac{\mu^2}{\mathcal{L}_{1,0}(\eta) + \mathcal{L}_{1,1}(\eta) + \mathcal{L}_{1,2}(\eta)}$$

$$= \lim_{k\to\infty} \left\{\frac{1}{2}\int_{-\infty}^{\infty} \big(-c(\widetilde{\eta}_{n_k}')^2 + \widetilde{\eta}^2\big) + \frac{\mu^2}{\mathcal{L}_{1,0}(\widetilde{\eta}_{n_k}) + \mathcal{L}_{1,1}(\widetilde{\eta}_{n_k}) + \mathcal{L}_{1,2}(\widetilde{\eta}_{n_k})}\right\} = c_0(\mu)$$

which shows that the minimum is attained at η. □

Denote $\eta(x)$ found in Theorem 4.8 for the critical point of $\mathcal{H}(\eta)$ in $\mathcal{B}_r$ by $\eta_0(x)$. Obviously, $\eta_0(x) \in \mathcal{B}_r$, since $\eta_0(x)$ is the limit of a weakly convergent sequence in $\mathcal{B}_r$. However, one problem is that $\eta_0(x)$ may lie on the boundary of $\mathcal{B}_r$ (i.e. the critical points η of $\mathcal{H}(\eta)$ satisfy $\|\eta\|_{H^2} = r$), which means that $c_0(\mu)$ may depend on r if we minimize $\mathcal{H}(\eta)$ in $\mathcal{B}_r$. We show that this will not happen, that is, if μ satisfies the conditions for two different r, $c_0(\mu)$ will be same, which implies that $c_0(\mu)$ is independent of r.

First, for $r > 0$ and $0 < \epsilon < 1/2$, we define

$$\mathcal{B}_{r,\epsilon} = \{\eta(x) \in H^2(\mathbb{R}) \quad \Big| \quad \epsilon\|\eta_{xx}\|_{L^2}^2 + \|\eta_x\|_{L^2}^2 + \|\eta\|_{L^2}^2 \le r^2\},$$

$$\mathcal{L}_\epsilon(\eta) = \epsilon\|\eta_{xx}\|_{L^2}^2 + \mathcal{L}_0(\eta),$$

and a real number

$$c_\epsilon(\mu) = \inf_{\{\eta\in\mathcal{B}_{r,\epsilon}, \|\eta\|_{H^2}\neq 0\}} \left\{\mathcal{H}_\epsilon(\eta) = \mathcal{L}_\epsilon(\eta) + \frac{\mu^2}{\mathcal{L}_1(\eta)}\right\}$$

with

$$C_\epsilon(\mu) = \{\eta \in \mathcal{B}_{r,\epsilon} \setminus \{0\} \mid \mathcal{H}_\epsilon(\eta) = c_\epsilon(\mu)\}$$

being the set of minimizers of $\mathcal{H}_\epsilon(\eta)$. A sequence $\{\eta_{n,\epsilon}\} \in \mathcal{B}_{r,\epsilon} \setminus \{0\}$ is called a minimizing sequence if $\lim_{n\to\infty} \mathcal{H}_\epsilon(\eta_{n,\epsilon}) = c_\epsilon(\mu)$. Notice that since $H^2(\mathbb{R}) \subset H^1(\mathbb{R})$ hence the following Lemma holds just like before with straightforward modifications.

Lemma 4.9. *Let*

$$\mathcal{H}_{\epsilon,0}(\eta) = \mathcal{L}_\epsilon(\eta) + \frac{\mu^2}{\mathcal{L}_{1,0}(\eta)}.$$

Then

(i) $\mathcal{H}_{\epsilon,0}(\eta) \ge 2\mu > 0$, *and*

(ii) *there exists a sequence* $\{\eta_\alpha\}$ *in* $H^2(\mathbb{R})$, $\lim_{\alpha\to 0} \|\eta_\alpha\|_{H^2}^2 = \lim_{\alpha\to 0} \|\eta_\alpha\|_{H^1}^2 = 2\mu$ *such that*

$$\lim_{\alpha\to 0} \mathcal{H}_{\epsilon,0}(\eta_\alpha) = 2\mu.$$

Now we show that if μ satisfies the conditions for two different r, $c_0(\mu)$ will be same, which implies that $c_0(\mu)$ is independent of r.

Theorem 4.10. $c_0(\mu)$ *is independent of* r *and the minimizer* $\eta \in \mathcal{B}_r$ *satisfies* $\|\eta\|_{H^2} \le C\sqrt{\mu}$ *for some constant* C *independent* μ, r. $c_0(\mu)$ *only depends upon* μ *if* μ *satisfies* $\mu \le \mu_0(r)$.

This theorem will be proved in Appendix 6.3.

Finally, we show that the minimization problem of $\mathcal{H}(\eta)$ in $\mathcal{B}_r$ is the same as the minimization problem of $\mathcal{H}(\eta, u)$ (defined in (1.4)) in $\mathcal{B}_r \times H^1(\mathbb{R})$. For $r \in (0, r_0)$ and $\mu \in (0, \mu_0(r))$, consider the following two problems

$$\inf\left\{\mathcal{H}(\eta, u) \;\Big|\; (\eta, u) \in \mathcal{B}_r \times H^1(\mathbb{R}),\ \mathcal{I}(\eta, u) = 2\mu\right\} = A, \tag{4.15}$$

and

$$\min_{\eta\in\mathcal{B}_r} \min_{u\in H^1(\mathbb{R})} \left\{\mathcal{H}(\eta, u) \;\Big|\; \mathcal{I}(\eta, u) = 2\mu\right\} = B. \tag{4.16}$$

We prove that $A = B$ and (4.15) is attained, namely $D(r, \mu)$ (defined in (1.9) with $\mathcal{H}_{r,\mu} = A$) is non-empty and the minimizing sequence converges to $D(r, \mu)$.

Clearly, $B \geq A$. Suppose now that $\mathcal{H}(\eta_i, u_i) \to \mathcal{H}(\eta, u) = A$ with $(\eta_i, u_i) \in \mathcal{B}_r \times H^1(\mathbb{R})$ and $\mathcal{H}(\eta_i, u_i)$ is nonincreasing, which is always possible by choosing a subsequence. For each fixed $\eta_i \in \mathcal{B}_r$, a non-trivial minimizer $u_i^* = u(\eta_i) \in H^1(\mathbb{R})$ for $\big\{\mathcal{H}(\eta_i, u)\big|\mathcal{I}(\eta_i, u) = 2\mu\big\}$ exists. Since for each i,

$$\mathcal{H}(\eta_i, u_i^*) \leq \mathcal{H}(\eta_i, u_i),$$

it yields

$$B \leq \min_{\eta_i \in \mathcal{B}_r} \mathcal{H}(\eta_i, u_i^*) \leq \inf_{(\eta_i, u_i) \in \mathcal{B}_r \times H^1(\mathbb{R})} \mathcal{H}(\eta_i, u_i) = A. \tag{4.17}$$

Therefore, it is concluded that $B = A$.

5 Stability for the Set of Minimizers

In this section, we show the set of minimizers $D(r, \mu)$ is stable under the small perturbation of initial data near $D(r, \mu)$.

Theorem 5.1. *$D(r, \mu)$ is non-empty and for every minimizing sequence $\{(\eta_k, u_k)\} \subset \mathcal{B}_r \times H^1(\mathbb{R})$, there is a subsequence, denoted again by $\{(\eta_k, u_k)\}$, such that*

$$dist\big((\eta_k, u_k), D(r, \mu)\big) \to 0.$$

Proof. For $r \in (0, r_0)$ and $\mu \in (0, \mu_0(r))$, let $\eta_n \in \mathcal{B}_r \setminus \{0\}$ be a minimizing sequence for $C_0(\mu)$. Due to Theorem 4.8, after possible translations and taking a subsequence if necessary, $\{\eta_n\}$ can be assumed to converge in $H^1(\mathbb{R})$ to some $\eta \in C_0(\mu)$. Let the Lagrange multiplier $\lambda = \lambda_{\eta_n}$ and $u_n^* = u(\eta_n)$ be defined as in (3.6) and (3.4) respectively. One wants to show first that

$$u_n^* \to u^* = u(\eta) \quad \text{in } H^1(\mathbb{R}).$$

As η_n converges to η strongly in $H^1(\mathbb{R})$, it follows that $\lim_{n\to\infty} \mathcal{L}_1(\eta_n) = \mathcal{L}_1(\eta)$ because of Lemma 3.6. Consequently, (3.6) implies that $\lim_{n\to\infty} \lambda_{\eta_n} = \lambda$. Using (3.4), one has

$$\begin{aligned}
&\|u^* - u_n^*\|_{H^1} \leq \|\lambda G^{-1}(\eta)(\eta - b\eta_{xx}) - \lambda G^{-1}(\eta_n)(\eta_n - b(\eta_n)_{xx})\|_{H^1} \\
&+ \|\lambda G^{-1}(\eta_n)(\eta_n - b(\eta_n)_{xx}) - \lambda_{\eta_n} G^{-1}(\eta_n)(\eta_n - b(\eta_n)_{xx})\|_{H^1} \\
&\leq \sup|\lambda| \|G^{-1}(\eta)(\eta - b\eta_{xx}) - G^{-1}(\eta_n)(\eta_n - b(\eta_n)_{xx})\|_{H^1} \\
&+ \|G^{-1}(\eta_n)(\eta_n - b(\eta_n)_{xx})\|_{H^1} |\lambda - \lambda_{\eta_n}|.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \|u_n^* - u^*\|_{H^1} = 0.$$

A straightforward calculation now confirms that

$$\begin{aligned}\big|\mathcal{H}(\eta,u^*)-\mathcal{H}(\eta_n,u_n^*)\big| = \Bigg|&\frac{1}{2}\int_{-\infty}^{\infty}(-c\eta_x^2 - cu_x^2 + \eta^2 + u^2 + \eta u^2)dx \\ &-\frac{1}{2}\int_{-\infty}^{\infty}(-c((\eta_n)')^2 - c((u_n^*)')^2 + \eta_n^2 + (u_n^*)^2 + \eta_n(u_n^*)^2)dx\Bigg| \to 0\end{aligned} \tag{5.1}$$

as $n \to \infty$. Thus $A = B$ in (4.15) and (4.16) implies $(\eta, u^*) \in D(r,\mu)$.

Moreover, for a minimizing sequence $\{(\eta_n, u_n)\}$ of $\mathcal{H}(\eta, u)$, we can construct u_n^* satisfying (4.17). Then, taking a subsequence if necessary, we have η, u^* such that $(\eta_n, u_n^*) \to (\eta, u^*)$ in $\mathcal{B}_r \times H^1(\mathbb{R})$ satisfying (5.1). By the definition of $G(\eta)$ in (3.5), the self-adjointness of $G(\eta)$ and (5.1), it is straightforward to check that there exists a $C > 0$ such that as $n \to +\infty$,

$$\begin{aligned}C\|u_n - u_n^*\|_{H^1}^2 \leq & \int_{-\infty}^{\infty}(u_n - u_n^*)G(\eta_n)(u_n - u_n^*)dx \\ =& \int_{-\infty}^{\infty}(u_nG(\eta_n)u_n + u_n^*G(\eta_n)u_n^*)dx - 2\int_{-\infty}^{\infty}u_nG(\eta_n)u_n^*dx \\ =& \int_{-\infty}^{\infty}(u_nG(\eta_n)u_n + u_n^*G(\eta_n)u_n^*)dx - 4\lambda_{\eta_n}\mu \\ =& \int_{-\infty}^{\infty}(u_nG(\eta_n)u_n - u_n^*G(\eta_n)u_n^*)dx \\ =& \mathcal{H}(\eta_n, u_n) - \mathcal{H}(\eta_n, u_n^*) \to 0\end{aligned}$$

which implies $(\eta_n, u_n) \to (\eta, u^*) \in D(r,\mu)$ or $dist\big((\eta_n, u_n), D(r,\mu)\big) \to 0$ as $n \to \infty$. □

One can now establish the stability of $D(r,\mu)$ based upon Theorem 5.1.

Theorem 5.2. *There exists an $r_0 > 0$ such that for $r \in (0, r_0)$ and $\mu \in (0, \mu_0(r))$, the following statement is true: For any $\epsilon > 0$, there exists a $\delta > 0$ such that if*

$$(\phi,\psi) \in \mathcal{B}_r \times H^2(\mathbb{R}),\ dist\big((\phi,\psi), D(r,\mu)\big) < \delta,$$

and if $\big(\eta(x,t), u(x,t)\big)$ is a solution of (1.1)-(1.2)-(1.6) satisfying $\eta(x,t) \in \mathcal{B}_r$ with all $t \in \mathbb{R}^+$ for the initial data (ϕ,ψ), then

$$dist\big((\eta(\cdot,t), u(\cdot,t)), D(r,\mu)\big) < \epsilon \quad \text{for all } t \in [0,\infty).$$

Proof. Suppose to the contrary that there exist a sequence $(\eta_{0,n}, u_{0,n}) \in \mathcal{B}_r \times H^2(\mathbb{R})$ and a sequence $(\mu_n) \in \mathbb{R}$ such that

$$dist\big((\eta_{0,n}, u_{0,n}), D(r,\mu)\big) < 1/n\,, \qquad \mathcal{I}(\eta_{0,n}, u_{0,n}) = 2\mu_n,$$

and

$$dist\big((\eta_n(\cdot,t_n), u_n(\cdot,t_n)), D(r,\mu)\big) \geq \epsilon \quad \text{and} \quad \mathcal{I}(\eta_n(\cdot,t_n), u_n(\cdot,t_n)) = 2\mu_n,$$

for some $t_n \in [0,\infty)$ and $\eta_n(\cdot,t_n) \in \mathcal{B}_r$. Since $dist\big((\eta_{0,n}, u_{0,n}), D(r,\mu)\big) \to 0$, using a subsequence if necessary, there is a sequence $(\widetilde{\eta}_n, \widetilde{u}_n) \in D(r,\mu)$ such that $\|\eta_{0,n} - \widetilde{\eta}_n\|_{H^1} + \|u_{0,n} - \widetilde{u}_n\|_{H^1} \to 0$ as $\to +\infty$. Since it is obvious that $(\widetilde{\eta}_n, \widetilde{u}_n)$ is a minimizing sequence of $\mathcal{H}(\eta, u)$ in $D(r,\mu)$, by Theorem 5.1, there is a subsequence, again denoted by the same notation, that converges to $(\eta, u^*) \in D(r,\mu)$ in H^1 norm. Therefore,

$$\|\eta_{0,n} - \eta\|_{H^1} + \|u_{0,n} - u^*\|_{H^1} \to 0,\ \mu_n \to \mu,\ \mathcal{I}(\eta_{0,n}, u_{0,n}) \to \mathcal{I}(\eta, u^*) = 2\mu$$

and

$$\lim_{n\to\infty} \mathcal{H}\big(\eta_n(\cdot,t_n), u_n(\cdot,t_n)\big) = \lim_{n\to\infty} \mathcal{H}(\eta_{0,n}, u_{0,n}) = \mathcal{H}(\eta, u^*) = \widetilde{c}(\mu).$$

Upon letting $\bar{\eta}_n :\equiv \eta_n(t_n)$ and $\bar{u}_n :\equiv (\mu/\mu_n) u_n(t_n)$, we obtain that for large n,

$$dist\big((\bar{\eta}_n, \bar{u}_n), D(r,\mu)\big) \geq \epsilon/2, \quad \lim_{n\to\infty} \mathcal{H}(\bar{\eta}_n, \bar{u}_n) = \widetilde{c}(\mu)$$

and $\mathcal{I}(\bar{\eta}_n, \bar{u}_n) = 2\mu$, a contradiction to Theorem 5.1. Thus, the proof is completed. □

6 Appendices

6.1 Proof of the Claim in the proof of Lemma 4.7

Here, we need to show that for $n \geq 1$ the sequences $\eta_n, \eta_{1,n}$ and $\eta_{2.n}$, (taking subsequences if necessary), satisfy

$$\big|\mathcal{L}_1(\eta_n) - \big(\mathcal{L}_1(\eta_{1,n}) + \mathcal{L}_1(\eta_{2,n})\big)\big| \leq C/n \tag{6.1}$$

for some constant C independent of n.

First, we prove the following:

Consider $G(0)u = f(x)$ where $f(x)$ satisfies

$$\int_{y_n-2s}^{y_n-s} |f(x)|^2 dx + \int_{y_n+s}^{y_n+2s} |f(x)|^2 dx \leq \epsilon\,. \tag{6.2}$$

If $f_1 = \phi_s(x-y_n)f(x) = \phi_s f(x)$, $f_2 = \psi_s(x-y_n)f(x) = \psi_s f(x)$ and $u_1 = G^{-1}(0)f_1$, $u_2 = G^{-1}(0)f_2$, then

$$\|\psi_s(x-y_n)u_1\|_{H^2} + \|\phi_s(x-y_n)u_2\|_{H^2} \le C(\epsilon + (\|f\|_{L^2}/|s|))$$

where $|s| \ge S_0$ for some fixed S_0 independent of ϵ, y_n and f.

Proof. Here, we denote $\phi_s = \phi_s(x-y_n), \psi_s = \psi_s(x-y_n)$ and C as a constant independent of ϵ, f, y_n. Note that $\|u_1\|_{H^2} + \|u_2\|_{H^2} \le C\|f\|_{L^2}$. By the definition of $G(0)$ in (3.5),

$$G(0)(\psi_s u_1) = \psi_s G(0)u_1 + a\psi_{sxx}u_1 + 2a\psi_{sx}u_{1x} = \psi_s f_1 + a\psi_{sxx}u_1 + 2a\psi_{sx}u_{1x}\,. \quad (6.3)$$

By a similar proof in Lemma 3.3, we have that

$$\|\psi_s u_1\|_{H^2} \le C(\|\psi_s f_1\|_{L^2} + \|\psi_{sxx}u_1\|_{L^2} + \|\psi_{sx}u_{1x}\|_{L^2}\,. \quad (6.4)$$

Since $\|\psi_{sx}\|_{L^2} \le C/|s|$ and $\psi_s f_1$ is zero for $|x-y_n| \ge 2s$ or $|x-y_n| \le s$, then by (6.2) the estimate follows immediately. A similar proof holds for $\phi_s u_2$. □

If $f = \eta - b\eta_{xx}$, then by $u_1 + u_2 = w_1$ defined in (3.15), we can multiply (3.15) by $\psi_s u_1$ and use integration by parts to obtain

$$\|\psi_s u_1\|_{H^1} \le C(\|\psi_s\eta_1\|_{H^1} + |s|^{-1})\,, \quad (6.5)$$

where $\eta_1 = \phi_s\eta, \eta_2 = \psi_s\eta$. A similar one holds for $\phi_s u_2$.

Now, we study $\mathcal{L}_1(\eta)$. Recall that

$$\mathcal{L}_1(\eta) = \mathcal{L}_{1,0}(\eta) + \mathcal{L}_{1,1}(\eta) + \mathcal{L}_{1,2}(\eta)$$

where

$$\begin{aligned}
\mathcal{L}_{1,0}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty} w_1 G(0) w_1 dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{(bk^2+1)^2}{(1-ak^2)}|\widehat{\eta}|^2 dk > 0,\\
\mathcal{L}_{1,1}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty} -\eta w_1^2 dx,\\
\mathcal{L}_{1,2}(\eta) &= \frac{1}{2}\int_{-\infty}^{\infty}(w_2G(0)w_2 + w_2G(0)w_3)dx = \frac{1}{2}\int_{-\infty}^{\infty}(w_2G(0)w_2 - \eta w_1 w_3)dx,
\end{aligned}$$

and w_1, w_2, w_3 are defined in (3.15). One proceeds to prove the Claim by taking care of $\mathcal{L}_1(\eta)$ term by term. Note that

$$w_1(\eta_n) = w_1(\eta_{1,n}) + w_1(\eta_{2,n}) = (\phi_s+\psi_s)w_1(\eta_{1,n}) + (\phi_s+\psi_s)w_1(\eta_{2,n})$$

$$=\phi_s w_1(\eta_{1,n}) + \psi_s w_1(\eta_{2,n}) + \psi_s w_1(\eta_{1,n}) + \phi_s w_1(\eta_{2,n})$$

where the last two terms are small in L^2 using (6.5). Since η_n and its first order derivative satisfy (6.2),

$$\|w_1(\eta_n) - (\phi_s w_1(\eta_{1,n}) + \psi_s w_1(\eta_{2,n}))\|_{L^2} \leq C(\epsilon + |s|^{-1})$$

Thus, by using a similar proof of Lemma 3.3, it is obtained that

$$\begin{aligned}
\mathcal{L}_{1,0}(\eta_n) &= \int_{-\infty}^{+\infty} w_1(\eta - b\eta_{xx})dx = \int_{-\infty}^{+\infty} (w_1\eta + w_{1x}b\eta_x)dx \\
&= \int_{-\infty}^{+\infty} ((\phi_s + \psi_s)(w_1(\eta_{1,n}) + w_1(\eta_{2,n}))(\eta_{1,n} + \eta_{2,n}) \\
&\quad + b(\phi_s + \psi_s)(w_1(\eta_{1,n}) + w_1(\eta_{2,n}))_x(\eta_{1,n} + \eta_{2,n})_x)dx \\
&= \int_{-\infty}^{+\infty} (w_1(\eta_{1,n})\eta_{1,n} + w_1(\eta_{2,n})\eta_{2,n} + bw_{1x}(\eta_{1,n})(\eta_{1,n})_x \\
&\quad + bw_{2x}(\eta_{2,n})(\eta_{2,n})_x)dx + O(\epsilon + |s|^{-1}) \\
&= \mathcal{L}_{1,0}(\eta_{1,n}) + \mathcal{L}_{1,0}(\eta_{2,n}) + O(\epsilon + |s|^{-1}) .
\end{aligned}$$

Here, the terms with a factor $\phi_s\psi_s$ in front of η_n or $\eta_{n,x}$ are of order ϵ because of (6.2) for η_n.

The next term to be considered is $\mathcal{L}_{1,1}$. Using a similar proof again, we have

$$\begin{aligned}
\mathcal{L}_{1,1}(\eta_n) &= \int_{-\infty}^{\infty} - \eta_n w_1^2(\eta_n)dx = -\int_{-\infty}^{\infty} (\eta_{1,n} + \eta_{2,n})[(\phi_s + \psi_s)(w_1(\eta_{1,n}) + w_1(\eta_{2,n}))]^2 dx \\
&= -\int_{-\infty}^{\infty} \left((\eta_{1,n} + \eta_{2,n})\left((\phi_s w_1(\eta_{1,n}))^2 + (\psi_s w_1(\eta_{2,n}))^2\right)\right) dx + O(\epsilon + |s|^{-1}) \\
&= -\int_{-\infty}^{\infty} \left(\eta_{1,n}(\phi_s w_1(\eta_{1,n}))^2 + \eta_{2,n}(\psi_s w_1(\eta_{2,n}))^2\right) dx + O(\epsilon + |s|^{-1}) \\
&= -\int_{-\infty}^{\infty} \left(\eta_{1,n} w_1(\eta_{1,n})^2 + \eta_{2,n} w_1(\eta_{2,n})^2\right) dx = \mathcal{L}_{1,1}(\eta_{1,n}) + \mathcal{L}_{1,1}(\eta_{2,n}) + O(\epsilon + |s|^{-1}) .
\end{aligned}$$

Considered next is the first term in $\mathcal{L}_{1,2}$. A straightforward calculation gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} w_2(\eta_n)G(0)w_2(\eta_n)dx \\
&= \int_{-\infty}^{\infty} \Bigg\{ - a\Big((\phi_s + \psi_s)G^{-1}(0)\big((\phi_s + \psi_s)\eta_n w_1(\eta_n)\big)\Big)_x^2 \\
&\quad + \Big((\phi_s + \psi_s)G^{-1}(0)\big((\phi_s + \psi_s)\eta_n w_1(\eta_n)\big)\Big)^2 \Bigg\} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \Big\{ -a\Big(\phi_s G^{-1}(0)\big(\phi_s \eta_n w_1(\eta_n)\big)\Big)_x^2 + \Big(\phi_s G^{-1}(0)\big(\phi_s \eta_n w_1(\eta_n)\big)\Big)^2 \Big\} dx \\
&\quad + \int_{-\infty}^{\infty} \Big\{ -a\Big(\psi_s G^{-1}(0)\big(\psi_s \eta_n w_1(\eta_n)\big)\Big)_x^2 + \Big(\psi_s G^{-1}(0)\big(\psi_s \eta_n w_1(\eta_n)\big)\Big)^2 \Big\} dx \\
&\qquad + O(\epsilon + |s|^{-1}) \\
&= \int_{-\infty}^{\infty} \Big\{ -a\Big(G^{-1}(0)\big(\phi_s \eta_n w_1(\phi_s\eta_n)\big)\Big)_x^2 + \Big(G^{-1}(0)\big(\phi_s \eta_n w_1(\phi_s\eta_n)\big)\Big)^2 \Big\} dx \\
&\quad + \int_{-\infty}^{\infty} \Big\{ -a\Big(G^{-1}(0)\big(\psi_s \eta_n w_1(\psi_s\eta_n)\big)\Big)_x^2 + \Big(G^{-1}(0)\big(\psi_s \eta_n w_1(\psi_s\eta_n)\big)\Big)^2 \Big\} dx \\
&\qquad + O(\epsilon + |s|^{-1}) \\
&= \int_{-\infty}^{\infty} w_2(\eta_{1,n}) G(0) w_2(\eta_{1,n}) dx + \int_{-\infty}^{\infty} w_2(\eta_{2,n}) G(0) w_2(\eta_{2,n}) dx + O(\epsilon + |s|^{-1}) \,. \quad (6.6)
\end{aligned}
$$

For the last term in $\mathcal{L}_{1,2}$, since $w_1(\eta_n) = w_1(\eta_{1,n}) + w_1(\eta_{2,n}) + O(\epsilon + |s|^{-1})$ and $\eta_n = \eta_{1,n} + \eta_{2,n}$, we have

$$
\int_{-\infty}^{+\infty} \eta_n w_1(\eta_n) w_3(\eta_n) dx = \int_{-\infty}^{+\infty} \left(\eta_{1,n} w_1(\eta_{1,n}) + \eta_{2,n} w_1(\eta_{2,n})\right) w_3(\eta_n) dx + O(\epsilon + |s|^{-1}) \,.
$$

Moreover, from the definition of $w_2(\eta)$, (6.4), (6.5) and the proof of Lemma 3.3, we obtain

$$
\begin{aligned}
w_2(\eta_n) =& G^{-1}(0)(-\eta_n w_1(\eta_n)) = G^{-1}(0)(-\eta_{1,n} w_1(\eta_{1,n}) - \eta_{2,n} w_1(\eta_{2,n})) + O(\epsilon + |s|^{-1}) \\
=& w_2(\eta_{1,n}) + w_2(\eta_{2,n}) + O(\epsilon + |s|^{-1}).
\end{aligned}
$$

By the definition of w_3 in (3.15), it is obtained that

$$
G(0) w_3(\eta_n) = -\eta_n w_2(\eta_n) - \eta_n w_3(\eta_n) \,, \text{ and } G(0) w_3(\eta_{jn}) = -\eta_{j,n} w_2(\eta_{j,n}) - \eta_{j,n} w_3(\eta_{j,n})
$$

for $j = 1, 2$. Also,

$$
\begin{aligned}
G(0)\phi_s w_3(\eta_{2,n}) =& \phi_s G(0) w_3(\eta_{2,n}) + O(|s|^{-1})(w_3(\eta_{2,n}) + w_{3x}(\eta_{2,n})) \\
=& -\phi_s \Big(\eta_{2,n} w_2(\eta_{2,n}) - \eta_{2,n} w_3(\eta_{2,n}) \Big) + O(|s|^{-1})(w_3(\eta_{2,n}) + w_{3x}(\eta_{2,n})) \,.
\end{aligned}
$$

Since the H^1-norm of $\phi_s \eta_{2,n}$ is less than ϵ, by a similar proof of Lemma 3.3 (iv), we have that

$$
\|\phi_s w_3(\eta_{2,n})\|_{H^1}^2 \le C(\epsilon + |s|^{-1}) \,.
$$

A similar calculation holds for $\psi_s w_3(\eta_{1,n})$. Let $w = w_3(\eta_n) - w_3(\eta_{1,n}) - w_3(\eta_{2,n})$. Then

$$
G(0) w = -\eta_n w_2(\eta_n) + \eta_{1,n} w_2(\eta_{1,n}) + \eta_{2,n} w_2(\eta_{2,n})
$$

$$
\begin{aligned}
&- \eta_n w_3(\eta_n) + \eta_{1,n} w_3(\eta_{1,n}) + \eta_{2,n} w_3(\eta_{2,n}) \\
= &-\eta_n w - \psi_s \eta_n w_3(\eta_{1,n}) - \phi_s \eta_n w_3(\eta_{2,n}) \\
&+ (-\eta_n w_2(\eta_n) + \eta_{1,n} w_2(\eta_{1,n}) + \eta_{2,n} w_2(\eta_{2,n})). \qquad (6.7)
\end{aligned}
$$

Note that last three terms in (6.7) are of order $O(\epsilon + |s|^{-1})$. Again, by using a similar proof of Lemma 3.3 (iv), we obtain

$$
\|w\|_{H^1}^2 \leq C(\epsilon + |s|^{-1}),
$$

which implies

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \eta_n w_1(\eta_n) w_3(\eta_n) dx = \int_{-\infty}^{+\infty} &\Big(\eta_{1,n} w_1(\eta_{1,n}) + \eta_{2,n} w_1(\eta_{2,n})\Big) \\
&\times \Big(w_3(\eta_{1,n}) + w_3(\eta_{2,n})\Big) dx + O(\epsilon + |s|^{-1}).
\end{aligned}
$$

Then, by a similar inequality as (6.4) again, we have

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \eta_n w_1(\eta_n) w_3(\eta_n) dx = \int_{-\infty}^{+\infty} &\Big(\eta_{1,n} w_1(\eta_{1,n}) w_3(\eta_{1,n}) + \eta_{2,n} w_1(\eta_{2,n}) w_3(\eta_{2,n}))\Big) dx \\
&+ O(\epsilon + |s|^{-1}). \qquad (6.8)
\end{aligned}
$$

Combining (6.6) and (6.8) yields

$$
\mathcal{L}_{1,2}(\eta_n) = \mathcal{L}_{1,2}(\eta_{1,n}) + \mathcal{L}_{1,2}(\eta_{2,n}) + O(\epsilon + |s|^{-1}).
$$

Now, if we choose s large enough and a subsequence of η_n if necessary, (6.1) is obtained.

6.2 Proof of (4.13) and (4.14)

Here, we first prove the following,

(1) $\lim_{n\to\infty} \|w_1(\eta_n) - w_1(\eta)\|_{H^1} = 0,$ (2) $\lim_{n\to\infty} \|w_2(\eta_n) - w_2(\eta)\|_{H^1} = 0,$

(3) $\lim_{n\to\infty} \|w_3(\eta_n) - w_3(\eta)\|_{L^2} = 0,$

where, for simplicity, we write $\eta_n = \tilde{\eta}_n$.

Recall that

$$
\begin{aligned}
&w_1(\eta) = G^{-1}(0)(\eta - b\eta_{xx}), \qquad w_2(\eta) = -G^{-1}(0)(\eta w_1(\eta)), \\
&w_3(\eta) = -G^{-1}(0)[\eta(w_2(\eta) + w_3(\eta))].
\end{aligned}
$$

To prove (1), for a minimizing sequence $\{\eta_n\} \in \mathcal{B}_r \setminus \{0\}$, Lemma 3.3 assures that $\|w_i(\eta_n)\|_{H^1} \le C_i$, $i = 1, 2, 3$. Notice now that if one lets $G^{-1}(0)(\eta_{nxx} - \eta_{xx}) = g_n$, then

$$\int_{-\infty}^{\infty} (-ag_{nx}^2 + g_n^2)dx = -\int_{-\infty}^{\infty} g_{nx}(\eta_{nx} - \eta_x)dx \le \|g_{nx}\|_{L^2}\|\eta_{nx} - \eta_x\|_{L^2} \to 0$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} \|G^{-1}(0)(\eta_{nxx} - \eta_{xx})\|_{H^1} = 0$. Similarly, one can show that $\lim_{n\to\infty} \|G^{-1}(0)(\eta_n - \eta)\|_{H^1} = 0$. Consequently,

$$\begin{aligned}\lim_{n\to\infty} \|w_1(\eta_n) - w_1(\eta)\|_{H^1} &= \lim_{n\to\infty} \|G^{-1}(0)[(\eta_n - \eta) - b(\eta_{nxx} - \eta_{xx})]\|_{H^1} \\ &\le \lim_{n\to\infty} \|G^{-1}(0)(\eta_n - \eta)\|_{H^1} + b \lim_{n\to\infty} \|G^{-1}(0)(\eta_{nxx} - \eta_{xx})\|_{H^1} = 0.\end{aligned}$$

For (2), let $G^{-1}(0)[\eta_n w_1(\eta_n) - \eta w_1(\eta)] = h_n$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} (-ah_{nx}^2 + h_n^2)dx &= -\int_{-\infty}^{\infty} h_n[\eta_n w_1(\eta_n) - \eta w_1(\eta)]dx \\ &\le \|h_n\|_{L^2}\|w_1(\eta_n)\|_{H^1}\|\eta_n - \eta\|_{L^2} + \|h_n\|_{L^2}\|\eta\|_{H^1}\|w_1(\eta_n) - w_1(\eta)\|_{L^2} \to 0\end{aligned}$$

as $n \to \infty$. Consequently, one concludes that

$$\lim_{n\to\infty} \|w_2(\eta_n) - w_2(\eta)\|_{H^1} = 0.$$

For (3), using the definition of w_3, one has that

$$\begin{aligned}\|w_3(\eta_n) - w_3(\eta)\|_{L^2} &= \Big\|G^{-1}(0)\Big([\eta_n w_2(\eta_n) - \eta w_2(\eta)] + [\eta_n w_3(\eta_n - \eta w_3(\eta)]\Big)\Big\|_{L^2} \\ &\le \|\eta_n w_2(\eta_n) - \eta w_2(\eta)\|_{L^2} + \|w_3(\eta_n)\|_{H^1}\|\eta_n - \eta\|_{L^2} + \|\eta\|_{H^1}\|w_3(\eta_n) - w_3(\eta)\|_{L^2}\end{aligned}$$

which implies that

$$\|w_3(\eta_n) - w_3(\eta)\|_{L^2} \le 2\Big(\|\eta_n w_2(\eta_n) - \eta w_2(\eta)\|_{L^2} + \|w_3(\eta_n)\|_{H^1}\|\eta_n - \eta\|_{L^2}\Big) \to 0$$

as $n \to \infty$, since

$$\|\eta_n w_2(\eta_n) - \eta w_2(\eta)\|_{L^2} \le \|w_2(\eta_n)\|_{H^1}\|\eta_n - \eta\|_{L^2} + \|\eta\|_{H^1}\|w_2(\eta_n) - w_2(\eta)\|_{L^2} \to 0$$

as $n \to \infty$. Thus (4.13) is proved.

Next, we show

$$\lim_{n\to\infty} \|G^{-1}(\eta)(\eta - b\eta_{xx}) - G^{-1}(\eta_n)(\eta_n - b\eta_{nxx})\|_{H^1} = 0.$$

Notice first that $\lim_{n\to\infty}\mathcal{H}(\eta_n) = \lim_{n\to\infty}\left(\mathcal{L}_0(\eta_n) + \dfrac{\mu^2}{\mathcal{L}_1(\eta_n)}\right) = \mathcal{H}(\eta)$. Let $w = G^{-1}(\eta)(\eta - b\eta_{xx})$. Since $\lim_{n\to\infty}\|\eta_n - \eta\|_{H^1} = 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\mathcal{L}_1(\eta_n) &= \lim_{n\to\infty}\int_{-\infty}^{\infty}\Big(-aw_x^2(\eta_n) + (1+\eta_n)w^2(\eta_n)\Big)dx\\ &= \int_{-\infty}^{\infty}\Big(-aw_x^2(\eta) + (1+\eta)w^2(\eta)\Big)dx = \mathcal{L}_1(\eta).\end{aligned}$$

From (3.14), (3.15) and (4.13), one can show that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}(1+\eta_n)w^2(\eta_n)dx = \int_{-\infty}^{\infty}(1+\eta)w^2(\eta)dx.$$

Thus, it follows that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}w_x^2(\eta_n)dx = \int_{-\infty}^{\infty}w_x^2(\eta)dx.$$

Therefore, $\lim_{n\to\infty}\|w(\eta_n)\|_{H^1} = \|w(\eta)\|_{H^1}$, which implies that $w(\eta_n)$ converges to $w(\eta)$ in H^1-norm. Consequently,

$$\lim_{n\to\infty}\|w(\eta_n) - w(\eta)\|_{H^1} = \lim_{n\to\infty}\|G^{-1}(\eta)(\eta - b\eta_{xx}) - G^{-1}(\eta_n)(\eta_n - b\eta_{nxx})\|_{H^1} = 0.$$

6.3 Proof of Theorem 4.10

One can use a concentration function $\rho_\epsilon(\eta) = \epsilon(\eta_{xx})^2 - c(\eta_x)^2 + \eta^2$ and apply a similar argument used in Lemmas 4.4–4.6 to show that there is a minimizing sequence $\{\eta_{n,\epsilon}(x)\}$ that is compact with $\mathcal{H}_\epsilon(\eta_{n,\epsilon}) \to c_\epsilon(\mu)$ for $\mu \le \mu_0(r)$ and $\int_{-\infty}^{+\infty}\rho_{n,\epsilon}(x)dx \to \beta_\epsilon$. By the same proof as in Theorem 4.8, one has $\eta_{n,\epsilon}(x) \to \eta_\epsilon(x)$ weakly in $H^2(\mathbb{R})$ and strongly in $H^1(\mathbb{R})$, which implies that as $n \to +\infty$,

$$\mathcal{L}_0(\eta_{n,\epsilon}) + \frac{\mu^2}{\mathcal{L}_1(\eta_{n,\epsilon})} \to \mathcal{L}_0(\eta_\epsilon) + \frac{\mu^2}{\mathcal{L}_1(\eta_\epsilon)}\,.$$

Now, for a fixed $\epsilon > 0$,

$$\mathcal{H}_\epsilon(\eta_\epsilon) \le \liminf_{n\to+\infty}\epsilon\|(\eta_{n,\epsilon})_{xx}\|_{L^2}^2 + \lim_{n\to+\infty}\left(\mathcal{L}_0(\eta_{n,\epsilon}) + \frac{\mu^2}{\mathcal{L}_1(\eta_{n,\epsilon})}\right) = \liminf_{n\to+\infty}\mathcal{H}_\epsilon(\eta_{n,\epsilon}) \le \mathcal{H}_\epsilon(\eta_\epsilon)$$

Thus, $(\eta_{n,\epsilon})_{xx} \to (\eta_\epsilon)_{xx}$ strongly in L^2, which yields $\eta_{n,\epsilon} \to \eta_\epsilon$ strongly in $H^2(\mathbb{R})$. Hence, $\mathcal{H}_\epsilon(\eta_\epsilon) = c_\epsilon(\mu)$ and $\eta_\epsilon \in C_\epsilon(\mu)$. Moreover, η_ϵ lies inside of $\mathcal{B}_{r,\epsilon}$ if μ is small enough. Therefore, η_ϵ satisfies that $\mathcal{L}_\epsilon(\eta_\epsilon) \le 2\mu$ and

$$\mathcal{L}'_\epsilon(\eta_\epsilon) = \frac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)}\mathcal{L}'_1(\eta_\epsilon)\,.$$

Multiply above equation by η_ϵ and integrate it to have

$$\int_{-\infty}^{\infty} \mathcal{L}_\epsilon'(\eta_\epsilon)\eta_\epsilon dx = \frac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)} \int_{-\infty}^{\infty} \mathcal{L}_1'(\eta_\epsilon)\eta_\epsilon dx$$

or

$$2\mathcal{L}_\epsilon(\eta_\epsilon) = \frac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)} \Big(2\mathcal{L}_1(\eta_\epsilon) + \|\eta_\epsilon\|_{H^1}^2 O(\sqrt{\mu}) \Big),$$

because $\|\eta_\epsilon\|_{H^1} \leq C\sqrt{\mu}$ with C independent of ϵ and μ. The above equation gives

$$2\mu \geq c_\epsilon(\mu) = \mathcal{L}_\epsilon(\eta_\epsilon) + \frac{\mu^2}{\mathcal{L}_1(\eta_\epsilon)} = \frac{\mu^2}{\mathcal{L}_1(\eta_\epsilon)} \left(1 + \frac{1}{\mathcal{L}_1(\eta_\epsilon)} (\mathcal{L}_1(\eta_\epsilon) + \|\eta_\epsilon\|_{H^1}^2 O(\sqrt{\mu})) \right),$$

which yields

$$\frac{\mu}{\mathcal{L}_1(\eta_\epsilon)} \leq 1 + C\sqrt{\mu} \tag{6.9}$$

for some constant C independent of ϵ and μ. From this, the following is obtained.

Claim: If $\eta_\epsilon \in C_\epsilon(\mu)$, then there exists a positive constant C independent of ϵ such that

$$\|\eta_\epsilon\|_{H^2} \leq C\sqrt{\mu}.$$

Proof. Because $\eta_\epsilon \in C_\epsilon(\mu)$, one has $\mathcal{L}_\epsilon'(\eta_\epsilon) = \dfrac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)} \mathcal{L}_1'(\eta_\epsilon)$ whence $\mathcal{L}_\epsilon'(\eta_\epsilon) = \epsilon\eta_{\epsilon,xxxx} + c\eta_{\epsilon,xx} + \eta_\epsilon$. For $f, g \in H^2(\mathbb{R})$, denote the inner-product of f and g by $\langle f, g \rangle$. Then

$$\begin{aligned} \left\langle \frac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)} \mathcal{L}_1'(\eta_\epsilon), c\eta_{\epsilon,xx} + \eta_\epsilon \right\rangle &= \left\langle \mathcal{L}_\epsilon'(\eta_\epsilon), c\eta_{\epsilon,xx} + \eta_\epsilon \right\rangle \\ &= \int_{-\infty}^{\infty} \left(-c\epsilon(\eta_{\epsilon,xxx})^2 + \epsilon(\eta_{\epsilon,xx})^2 + (c\eta_{xx} + \eta)^2 \right) dx. \end{aligned} \tag{6.10}$$

As $\eta_\epsilon \in C_\epsilon(\mu)$, it follows that $\|\eta_\epsilon\|_{H^1}^2 \sim O(\mu)$, which implies by Lemmas 3.1–3.3 that

$$\begin{aligned} \left\langle \mathcal{L}_1'(\eta_\epsilon), c\eta_{\epsilon,xx} + \eta_\epsilon \right\rangle &= \left\langle \mathcal{L}_{1,0}'(\eta_\epsilon), c\eta_{\epsilon,xx} + \eta_\epsilon \right\rangle + \|\eta_\epsilon\|_{H^2}^2 O(\sqrt{\mu}) \\ &= \int_{-\infty}^{\infty} \frac{(bk^2+1)^2(-ck^2+1)}{(1-ak^2)} |\widehat{\eta}_\epsilon|^2 dk + \|\eta_\epsilon\|_{H^2}^2 O(\sqrt{\mu}) \end{aligned}$$

Therefore, from (6.10) we obtain that

$$\int_{-\infty}^{\infty} \left(-c\epsilon k^6 + \epsilon k^4 + (-ck^2+1)^2 \right) |\widehat{\eta}_\epsilon|^2 dk$$

$$= \frac{\mu^2}{\mathcal{L}_1^2(\eta_\epsilon)} \left(\int_{-\infty}^{\infty} \frac{(bk^2+1)^2(-ck^2+1)}{(1-ak^2)} |\widehat{\eta}_\epsilon|^2 dk + \|\eta_\epsilon\|_{H^2}^2 O(\sqrt{\mu}) \right)$$

From above equality and (6.9), it is obtained that

$$\int_{-\infty}^{\infty} \left(\frac{(-ck^2+1)}{(1-ak^2)} \left((-ck^2+1)(1-ak^2) - (bk^2+1)^2 \right) \right) |\widehat{\eta}_\epsilon|^2 dk$$
$$\leq C\|\eta_\epsilon\|_{H^1}^2 + \|\eta_{\epsilon,xx}\|_{L^2}^2 O(\sqrt{\mu})$$

which gives

$$\left(ac - b^2 - O(\sqrt{\mu})\right) \|\eta_{\epsilon,xx}\|_{L^2}^2 \leq C\|\eta_\epsilon\|_{H^1}^2 \leq C\mu\,.$$

Since $ac - b^2 > 0$, we can choose μ small enough so that $ac - b^2 - O(\sqrt{\mu}) > 0$ and then $\|\eta_{\epsilon,xx}\|_{L^2}^2 \leq C\mu$. By combining the estimates of η_ϵ in $H^1(\mathbb{R})$, we have $\|\eta_\epsilon\|_{H^2} \leq C\sqrt{\mu}$. □

Now, let η_ϵ be the minimizer with $\|\eta_\epsilon\|_{H^2} \leq C\sqrt{\mu}$ of $\mathcal{H}_\epsilon(\eta)$ in $\mathcal{B}_{r,\epsilon}$ found above. Then, for μ small enough, $\eta_\epsilon \in \mathcal{B}_r$. Also, $\mathcal{H}_\epsilon(\eta_\epsilon) = c_\epsilon(\mu)$ or $\mathcal{H}_0(\eta_\epsilon) = c_\epsilon(\mu) - \epsilon\|\eta_{\epsilon,xx}\|_{L^2}^2 \geq c_0(\mu)$. On the other hand, since $\eta_0 \in \mathcal{B}_r \subset \mathcal{B}_{r,\epsilon}$ for small ϵ, $\mathcal{H}_\epsilon(\eta_0) \geq c_\epsilon(\mu)$, which implies that $c_0(\mu) = \mathcal{H}_0(\eta_0) \geq c_\epsilon(\mu) - \epsilon\|\eta_{0,xx}\|_{L^2}^2$. Therefore, $\lim_{\epsilon\to 0} c_\epsilon(\mu) = c_0(\mu)$. Hence, as $\epsilon \to 0$, η_ϵ is a minimizing sequence of $\mathcal{H}_0(\eta)$ and $\|\eta_\epsilon\|_{H^2} \leq C\sqrt{\mu} \leq r/2$ for μ small. By a similar argument as in the proof of Theorem 4.9, it is obtained that for some subsequence of η_ϵ (still denoted by η_ϵ) $\eta_\epsilon \to \tilde{\eta}_0$, as some sequence of ϵ goes to zero, with $\|\tilde{\eta}_0\|_{H^2} \leq r/2$ and $\mathcal{H}_0(\tilde{\eta}_0) = \mathcal{H}(\tilde{\eta}_0) = c_0(\mu)$. Thus, after $r > 0$ is small and chosen, then $c_0(\mu)$ is independent of r if $\mu \leq \mu_0(r)$ and at least one minimizer $\tilde{\eta}_0$ of $\mathcal{H}(\eta)$ is a true minimizer and does not lie on the boundary of $\mathcal{B}_r$ with $\|\tilde{\eta}_0\|_{H^2} \leq C\sqrt{\mu}$, which implies that $c_0(\mu)$ is independent of r if $\mu \leq \mu_0(r)$.

Now, for an arbitrary minimizer η of $\mathcal{H}_0(\eta)$ in $\mathcal{B}_r$, since $c_0(\mu)$ is dependent of r, we can assume that η is an interior point of $\mathcal{B}_r$. Therefore, by a similar proof as that of the above claim with $\epsilon = 0$, we can have $\|\eta\|_{H^2} \leq C\sqrt{\mu}$. The proof is completed.

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