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EXISTENCE OF TRAVELING-WAVE SOLUTIONS TO BOUSSINESQ SYSTEMS

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Abstract. In this manuscript, the existence of traveling-wave solutions to Boussinesq systems

$$\begin{cases} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{cases}$$

is established. We prove that all the systems with $a < 0$, $c < 0$ and $b = d$ exhibit traveling-wave solutions with small propagation speeds. The result complements our earlier work [6] on a restricted family of the systems where both existence and stability of traveling-wave solutions were established in the presence of large surface tension, namely when $a + b + c + d < 0$.

1. Introduction

The four-parameter family of Boussinesq systems

$$\begin{cases} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0, \end{cases} \tag{1.1}$$

is introduced in [2] (generalized to include the surface tension in [7]) to describe the motion of small-amplitude long waves on the surface of an ideal fluid under the force of gravity. All the variables are scaled with length scale h_0 and time scale $\sqrt{h_0/g}$, where g is the gravitational constant and h_0 (scaled to 1) the undisturbed average water depth. The quantity $\eta(x,t)$ is the deviation of free surface with respect to the undisturbed state, so

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$\eta(x,t)+1$ corresponds to the total depth of the liquid at (x,t), while $u(x,t)$ is the dimensionless horizontal velocity field at height θ, where $0 \leq \theta \leq 1$. From the derivation of (1.1), the parameters a, b, c, d are not independently specified but satisfy the consistency condition

$$a + b + c + d = \tfrac{1}{3} - \tau, \tag{1.2}$$

where τ is the non-dimensional surface tension coefficient. In this paper, we assume that τ is any fixed non-negative number (including zero) and

$$a < 0,\ c < 0 \ \text{ and } \ b = d. \tag{1.3}$$

If a_0 connotes a typical wave amplitude and λ a typical wavelength, the condition of "small amplitude and long wavelength" just mentioned amounts to

$$\alpha = \frac{a_0}{h_0} << 1, \quad \beta = \frac{h_0^2}{\lambda^2} << 1, \quad \frac{\alpha}{\beta} = \frac{a_0 \lambda^2}{h_0^3} \approx 1. \tag{1.4}$$

Systems (1.1) are first-order approximations in α and β to Euler's equations, justified rigorously by Bona, Colin and Lannes in [4]. We refer the readers to the papers [2] and [3] for further discussion about the derivation and well posedness of these systems.

These systems are free of the presumption of uni-directionality that is the hallmark of KdV-type equations. One therefore expects that these Boussinesq systems will have more intrinsic interest than the one-way models on account of their considerably wider range of potential applicability. Because dissipation is ignored in the derivation of (1.1) and the overlying Euler equations are Hamiltonian, it is expected that some of the systems in (1.1) will likewise possess a Hamiltonian form. One finds indeed that, whenever $b = d$, the functional

$$\mathcal{H}(\eta, u) = \frac{1}{2} \int_{-\infty}^{\infty} \Big(- c\eta_x^2 - a u_x^2 + \eta^2 + (1+\eta) u^2 \Big) dx \tag{1.5}$$

serves as a Hamiltonian and the systems have the conserved quantities

$$\int_{-\infty}^{\infty} u(x,t) dx, \quad \int_{-\infty}^{\infty} \eta(x,t) dx, \quad \mathcal{I}(\eta, u) = \int_{-\infty}^{\infty} \Big(\eta u + b \eta_x u_x \Big) dx \tag{1.6}$$

along with $\mathcal{H}(\eta, u)$ (see Remark 4.1 in [3]).

By a traveling-wave solution we shall mean a solution (η, u) of (1.1) of the form

$$\eta(x,t) = \eta(x - \omega t) \quad \text{ and } \quad u(x,t) = u(x - \omega t) \tag{1.7}$$

where ω denotes the phase speed of the wave. Notice that if $\big(\eta(x-\omega t), u(x-\omega t)\big)$ is a solution, then $\big(\eta(x+\omega t), -u(x+\omega t)\big)$ is also a solution; thus the existence of a traveling-wave solution for $\omega > 0$ will imply the existence of a traveling-wave solution for $\omega < 0$. When the traveling speed ω is zero, the solution is time independent which usually is referred to as a standing wave solution. In what follows, we require that η, $u \in H^1(\mathbb{R})$ and restrict ourselves to the cases with (1.3). Let $\xi = x - \omega t$ and substitute the form of the solution (1.7) into (1.1), integrate once and evaluate the constants of the integrations using the fact that $\eta, u \in H^1(\mathbb{R})$. Then, one sees that (η, u) must satisfy

$$\begin{aligned} c\eta_{\xi\xi} + \eta - \omega u + b\omega u_{\xi\xi} + \tfrac{1}{2}u^2 &= 0, \\ a u_{\xi\xi} + u - \omega\eta + b\omega\eta_{\xi\xi} + \eta u &= 0. \end{aligned} \tag{1.8}$$

Local existence and continuous dependence on initial data have been studied in [3] for numerous cases of (1.1). In order to extend the local result to a global one, some kind of control on the norms is needed in the energy estimates. Whenever $b = d$, the systems (1.1) admit the conservation laws (1.5) and (1.6) which allow one to obtain the control needed. Moreover, in this case, the systems (1.1) with (1.3) can be written as

$$\partial_t \begin{bmatrix} \eta \\ u \end{bmatrix} = J \text{ grad } \mathcal{H}(\eta, u), \tag{1.9}$$

where the operator J is defined as

$$J = \begin{bmatrix} 0 & (I - b\partial_x^2)^{-1}\partial_x \\ (I - b\partial_x^2)^{-1}\partial_x & 0 \end{bmatrix},$$

and grad $\mathcal{H}$ stands for the gradient or Euler derivative, computed with respect to the $L^2 \times L^2$-inner product, of the functional $\mathcal{H}$. Because the operator J is skew-adjoint, $\mathcal{H}$ can be seen as a Hamiltonian for the systems.

Because none of the conserved quantities is composed only of positive terms, they do not on their own provide the *a priori* information one needs to conclude the global existence of solutions to the initial-value problem. However, a time-dependent relationship can be coupled with the invariance of the Hamiltonian to give suitable information leading to a global existence theory. The global existence has been established in [3].

In this manuscript, the existence of traveling waves of the systems (1.1) with (1.3) is studied, namely the existence of traveling-wave solutions with small propagation speeds is guaranteed. The special properties of this class of systems include established global well posedness and previously stated conserved quantities which enable the use of the technique of constrained global minimization. However, unlike the case with large surface tension investigated in [6], the stability of traveling-wave solutions cannot be obtained for this general case by this method (see item 4 of Remark 2.2).

The precise statement of the result is as follows.

Theorem 1.1. *Let* $a, c < 0$, $b = d$ *and* $|\mu| < \min\{1, \sqrt{ac}/|b|\} = \mu_0$ *(here, if* $b = 0$*, then* $\mu_0 = 1$*). Then* (1.1) *exhibits traveling-wave solutions with propagation speed* $\omega = \mu$.

Here, the propagation speed ω can be zero, which implies that the system (1.1) has a time-independent solution (or standing wave solution). Moreover, we note that the KdV equation

$$\eta_t + \lambda_1 \eta_x + \eta\eta_x + \eta_{xxx} = 0$$

also has standing wave solutions if λ_1 is negative. If λ_1 is arbitrary, then for the existence of traveling-wave solutions $\eta(x - \omega t)$, ω must satisfy $\omega > \lambda_1$.

The manuscript is organized as follows. In Section 2, some necessary estimates for functionals are given and the existence proof of minimizers to a variational problem is provided. The existence of traveling-wave solutions with small propagation speed is established in Section 3.

The standard notations are used. For $1 \le p < \infty$, L^p is the usual Banach space of measurable functions on $\mathbb{R}$ with norm given by $\|f\|_{L^p} = (\int_{-\infty}^{\infty} |f|^p \, dx)^{1/p}$. The space L^∞ consists of the measurable, essentially bounded functions f on $\mathbb{R}$ with norm $|f|_\infty = \text{ess sup}_{x\in\mathbb{R}}|f(x)|$. For $s \in \mathbb{R}$, the L^2-based Sobolev space $H^s = H^s(\mathbb{R})$ (see [1]) is the set of all tempered distributions f on $\mathbb{R}$ whose Fourier transforms $\widehat{f}$ are measurable functions on $\mathbb{R}$ satisfying

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + |k|^2 + \cdots + |k|^{2s}) |\widehat{f}(k)|^2 \, dk < \infty. \tag{1.10}$$

We denote the spaces $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by X and Y respectively.

2. VARIATIONAL PROBLEM

To prove the existence of traveling-wave solutions, we use the method of concentration compactness introduced by Lions [10, 9]. The two quantities associated with the systems, $H_\mu(\eta, u)$ and $P(\eta, u)$ are used, where for a fixed μ (zero included) and $\eta, u \in X$,

$$H_\mu(\eta, u) = \frac{1}{2}\int_{-\infty}^{\infty} (-c\eta_x^2 - au_x^2 + \eta^2 + u^2)dx - \mu \int_{-\infty}^{\infty} (\eta u + b\eta_x u_x)dx \tag{2.1}$$

and

$$P(\eta, u) = \frac{1}{2}\int_{-\infty}^{\infty} \eta u^2 dx. \tag{2.2}$$

For $p > 0$, define the real number $m_p(\mu)$ by

$$m_p(\mu) = \inf\{H_\mu(\eta, u) : (\eta, u) \in X,\ P(\eta, u) = p\}. \tag{2.3}$$

The set of minimizers for $m_p(\mu)$ is

$$G(p) = \{(\eta, u) \in X : H_\mu = m_p(\mu), P(\eta, u) = p\}, \tag{2.4}$$

and a minimizing sequence for $m_p(\mu)$ is any sequence $\{(\eta_n, u_n)\}$ of functions in X satisfying

$$P(\eta_n, u_n) = p \quad \forall n, \qquad \text{and} \qquad \lim_{n\to\infty} H_\mu(\eta_n, u_n) = m_p(\mu). \tag{2.5}$$

Lemma 2.1. *For $a, c < 0$, $b = d$ and μ satisfying $|\mu| < \mu_0 = \min\{1, \sqrt{ac}/|b|\}$ (here, if $b = 0$, then $\mu_0 = 1$), one has*

$$H_\mu(\eta, u) \geq C_0 \int_{-\infty}^{\infty} \left(\eta_x^2 + u_x^2 + \eta^2 + u^2\right) dx,$$

where

$$C_0 = \frac{1}{2}\min\left\{\left(1 - \frac{|\mu b|}{\sqrt{ac}}\right)|a|, \left(1 - \frac{|\mu b|}{\sqrt{ac}}\right)|c|, (1 - |\mu|)\right\}.$$

Thus, if $|\mu| < \mu_0$, then $C_0 > 0$.

Proof.

$$\begin{aligned}
H_\mu(\eta, u) &= \frac{1}{2}\int_{-\infty}^{\infty}\Big((\sqrt{|c|}\eta_x)^2 + (\sqrt{|a|}u_x)^2 - 2\mu b\eta_x u_x + \eta^2 + u^2 - 2\mu\eta u\Big)dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\bigg(\Big(1 - \frac{|\mu b|}{\sqrt{ac}}\Big)\Big((\sqrt{|c|}\eta_x)^2 + (\sqrt{|a|}u_x)^2\Big) + \frac{|\mu b|}{\sqrt{ac}}\Big(\sqrt{|c|}\eta_x \pm \sqrt{|a|}u_x\Big)^2 \\
&\quad + (1 - |\mu|)(\eta^2 + u^2) + |\mu|(\eta \pm u)^2\bigg)dx \\
&\geq \frac{1}{2}\int_{-\infty}^{\infty}\bigg(\Big(1 - \frac{|\mu b|}{\sqrt{ac}}\Big)\Big((\sqrt{|c|}\eta_x)^2 + (\sqrt{|a|}u_x)^2\Big) + (1 - |\mu|)(\eta^2 + u^2)\bigg)dx \\
&\geq C_0 \int_{-\infty}^{\infty}\left(\eta_x^2 + u_x^2 + \eta^2 + u^2\right) dx,
\end{aligned}$$

where $\pm$ means that for positive $b\mu$ (or μ), $-$ is used, while for negative $b\mu$ (or μ), $+$ is used. □

Remark 2.2. 1) It will be shown momentarily that indeed $m_p(\mu) > 0$.

2) Because of the homogeneity of the functionals

$$\inf\{H_\mu(\eta, u) : P(\eta, u) = 1\} = \inf\left\{\frac{1}{p^{2/3}}H_\mu(\eta, u) : P(\eta, u) = p\right\},$$

it follows that for any $p > 0$, $m_p(\mu) = p^{2/3}m_1(\mu)$. Thus, we consider instead the problem of investigating the minimizers for $m_1(\mu)$, where

$$m_1(\mu) = \inf\{H_\mu(\eta, u) : (\eta, u) \in X, P(\eta, u) = 1\}.$$

3) Notice that, if $|\mu| < \mu_0$, then $H_\mu(\eta, u)$ is equivalent to the X-norm of (η, u). Therefore, any minimizing sequence $\{(\eta_n, u_n)\}$ is uniformly bounded in X.

4) In our previous paper [6], a different variational set up was used where we minimized $\mathcal{H}(\eta, u)$ while holding $\mathcal{I}(\eta, u)$ constant. As both of the functionals $\mathcal{H}$ and $\mathcal{I}$ are time independent, the stability of traveling wave solutions is a direct consequence. However, that result came at the cost of having to require $\tau > 1/3$, *i.e.,* the presence of large surface tension. In the present manuscript, the time-dependent functionals $H_\mu(\eta, u)$ and $P(\eta, u)$ are used, where $H_\mu + P = \mathcal{H} + \mathcal{I}$ is conserved.

Let $\{(\eta_n, u_n)\}$ be a minimizing sequence and consider the concentration function $\rho_n = (\eta_n')^2 + \eta_n^2 + (u_n')^2 + u_n^2$. As $\|(\eta_n, u_n)\|_X \le C$ for all n, we can extract a convergent subsequence which we again denote as $\{(\eta_n, u_n)\}$, so that

$$\lambda = \lim_{n\to\infty}\int_{-\infty}^{\infty}\rho_n(x)dx$$

exists. Define a sequence of non-decreasing functions $M_n : [0, \infty) \to [0, \lambda]$ as follows:

$$M_n(r) = \sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}\rho_n(x)dx.$$

As $M_n(r)$ is a uniformly bounded sequence of non-decreasing function in r, one can show that it has a subsequence, which we still denote as M_n, that converges pointwise to a non-decreasing limit function $M(r) : [0, \infty) \to [0, \lambda]$. Let

$$\lambda_0 = \lim_{r\to\infty} M(r) :\equiv \lim_{r\to\infty}\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}\rho_n(x)dx.$$

Then $0 \le \lambda_0 \le \lambda$. From Lions' concentration compactness lemma, there are 3 possibilities to consider:

(a) Case 1: (*Vanishing*) $\lambda_0 = 0$. Since $M(r)$ is non-negative and non-decreasing, this is equivalent to

$$M(r) = \lim_{n\to\infty} M_n(r) = \lim_{n\to\infty} \sup_{y\in R} \int_{y-r}^{y+r} \rho_n(x)dx = 0$$

for all $r < \infty$, or

(b) Case 2: (*Dichotomy*) $\lambda_0 \in (0, \lambda)$, or

(c) Case 3: (*Compactness*) $\lambda_0 = \lambda$, which means there exists $\{y_n\}_{n=1} \in \mathbb{R}$ such that $\rho_n(\cdot + y_n)$ is tight; that is, for all $\epsilon > 0$, there exists $r < \infty$ such that

$$\int_{y-r}^{y+r} \rho_n(x)dx \geq \lambda - \epsilon.$$

It is our purpose to show that the only possibility is case 3, so that the minimizing sequence $\{(\eta_n, u_n)\}$ has a subsequence which, up to translations in the underlying spatial domain, converges strongly in X to an element of $G(1)$.

Lemma 2.3 ((Non-vanishing of the sequence $\{\rho_n\}$))**.** *There exists a $\gamma > 0$ such that*

$$\lim_{n\to\infty} M_n\left(\tfrac{1}{2}\right) = \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \geq \gamma. \tag{2.6}$$

Therefore, $\lambda_0 \geq \gamma > 0$.

Proof. Suppose that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx = 0.$$

Let $I_j = [j - 1/2, j + 1/2]$. On I_j, one can see that

$$(\sup_{x\in I_j} |\eta_n(x)|)^2 \leq C \int_{I_j} [\left(\eta_n'(s)\right)^2 + \left(\eta_n(s)\right)^2]ds \leq C \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx.$$

From the expression for $P(\eta, u)$, it is deduced that

$$\begin{aligned} 2|P(\eta_n, u_n)| &= \Big|\int_{-\infty}^{\infty} \eta_n(x)u_n^2(x)dx\Big| = \Big|\sum_{j=-\infty}^{\infty} \int_{I_j} \eta_n(x)u_n^2(x)dx\Big| \\ &\leq \sum_{j=-\infty}^{\infty} \sup_{x\in I_j} |\eta_n| \int_{I_j} u_n^2(x)dx \leq C \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \int_{-\infty}^{\infty} u_n^2(x)dx \end{aligned}$$

$$\leq C\|(\eta_n, u_n)\|_X^2 \sup_{y\in\mathbb{R}} \int_{y-1/2}^{y+1/2} \rho_n(x)dx \longrightarrow 0$$

as $n \to \infty$, a contradiction. Hence, it follows that

$$\lambda_0 = \lim_{r\to\infty} M(r) \geq M\left(\tfrac{1}{2}\right) = \lim_{n\to\infty} M_n\left(\tfrac{1}{2}\right) \geq \gamma > 0. \qquad \square$$

Remark 2.4. Notice that the above argument shows more than just the non-vanishing of the sequence $\{\rho_n\}$. It actually says that neither η_n nor u_n can vanish.

Corollary 2.5. *For any $p > 0$, $m_p(\mu) = p^{\frac{2}{3}} m_1(\mu) > 0$.*

Given any $\epsilon > 0$, for all sufficiently large values of r, one has

$$\lambda_0 - (\epsilon/2) < M(r) \leq M(2r) \leq \lambda_0. \tag{2.7}$$

Suppose for the moment that a large value of r has been chosen so that (2.7) holds. Then one can choose N large enough that

$$\lambda_0 - (2\epsilon/3) \leq M_n(r) \leq M_n(2r) \leq \lambda_0 + (2\epsilon/3)$$

for all $n \geq N$. Hence, for each $n \geq N$, one can find y_n such that

$$\int_{y_n-r}^{y_n+r} \rho_n(x)dx > \lambda_0 - \epsilon \qquad \text{and} \qquad \int_{y_n-2r}^{y_n+2r} \rho_n(x)dx < \lambda_0 + \epsilon.$$

Now, choose $\phi \in C_0^\infty[-2, 2]$ such that $\phi = 1$ on $[-1, 1]$, and let $\psi \in C^\infty(\mathbb{R})$ be such that $\phi^2 + \psi^2 = 1$ on $\mathbb{R}$. For each $r \in \mathbb{R}$, let $\phi_r(x) = \phi(\frac{x}{r})$ and $\psi_r(x) = \psi(\frac{x}{r})$ and define

$$\begin{aligned} g_n(x) &= \phi_r(x - y_n)\eta_n(x), \qquad & \tilde{g}_n(x) &= \psi_r(x - y_n)\eta_n(x), \\ h_n(x) &= \phi_r(x - y_n)u_n(x), \qquad & \tilde{h}_n(x) &= \psi_r(x - y_n)u_n(x). \end{aligned} \tag{2.8}$$

Set

$$\rho_{1,n} = (g_n')^2 + g_n^2 + (h_n')^2 + h_n^2 \qquad \text{and} \qquad \rho_{2,n} = (\tilde{g}_n')^2 + \tilde{g}_n^2 + (\tilde{h}_n')^2 + \tilde{h}_n^2.$$

Notice that $g_n, \tilde{g}_n, h_n$ and $\tilde{h}_n$ depend on r (which has been chosen for the moment large enough so that (2.7) holds) and hence so do $\rho_{1,n}$ and $\rho_{2,n}$. One can therefore establish the following lemma.

Lemma 2.6. *For every $\epsilon > 0$, there exist R and N large enough such that, for $n \geq N$ and $r \geq R$,*

a) $H_\mu(\eta_n, u_n) = H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon)$,

b) $P(\eta_n, u_n) = P(g_n, h_n) + P(\tilde{g}_n, \tilde{h}_n) + O(\epsilon)$.

Proof. From the definitions of $g_n, \tilde{g}_n, h_n$ and $\tilde{h}_n$, it follows that

$$
\begin{aligned}
&H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n)\\
&= H_\mu(\eta_n, u_n) + \frac{1}{2}\int_{-\infty}^{\infty}\Big[- c(\phi_r')^2\eta_n^2 - c(\psi_r')^2\eta_n^2 - 2c\phi_r\phi_r'\eta_n\eta_n'\\
&\quad - 2c\psi_r\psi_r'\eta_n\eta_n' - a(\phi_r')^2u_n^2\Big]dx + \frac{1}{2}\int_{-\infty}^{\infty}\Big[- a(\psi_r')^2u_n^2 - 2a\phi_r\phi_r'u_nu_n'\\
&\quad - 2a\psi_r\psi_r'u_nu_n' - 2\mu b(\phi_r')^2\eta_nu_n - 2\mu b(\psi_r')^2\eta_nu_n\Big]dx\\
&+ \frac{1}{2}\int_{-\infty}^{\infty}\Big[-2\mu b\phi_r\phi_r'\eta_nu_n' - 2\mu b\psi_r\psi_r'\eta_nu_n - 2\mu b\phi_r\phi_r'\eta_n'u_n - 2\mu b\psi_r\psi_r'\eta_n'u_n\Big]dx,
\end{aligned}
$$

where for ease of notation, we have written simply ϕ_r and ψ_r for the functions $\phi_r(x-y_n)$ and $\psi_r(x-y_n)$. Using the facts that $\|(\eta_n, u_n)\|_X \le C$ for all n, and $\phi_r^2 + \psi_r^2 \equiv 1$, $|\phi_r'|_{L^\infty} \sim O(1/r)$ and $|\psi_r'|_{L^\infty} \sim O(1/r)$, one can see that

$$H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) = H_\mu(\eta_n, u_n) + O\left(\tfrac{1}{r}\right)$$

where $O(\frac{1}{r})$ denotes terms bounded in absolute value by A_1/r with A_1 independent of r and n. Similarly, for $P(\eta_n, u_n)$ one can see that

$$
\begin{aligned}
P(g_n, h_n) + P(\tilde{g}_n, \tilde{h}_n) &= \frac{1}{2}\int_{-\infty}^{\infty}\Big(\eta_nu_n^2 + (\phi_r^3 + \psi_r^3 - 1)\eta_nu_n^2\Big)dx\\
&= P(\eta_n, u_n) + A_2\epsilon
\end{aligned}
$$

because $\phi_r = 1, \psi_r = 0$ for $|x - y_n| \le r$ and $\phi_r = 0, \psi_r = 1$ for $|x - y_n| \ge 2r$ which gives

$$\Big|\int_{-\infty}^{\infty}(\phi_r^3 + \psi_r^3 - 1)\eta_nu_n^2dx\Big| \le \|\eta_n\|_{L^\infty}\Big(2\int_{r\le|x-y_n|\le 2r}\rho_n\, dx\Big) \le A_2\epsilon$$

where again A_2 is independent of r and n. Now, one can choose r large enough so that $1/r \le \epsilon$. Consequently, for all $n \ge N$, one has

$$
\begin{aligned}
H_\mu(\eta_n, u_n) &= H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon),\\
P(\eta_n, u_n) &= P(g_n, h_n) + P(\tilde{g}_n, \tilde{h}_n) + O(\epsilon),
\end{aligned}
$$

which proves the lemma. □

Notice that, for any $(\eta, u) \ne (0,0)$, $H_\mu > 0$ for any μ satisfying $|\mu| < \mu_0$. With the above Lemma 2.6 in hand, one can now proceed to rule out the dichotomy case as follows.

Proposition 2.7. $\lambda_0 \notin (0, \lambda)$ *and dichotomy cannot occur.*

Proof. The following argument is adapted from [8]. Suppose dichotomy happens. Let $\{(\eta_n, u_n)\}$ be a minimizing sequence and consider two sequences $\{(g_n, h_n)\}$ and $\{(\tilde{g}_n, \tilde{h}_n)\}$ as defined in (2.8). Then, for large r, Lemma 2.6 ensures that

$$H_\mu(\eta_n, u_n) = H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon),$$

$$P(\eta_n, u_n) = P(g_n, h_n) + P(\tilde{g}_n, \tilde{h}_n) + O(\epsilon).$$

As $\{(\eta_n, u_n)\}$ is bounded uniformly in X, it follows that $\|(g_n, h_n)\|_X$ and $\|(\tilde{g}_n, \tilde{h}_n)\|_X$ are also bounded independently of n and ϵ. Consequently, $P(g_n, h_n)$ and $P(\tilde{g}_n, \tilde{h}_n)$ are bounded and we can pass to subsequences to define

$$\sigma(\epsilon) = \lim_{n\to\infty} P(g_n, h_n) \quad \text{and} \quad \tilde{\sigma}(\epsilon) = \lim_{n\to\infty} P(\tilde{g}_n, \tilde{h}_n).$$

As $\sigma(\epsilon)$ and $\tilde{\sigma}(\epsilon)$ are bounded independently of ϵ, we can pick a sequence $\{\epsilon_j\} \to 0$ (here, we also need to choose large $r_j \to +\infty$) such that both the limits

$$\lim_{j\to\infty} \sigma(\epsilon_j) = \sigma \quad \text{and} \quad \lim_{j\to\infty} \tilde{\sigma}(\epsilon_j) = \tilde{\sigma}$$

exist. Certainly, $\sigma + \tilde{\sigma} = 1$ and there are only three cases to consider now.

Case 1: When $\sigma \in (0, 1)$, then

$$\begin{aligned} H_\mu(\eta, u) &= H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon_j) \\ &\geq m_{P(g_n,h_n)}(\mu) + m_{P(\tilde{g}_n,\tilde{h}_n)}(\mu) + O(\epsilon_j) \\ &= \left[P^{\frac{2}{3}}(g_n, h_n) + P^{\frac{2}{3}}(\tilde{g}_n, \tilde{h}_n)\right] m_1(\mu) + O(\epsilon_j). \end{aligned}$$

We first let $n \to \infty$ to obtain $m_1(\mu) \geq [\sigma^{\frac{2}{3}}(\epsilon_j) + \tilde{\sigma}^{\frac{2}{3}}(\epsilon_j)] m_1(\mu) + O(\epsilon_j)$. Letting $j \to \infty$ next we arrive at $m_1(\mu) \geq (\sigma^{\frac{2}{3}} + \tilde{\sigma}^{\frac{2}{3}}) m_1(\mu) > m_1(\mu)$, a contradiction.

Case 2: When $\sigma = 0$ (or when $\sigma = 1$), we have

$$\begin{aligned} H_\mu(g_n, h_n) &\geq C \int_{-\infty}^{\infty} \left((g_n')^2 + g_n^2 + (h_n')^2 + h_n^2\right) dx \\ &= C \int_{|x-y_n|\leq 2r} \left((u_n')^2 + u_n^2 + (\eta_n')^2 + \eta_n^2\right) dx + O(\epsilon_j) \geq C\lambda_0 + O(\epsilon_j). \end{aligned}$$

Thus,

$$\begin{aligned} H_\mu(\eta_n, u_n) &= H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon_j) \\ &\geq C\lambda_0 + O(\epsilon_j) + P^{\frac{2}{3}}(\tilde{g}_n, \tilde{h}_n) m_1(\mu). \end{aligned}$$

Again, letting n and $j \to \infty$ respectively, we obtain

$$m_1(\mu) \geq C\lambda_0 + m_1(\mu) > m_1(\mu),$$

a contradiction.

Case 3: When $\sigma > 1$ (or when $\sigma < 0$), we have

$$\begin{aligned} H_\mu(\eta_n, u_n) &= H_\mu(g_n, h_n) + H_\mu(\tilde{g}_n, \tilde{h}_n) + O(\epsilon_j) \geq H_\mu(g_n, h_n) + O(\epsilon_j) \\ &\geq P^{\frac{2}{3}}(g_n, h_n) m_1(\mu) + O(\epsilon_j). \end{aligned}$$

As before, letting n and $j \to \infty$ respectively, we arrive at the contradiction

$$m_1(\mu) \geq \sigma^{\frac{2}{3}} m_1(\mu) > m_1(\mu).$$

Thus, each case gives a contradiction, which implies that $\lambda_0 \notin (0, \lambda)$. $\square$

As we have ruled out both vanishing and dichotomy, Lions' concentration compactness lemma guarantees that the sequence $\{\rho_n\}$ is tight; i.e., there exists a sequence of real numbers $\{y_n\}$ such that, for any $\epsilon > 0$, there exists $r = r(\epsilon)$ so that

$$\int_{y_n - r}^{y_n + r} \left((\eta_n')^2 + \eta_n^2 + (u_n')^2 + u_n^2\right) dx > \lambda - \epsilon$$

for all sufficiently large n. Consequently, one arrives at the following.

Theorem 2.8. *Let $a, c < 0$, $b = d$ and $|\mu| < \mu_0$. Then the minimizing set $G(1)$ is non-empty. Moreover, any minimizing sequence $\{(\eta_n, u_n)\}$ is compact in X up to translation; that is, there exist a sequence of points $\{y_{n_k}\} \in \mathbb{R}$ and $(\eta, u) \in G(1)$ such that $\big(\eta_{n_k}(\cdot + y_{n_k}), u_{n_k}(\cdot + y_{n_k})\big)$ has a subsequence converging to (η, u) strongly in X.*

Proof. Since the minimizing sequence $\{(\eta_n, u_n)\}$ is bounded uniformly in X, there exists a subsequence which, for ease of reading we again denote as $\{(\eta_n, u_n)\}$, that converges strongly in $(L^2 \times L^2)$-locally to a pair of limiting functions (η, u). We next show that $(\eta_n, u_n) \to (\eta, u)$ strongly in $Y = L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Indeed, for any given $\epsilon > 0$, we first choose r_0 so large that

$$\int_{|x| \geq r_0} \big(\eta^2(x) + u^2(x)\big) dx < \epsilon.$$

By tightness of the minimizing sequence (η_n, u_n), there exist an $N = N(\epsilon)$ and $r = r(\epsilon) > r_0$ such that

$$\int_{|x| \geq r} \big(\eta_n^2(x + y_n) + u_n^2(x + y_n)\big) dx < \epsilon,$$

for all $n \geq N$. From the strong convergence in $(L^2 \times L^2)$-locally of (η_n, u_n), there exists an $\tilde{N} = \tilde{N}(\epsilon) \geq N$ such that

$$\|(\eta_n, u_n) - (\eta, u)\|^2_{L^2(-r,r)\times L^2(-r,r)} < \epsilon$$

for all $n \geq \tilde{N}$. Consequently, $\|(\eta_n, u_n) - (\eta, u)\|_{L^2(R)\times L^2(R)} < 8\epsilon$.

By the uniform boundedness of u_n, η_n in H^1 and $C^0(R)$, we also have

$$\begin{aligned}
\int_{-\infty}^{\infty} |\eta_n u_n - \eta u|\, dx &\leq \int_{-\infty}^{\infty} |\eta_n u_n - \eta_n u|\, dx + \int_{-\infty}^{\infty} |\eta_n u - \eta u|\, dx \\
&\leq \|\eta_n\|_{L^2}\|u_n - u\|_{L^2} + \|u\|_{L^2}\|\eta_n - \eta\|_{L^2}; \\
\int_{-\infty}^{\infty} |\eta_n u_n^2 - \eta u^2|\, dx &\leq \int_{-\infty}^{\infty} |\eta_n u_n^2 - \eta u_n^2|\, dx + \int_{-\infty}^{\infty} |\eta u_n^2 - \eta u^2|\, dx \\
&\leq C(\|\eta_n - \eta\|_{L^2} + \|u_n - u\|_{L^2}.
\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} \eta_n u_n\, dx \to \int_{-\infty}^{\infty} \eta u\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \eta_n u_n^2\, dx \to \int_{-\infty}^{\infty} \eta u^2\, dx \quad \text{as } n \to \infty.$$

Since

$$\frac{1}{2}\int_{-\infty}^{\infty} \eta_n u_n^2\, dx = 1,$$

it follows that

$$\frac{1}{2}\int_{-\infty}^{\infty} \eta u^2\, dx = 1.$$

Furthermore, from the weak compactness of the unit sphere, we have that $\{(\eta_n, u_n)\}$ converges weakly to (η, u) in a Hilbert space $X = H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Thus, by Lemma 2.1, as $n \to \infty$, we obtain

$$\begin{aligned}
&C\big(\|\eta_n - \eta\|^2_{H^1} + \|u_n - u\|^2_{H^1}\big) \leq H_\mu(\eta_n - \eta, u_n - u) = H_\mu(\eta_n, u_n) - H_\mu(\eta, u) \\
&- \int_{-\infty}^{\infty} \Big(- c(\eta_{n,x} - \eta_x)\eta_x - a(u_{n,x} - u_x)u_x + (\eta_n - \eta)\eta + (u_n - u)u \\
&\qquad - \mu\left((\eta_n - \eta)u + (u_n - u)\eta + b(\eta_{n,x} - \eta_x)u_x + b(u_{n,x} - u_x)\eta_x\right) \Big) dx \\
&\to m_1(\mu) - H_\mu(\eta, u) \leq 0.
\end{aligned}$$

Consequently, it follows that (η_n, u_n) converges strongly to (η, u) in X-norm and

$$H_\mu(\eta, u) = \lim_{n\to\infty} H_\mu(\eta_n, u_n) = m_1(\mu)$$

with $(\eta, u) \in G(1)$. The theorem is proved. □

3. Existence of Traveling-Wave Solutions

The pair of functions (η, u) is a minimizer of H_μ subject to the constraint $P = 1$ and therefore is a weak solution of the Euler-Lagrange equation $\nabla H_\mu = \kappa \nabla P$; i.e.,

$$\begin{cases} c\eta_{xx} + \eta - \mu u + b\mu u_{xx} = \kappa u^2 \\ a u_{xx} + u - \mu\eta + b\mu\eta_{xx} = 2\kappa\eta u, \end{cases} \tag{3.1}$$

for some multiplier $\kappa \in \mathbb{R}$. Indeed, one has $\kappa > 0$ as follows.

Proposition 3.1. *The Lagrange multiplier κ is positive.*

Proof. Multiplying the first and second equations in (3.1) by η and u respectively and integrating over the real line, we obtain

$$\int_{-\infty}^{\infty} (-c\eta_x^2 + \eta^2 - \mu\eta u - b\mu\eta_x u_x)dx = \frac{\kappa}{2}\int_{-\infty}^{\infty} \eta u^2 dx$$
$$\int_{-\infty}^{\infty} (-a u_x^2 + u^2 - \mu\eta u - b\mu\eta_x u_x)dx = \kappa\int_{-\infty}^{\infty} \eta u^2 dx.$$

Adding the two equations and using the facts that $H_\mu(\eta, u) = m_1(\mu)$ and $P(\eta, u) = 1$, one arrives at

$$\kappa = 4m_1(\mu)/3 > 0. \tag{3.2}$$

Because of the homogeneity of the functionals H_μ and P, if (η, u) is a solution of (3.1), then $(\phi, \psi) = -\kappa(\eta, u)$ is a solution of (1.8). We will call such a solution a **ground state** solution. This weak ground state solution is indeed a classical solution of (1.8).

Proposition 3.2. *Suppose $(\phi, \psi) \in X$ is a weak ground state solution of*

$$\begin{cases} c\phi_{xx} + \phi - \mu\psi + b\mu\psi_{xx} + \frac{1}{2}\psi^2 = 0, \\ a\psi_{xx} + \psi - \mu\phi + b\mu\phi_{xx} + \phi\psi = 0. \end{cases} \tag{3.3}$$

Then (ϕ, ψ) is actually a classical solution; that is, $(\phi, \psi) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})$.

Proof. Since the solution is in X, then $\frac{1}{2}\psi^2$ and $\phi\psi$ are in L^2. Take the Fourier transform of (3.3) to obtain

$$(1 - ck^2)\hat{\phi} - \mu(1 + bk^2)\hat{\psi} + \tfrac{1}{2}\widehat{\psi^2} = 0,$$
$$(1 - ak^2)\hat{\psi} - \mu(1 + bk^2)\hat{\phi} + \widehat{\phi\psi} = 0,$$

where $\hat{f}$ is denoted as the Fourier transform of f. Thus,

$$\hat{\psi} = \frac{\mu(1 + bk^2)\hat{\phi} - \widehat{\phi\psi}}{(1 - ak^2)}$$

and

$$\frac{(1-ck^2)(1-ak^2)-\mu^2(1+bk^2)^2}{(1-ak^2)}\hat{\phi} = -\frac{\mu(1+bk^2)\hat{\phi\psi}}{(1-ak^2)} - \frac{1}{2}\hat{\psi^2}.$$

By $|\mu| < \mu_0 = \min\{1, \sqrt{ac}/|b|\}$, it is straightforward to show that

$$(1-ck^2)(1-ak^2)-\mu^2(1+bk^2)^2 \geq C(1+k^4)$$

for some constant $C > 0$. Thus, $(1+k^2)\hat{\phi} \in L^2$ or $\phi \in H^2$, which gives $\psi \in H^2$. Then, the proposition follows by a standard bootstrapping argument. □

Remark 3.3. The variational problem shows that traveling-wave solutions are thus global minimizers of H_μ subject to the constraint $P = 1$ and that the speed of propagation is μ.

One now arrives at the advertised existence result of traveling-wave solutions of the systems (1.1).

Theorem 3.4. *Let $a, c < 0$, $b = d$ and $|\mu| < \mu_0$. Then* (1.1) *exhibits traveling-wave solutions with propagation speed $\omega = \mu$.*

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