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Standing waves for a two-way model system for water waves

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Abstract

In this paper, we prove the existence of a large family of nontrivial bifurcating standing waves for a model system which describes two-way propagation of water waves in a channel of finite depth or in the near shore zone. In particular, it is shown that, contrary to the classical standing gravity wave problem on a fluid layer of finite depth, the Lyapunov–Schmidt method applies to find the bifurcation equation. The bifurcation set is formed with the discrete union of Whitney's umbrellas in the three-dimensional space formed with 3 parameters representing the time-period and the wave length, and the average of wave amplitude.

1. Introduction

There are many models for studying weakly nonlinear dispersive water waves in a channel or in the near shore zone. For one-way waves, namely when the wave motion occurs in one-direction, the well known KdV (Korteweg–de Vries) and BBM (Benjamin–Bona–Mahoney) equations are the most studied. For two-way waves, a four parameter class of model equations (which are called Boussinesq-type systems)

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{1}$$

was put forward by Bona, Chen and Saut [1] for small-amplitude and long wavelength gravity waves of an ideal, incompressible liquid. Systems (1) are first-order approximations to the two-dimensional Euler equation in the small parameters $\epsilon_1 = A/h_0$ and $\epsilon_2 = h_0^2/L^2$, where h_0 is the depth of water in its quiescent state, A is a typical wave amplitude and L is a typical wavelength. The dependent variables $\eta(x,t)$ and $u(x,t)$, scaled by h_0 and $c_0 = \sqrt{gh_0}$ respectively with g being the acceleration of gravity, represent the dimensionless deviation of the water surface from its undisturbed position and the horizontal velocity at the level of θh_0 of the depth of the undisturbed fluid with $0 \leqslant \theta \leqslant 1$, respectively. The coordinate x which measures the distance along the channel is scaled by h_0 and time t is scaled by $\sqrt{h_0/g}$. The dispersive parameters a, b, c and d are not independently specifiable parameters, but have to satisfy certain physical relevant conditions [1]. Systems in (1) are not only formally approximations to Euler's equation, but also recently further justified by Bona, Colin and Lannes (cf. [2]).

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In this work, attention will be directed to (x,t) – periodic solutions invariant under the reflexion symmetry $x \mapsto -x$, of the following system of partial differential equations (which we refer as BBM system since it has certain common properties as the BBM equation)

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &u_t + \eta_x + u u_x - \frac{1}{6}u_{xxt} = 0, \end{aligned} \tag{2}$$

which is a member of (1) where $\theta = \sqrt{2/3}$. One of the advantages that (2) has over alternative Boussinesq-type systems in (1) (see Bona, Chen and Saut [1]) is the ease with which it may be integrated numerically. Furthermore, it was proved in [3] and [4] that the initial value problem either for $x \in \mathbb{R}$ or with boundary conditions ($x \in [a,b]$) for (2) is well posed in certain natural function classes. It was also proved that the solution of (2) approximates the solution of Euler's equation with the order of accuracy of the equation (cf. [2,5–8]), namely, for any initial value $(\eta_0, u_0) \in H^\sigma(\mathbb{R})^2$ with $\sigma \geqslant s \geqslant 0$ large enough, there exists a unique solution $(\eta_{\text{euler}}, u_{\text{euler}})$ of Euler equations, such that

$$\|u - u_{\text{euler}}\|_{L^\infty(0,t;H^s)} + \|\eta - \eta_{\text{euler}}\|_{L^\infty(0,t;H^s)} = \mathrm{O}(\epsilon_1^2 t, \epsilon_2^2 t, \epsilon_1\epsilon_2 t)$$

for $0 \leqslant t \leqslant \mathrm{O}(\epsilon_1^{-1}, \epsilon_2^{-1})$.

Since we look for periodic solutions in (x,t), let us introduce the scaled variables $\tilde{x} = \frac{2\pi\sqrt{6}}{\lambda}x$, $\tilde{t} = \frac{2\pi\sqrt{6}}{T}t$, with $T/\sqrt{6}$ and $\lambda/\sqrt{6}$ being the time period and the wave length. One obtains the rescaled BBM system (the tilde is dropped for simplicity in notation)

$$\eta_t + \beta u_x - \alpha\eta_{xxt} + \beta(u\eta)_x = 0, \tag{3}$$

$$u_t + \beta\eta_x - \alpha u_{xxt} + \beta(u^2/2)_x = 0, \tag{4}$$

where α and β are positive parameters defined by

$$\alpha = (2\pi)^2/\lambda^2, \qquad \beta = T/\lambda.$$

The standing waves we are looking for are solutions (η, u) doubly 2π-periodic functions of (x,t), with u odd and η even in x. This fixes the origin in x, but leaves the time shift invariance.

Defining the average of η by A, we have now a 3-dimensional parameter space, where only the quarter $\alpha > 0, \beta > 0$ is physically relevant. We prove below (see Theorem 6) that, roughly speaking, there is a discrete set of surfaces (Whitney's umbrellas) in the space (α, β, A), which constitutes the bifurcation set of standing waves, solutions of the system (3), (4). It is worth noting that the situation here is extremely different from the standing gravity waves problem for the full water waves equations solved for the finite depth case by Plotnikov and Toland [9], and in the infinite depth case by Iooss, Plotnikov and Toland [10]. In the finite depth case, there is a small divisor problem in the inversion of the linearized operator, which necessitates the use of the Nash–Moser theorem for solving the nonlinear problem. There is also this necessity in the infinite depth case, with the additional difficulty of complete resonance, i.e. the occurrence of an infinite dimensional kernel for the linearized operator for the critical value of the parameter. Similar difficulties also occur in the study of model systems (1) when $abcd > 0$, or when $b = c = d = 0$, $a < 0$. In the present case, we show (see Lemma 4) that in general the kernel of the linearized operator is three-dimensional and that there is *no small divisor problem*. It is then possible, using a $O(2)$ invariance of the system (see below), to adapt the Lyapunov–Schmidt method to reduce the bifurcation problem to a one-dimensional bifurcation equation and the precise result is set at Theorem 6. The same method might apply on the model system (1) when $abcd < 0$, or $b = c = d = 0$, $a > 0$ (the Kaup system [11,12]), or when $a = b = c = 0$, $d > 0$ (the so called classical Boussinesq system [13–16]) or when $a = 0$, $b = d > 0$, $c < 0$ (such as the Bona–Smith system [17]).

2. Study of the linearized operator

We start the study by an examination of the linearized operator. The bifurcation set, in the space of parameters, is in general such that this operator has a nontrivial kernel. Let us study the following linear system

$$\begin{aligned} &\eta_t + \beta u_x - \alpha\eta_{xxt} = f_x, \\ &u_t + \beta\eta_x - \alpha u_{xxt} = g_x, \end{aligned} \tag{5}$$

where we look for solutions (η, u) with η even in x, and u odd in x, and with f given odd in x, and g given even in x. Let us write the Fourier series

$$\eta(x,t)=\sum_{p\geqslant 0,q\in\mathbb{Z}}\eta_{pq}(\cos px)\mathrm{e}^{\mathrm{i}qt},$$
$$u(x,t)=\sum_{p>0,q\in\mathbb{Z}}u_{pq}(\sin px)\mathrm{e}^{\mathrm{i}qt},$$
$$f(x,t)=\sum_{p>0,q\in\mathbb{Z}}f_{pq}(\sin px)\mathrm{e}^{\mathrm{i}qt},$$
$$g(x,t)=\sum_{p\geqslant 0,q\in\mathbb{Z}}g_{pq}(\cos px)\mathrm{e}^{\mathrm{i}qt}.$$

Then we get for $p>0$, $q\in\mathbb{Z}$

$$\mathrm{i}q(1+\alpha p^2)\eta_{pq}+p\beta u_{pq}=pf_{pq},$$
$$p\beta\eta_{pq}-\mathrm{i}q(1+\alpha p^2)u_{pq}=pg_{pq},$$

and for $p=0$, $q\in\mathbb{Z}$

$$\eta_{0q}=0,\quad \text{when } q\neq 0 \quad \text{and}$$
$$\eta_{00} \text{ is arbitrary.}$$

Let us define

$$\Delta(p,q)=q^2(1+\alpha p^2)^2-p^2\beta^2 \tag{6}$$

then if $\Delta\neq 0$, we obtain

$$\eta_{pq}=-\Delta^{-1}p\big[\mathrm{i}q(1+\alpha p^2)f_{pq}+p\beta g_{pq}\big],$$
$$u_{pq}=-\Delta^{-1}p\big[p\beta f_{pq}-\mathrm{i}q(1+\alpha p^2)g_{pq}\big].$$

The linearized operator has a nontrivial kernel if there exists a pair (p_0,q_0) satisfying

$$q_0^2(1+\alpha p_0^2)^2-p_0^2\beta^2=0. \tag{7}$$

One can then give estimates for (η_{pq},u_{pq}) in terms of (f_{pq},g_{pq}) in cases of $(p,|q|)\neq(p_0,q_0)$. These estimates allow to give a bound of the pseudo-inverse of the linearized operator on its range. We first observe that

$$\Delta(p,q)=\big\{q(1+\alpha p^2)-p\beta\big\}\big\{q(1+\alpha p^2)+p\beta\big\}, \tag{8}$$

hence for $\Delta\neq 0$ we have

$$|\eta_{pq}|+|u_{pq}|\leqslant\frac{p}{||q|(1+\alpha p^2)-p\beta|}\big\{|f_{pq}|+|g_{pq}|\big\}. \tag{9}$$

We have now the following useful precision on the couples (α,β) solving (7):

Lemma 1. *Given α and β positive real numbers, the subset*

$$\Sigma_{(\alpha,\beta)}:=\big\{(p,q)\in\mathbb{N}^2,\,q(1+\alpha p^2)-p\beta=0\big\}$$

of $\mathbb{N}^2$ is either empty, or finite. When there exists $(p_0,q_0)\in\Sigma_{(\alpha,\beta)}$, then (p_0,q_0) is the only element of $\Sigma_{(\alpha,\beta)}$ if one of the following conditions is realized:

(i) *α is irrational*;

(ii) *α is rational and $1/(\alpha p_0)$ is not an integer, and the numbers $\beta^2-4\alpha q_j^2$ are not squares of rational numbers for $q_j=1,2,\ldots,q_m$, $q_j\neq q_0$, $q_m=[\beta/(2\sqrt{\alpha})]$, where $[\cdot]$ means the integer part.*

Proof. (i) If α is irrational, then β is irrational, *rationally related to* α by

$$\beta-\alpha p_0q_0=q_0/p_0.$$

Another solution $(p,q)\in\mathbb{N}^2$ of the above equation would imply

$$\alpha(p_0q_0-pq)+q_0/p_0-q/p=0$$

which implies that α is rational. Hence there is a unique element in $\Sigma_{(\alpha,\beta)}$ when α is irrational.

(ii) If α is rational, then β is also rational. Set

$$X = p\sqrt{\alpha}, \qquad Y = q\frac{\sqrt{\alpha}}{\beta}$$

then a solution $(p,q) \in \mathbb{N}^2$ of $q(1+\alpha p^2) - p\beta = 0$, leads to

$$Y(1+X^2) - X = 0.$$

This leads immediately to $Y \leqslant \frac{1}{2}$ which yields

$$q \leqslant q_m = \left[\beta/(2\sqrt{\alpha})\right].$$

Hence, the only possible values for q are

$$q = 1, 2, \ldots, q_m$$

and q_0 is in this set. For each value q_j of q we have

$$p_j^{\pm} = \frac{\beta}{2q_j\alpha}\left(1 \pm \sqrt{1-4Y_j^2}\right),$$

where

$$Y_j = q_j\frac{\sqrt{\alpha}}{\beta}.$$

A necessary condition for $p_j^{\pm}$ to be an integer is that $1-4Y_j^2$ is the square of a rational number. This is in particular true for $q_j = q_0$ since

$$1-4Y_j^2 = \frac{(\beta^2 - 4\alpha q_j^2)}{\beta^2},$$

$$1-4Y_0^2 = \frac{(\beta^2 - 4\alpha q_0^2)}{\beta^2} = \frac{q_0^2(1-\alpha p_0^2)^2}{p_0^2\beta^2},$$

which gives $p_0^+ = 1/(\alpha p_0)$, $p_0^- = p_0$. For $q_j \neq q_0$ the number $1-4Y_j^2$ is in general not the square of a rational, hence $p_j^{\pm}$ are not integers. □

We then have the following (denote by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$)

Proposition 2. *For all positive parameter values* (α,β), *there exists a constant* $M > 0$ (*depending on* (α,β)) *such that for any pair* $(p,q) \in \mathbb{N}_0 \times \mathbb{Z}$, $(p,|q|) \notin \Sigma_{(\alpha,\beta)}$ *and* $(p,q) \neq (0,0)$ *we have*

$$\frac{p + p^2|q|}{||q|(1+\alpha p^2) - p\beta|} \leqslant M. \tag{10}$$

Proof. Let us first consider pairs (p,q) satisfying $p|q| \geqslant 2\beta/\alpha$, then

$$|q|(1+\alpha p^2) - p\beta \geqslant \begin{cases} p\beta + |q|, \\ |q| + \alpha p^2|q|/2 \end{cases}$$

hence

$$\frac{p + p^2|q|}{||q|(1+\alpha p^2) - p\beta|} \leqslant \frac{1}{\beta} + \frac{2}{\alpha}.$$

Now for $p = 0, q \neq 0$ we have

$$\frac{0}{||q|(1+\alpha 0^2) - 0\beta|} = 0,$$

and for $q = 0$, $p \neq 0$

$$\frac{p+0}{|0(1+\alpha p^2) - p\beta|} = \frac{1}{\beta}.$$

Now the set of pairs (p,q) such that $1 \leqslant p|q| < 2\beta/\alpha$, is *finite*, hence a finite bound exists once the denominator is not equal to zero. Such a situation would imply

$$|q|(1+\alpha p^2) - p\beta = 0,$$

i.e., $(p,|q|) \in \Sigma_{(\alpha,\beta)}$. $\square$

Remark. Let us give a geometric interpretation of Lemma 1. In the (α,β) plane the equation

$$q^2(1+\alpha p^2)^2 - p^2\beta^2 = 0$$

defines for a fixed $(p,q) \in \mathbb{N}^2$ a couple of straight lines, intersecting at $(\alpha,\beta) = (-1/p^2, 0)$. Only the line

$$q(1+\alpha p^2) - p\beta = 0$$

is relevant in the quarter of plane $(\alpha,\beta) \in (\mathbb{R}^+)^2$. Lemma 1 shows that if (α,β) belongs to such a line for $(p,q) = (p_0,q_0)$, then it belongs to at most a finite number of such lines for $(p,q) \in \mathbb{N}^2$, this number being one in general. Moreover, in the region $\beta^2 - 4\alpha < 0$ of $(\mathbb{R}^+)^2$, there is none of these lines, and in the rest of the quarter plane, the union of this discrete set of lines is not dense.

Let us now introduce the Sobolev spaces

$$H^k_{\natural\natural} = H^k\{(\mathbb{R}/2\pi\mathbb{Z})^2\}, \qquad H^{k,e}_{\natural\natural} = \{w \in H^k_{\natural\natural};\ w \text{ is even in } x\}$$

and similarly $H^{k,o}_{\natural\natural} = \{w \in H^k_{\natural\natural};\ w \text{ is odd in } x\}$. We also define the operator π_0 by

$$(\pi_0 g)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x,t)\,\mathrm{d}x,$$

and D_x^{-1} by

$$D_x^{-1}\cos px = p^{-1}\sin px, \quad p \neq 0,$$
$$D_x^{-1}\sin px = -p^{-1}\cos px, \quad D_x^{-1}1 = 0.$$

Notice that the operator D_x^{-1} consists in first suppressing the average and then take the primitive which has a 0 average. This guarantees the periodicity of $D_x^{-1}f$ for any periodic $f \in L^2$. In particular one has for any $f \in H^1_\natural$

$$D_x^{-1}\partial_x f = \partial_x D_x^{-1} f = (\mathbb{I} - \pi_0)f.$$

Taking into account the evenness of η and oddness of u, let us define the linear operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}(\eta,u) &= D_x^{-1}(u_t + \beta\eta_x - \alpha u_{xxt}, \eta_t + \beta u_x - \alpha\eta_{xxt})\\ &= \big(D_x^{-1}u_t + \beta(\mathbb{I}-\pi_0)\eta - \alpha u_{xt}, D_x^{-1}\eta_t + \beta u - \alpha\eta_{xt}\big),\end{aligned} \tag{11}$$

and notice that the structure of system (2) leads to $\eta_{0q} = 0,\ q \neq 0$, for Fourier coefficients of η. We now look for η in the corresponding invariant subspace

$$H^{k,e}_{\natural\natural,0} = \{\eta \in H^{k,e}_{\natural\natural};\ \eta_{0q} = 0,\ q \neq 0\}.$$

Let us introduce the symmetry operator $\mathcal{S}$, representing the reflexion symmetry $x \mapsto -x$, defined by

$$\{\mathcal{S}(\eta,u)\}(x,t) = \big(\eta(-x,t), -u(-x,t)\big). \tag{12}$$

We have then the following straightforward results:

Lemma 3. *The linear operator $\mathcal{L}$ defined in* (11) *is symmetric in $L^{2,e}_{\natural\natural,0} \times L^{2,o}_{\natural\natural}$ and looking for (η,u) in $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$, $k \geqslant 0$, the problem of solving* (5) *where (g,f) is given in $L^{2,e}_{\natural\natural} \times L^{2,o}_{\natural\natural}$, is equivalent to solve*

$$\mathcal{L}(\eta,u) = \big((\mathbb{I}-\pi_0)g, f\big). \tag{13}$$

The linear operator $\mathcal{L}$ commutes with $\mathcal{S}$:

$$\mathcal{S}\mathcal{L}U = \mathcal{L}\mathcal{S}U, \quad \text{for any } U \in \text{domain } (\mathcal{L}).$$

We can now give precisions on the kernel and range of $\mathcal{L}$ through the following

Lemma 4. *Assume that* (α_0, β_0) *is such that* $\Sigma_{(\alpha_0,\beta_0)}$ *has a unique element* (p_0, q_0) (*see the above Lemma* 1), *then denote by* $\mathcal{L}_0$ *the corresponding linear operator* (11). *The kernel of* $\mathcal{L}_0$ *in* $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$, $k \geqslant 0$ *is the* 3*-dimensional subspace spanned by* $\xi_0 = \{1, 0\}$ *and* ζ_0 *and* $\overline{\zeta_0}$ *where* $\mathcal{S}\xi_0 = \xi_0$, $\mathcal{S}\zeta_0 = \overline{\zeta_0}$ *and*

$$\zeta_0 = (\mathrm{e}^{\mathrm{i}q_0t} \cos p_0x, -\mathrm{i}\mathrm{e}^{\mathrm{i}q_0t} \sin p_0x).$$

The range of $\mathcal{L}_0$ *in* $H^{k,e}_{\natural\natural} \times H^{k,o}_{\natural\natural}$ *is closed, defined as the intersection of* $\ker(\pi_0 \otimes 0)$ *with the orthogonal complement of the two-dimensional subspace spanned by* ζ_0 *and* $\overline{\zeta_0}$. *Furthermore, the equation*

$$\mathcal{L}_0(\eta, u) = (g, f) \tag{14}$$

has a unique solution (η, u), *denoted by* $\widetilde{\mathcal{L}}_0^{-1}(g, f)$, *which belongs to* $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$ *orthogonal in* $(L^2_{\natural\natural})^2$ *to* ξ_0, ζ_0 *and* $\overline{\zeta_0}$, *and satisfies*

$$\|(\eta, u)\|_{H^k} \leqslant M\|(f, g)\|_{H^k}. \tag{15}$$

For $(g, f) = (\psi_{xt}, \phi_{xt})$ *orthogonal to* ζ_0 *and* $\overline{\zeta_0}$, *with* $(\phi, \psi) \in H^{k,e}_{\natural\natural} \times H^{k,o}_{\natural\natural}$ *the solutions* (η, u) *of* (14) *still lie in* $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$ *and, the equation leads to a unique solution* $(\eta, u) = \widetilde{\mathcal{L}}_0^{-1}(\phi_{xt}, \psi_{xt}) \in H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$, *orthogonal in* $(L^2_{\natural\natural})^2$ *to* ξ_0, ζ_0 *and* $\overline{\zeta_0}$, *which satisfies*

$$\|(\eta, u)\|_{H^k} \leqslant M\|(\phi, \psi)\|_{H^k}. \tag{16}$$

Corollary 5. *The operator* $\mathcal{L}$ *defined by* (11) *is selfadjoint in* $L^{2,e}_{\natural\natural,0} \times L^{2,o}_{\natural\natural}$.

Proof. We noticed at Lemma 3 that the system (5) for $(\alpha, \beta) = (\alpha_0, \beta_0)$, with the condition $\pi_0(\eta_t) = 0$, is equivalent to

$$\mathcal{L}(\eta, u) = (f, \tilde{g}),$$

where $\tilde{g} = g - \pi_0 g$ satisfies $\pi_0\tilde{g} = 0$. Then we have $\tilde{g}_{pq} = g_{pq}$ for $p \neq 0$, hence for $(p, |q|) \neq (p_0, q_0)$ and $p > 0$

$$\eta_{pq} = -\frac{1}{\Delta_0(p,q)} p\big[\mathrm{i}q(1 + \alpha_0 p^2) f_{pq} + p\beta_0 \tilde{g}_{pq}\big],$$
$$u_{pq} = -\frac{1}{\Delta_0(p,q)} p\big[p\beta_0 f_{pq} - \mathrm{i}q(1 + \alpha_0 p^2)\tilde{g}_{pq}\big],$$

where

$$\Delta_0(p, q) = q^2(1 + \alpha_0 p^2)^2 - p^2\beta_0^2$$

and

$$\eta_{0q} = 0, \quad \text{for } q \neq 0, \ \text{(by construction)}$$
$$\eta_{00} \ \text{arbitrary},$$

and, for $(p, |q|) = (p_0, q_0)$, provided that

$$(g, f) \in \{\zeta_0, \overline{\zeta_0}\}^{\perp},$$

which means that

$$\begin{aligned} &g_{p_0,q_0} - \mathrm{i} f_{p_0,q_0} = 0,\\ &g_{p_0,-q_0} - \mathrm{i} f_{p_0,-q_0} = 0 \end{aligned} \tag{17}$$

is satisfied, then

$$\eta_{p_0q_0} = -\frac{\mathrm{i} f_{p_0q_0}}{2\beta_0} + \mathrm{i}a, \qquad \eta_{p_0,-q_0} = \frac{\mathrm{i} f_{p_0,-q_0}}{2\beta_0} - \mathrm{i}b,$$
$$u_{p_0q_0} = \frac{f_{p_0q_0}}{2\beta_0} + a, \qquad u_{p_0,-q_0} = \frac{f_{p_0,-q_0}}{2\beta_0} + b,$$

where a and b are arbitrary. Orthogonality in $(L^2_{\natural\natural})^2$ to ξ_0, ζ_0 and $\overline{\zeta_0}$ leads to

$$\eta_{00} = 0,$$
$$\eta_{p_0q_0} = -\frac{\mathrm{i} f_{p_0q_0}}{2\beta_0}, \qquad \eta_{p_0,-q_0} = \frac{\mathrm{i} f_{p_0,-q_0}}{2\beta_0},$$
$$u_{p_0q_0} = \frac{f_{p_0q_0}}{2\beta_0}, \qquad u_{p_0,-q_0} = \frac{f_{p_0,-q_0}}{2\beta_0},$$

which insures the uniqueness of the solution. The estimate obtained in (9)–(10) leads to $(\eta, u) \in H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$ satisfying (15) or (16) as soon as $(g, f) \in H^{k,e}_{\natural\natural} \times H^{k,o}_{\natural\natural}$ or $(\phi, \psi) \in H^{k,e}_{\natural\natural} \times H^{k,o}_{\natural\natural}$, and the compatibility condition (17) is satisfied. This gives the range of $\mathcal{L}$. The result on the kernel is a direct consequence of the above formulas. The selfadjointness of $\mathcal{L}$ in $L^{2,e}_{\natural\natural,0} \times L^{2,o}_{\natural\natural}$ results from its symmetry (see Lemma 3) and from the boundedness of its inverse. □

Remark on other models in (1). If one considers the linearized system associated with (1), the determinant Δ corresponding to (8) is as follows

$$\Delta = q^2(1 + \alpha b p^2)(1 + \alpha d p^2) - \beta^2 p^2(\alpha a p^2 - 1)(\alpha c p^2 - 1),$$

where $\alpha = (2\pi/\lambda)^2$, $\beta = T/\lambda$, λ and T being the wave length and the time period. The equation $\Delta = 0$ gives the couples (p, q) corresponding to the kernel of the linearized operator, and for noncritical couples we need to bound $1/\Delta$ independently of (p, q). We then observe that if $abcd > 0$, there is a *small divisor problem*. However, in making Diophantine assumptions on the coefficients $a, b, c, d, \alpha, \beta$, it is possible to give a bound for the pseudo-inverse of $\mathcal{L}$, but in loosing some regularity between (g, f) and (η, u). Then, it is in general not possible to use the standard Lyapunov–Schmidt method since we do not regain enough regularity when this pseudo-inverse is applied to nonlinear terms. The use of Nash–Moser theorem becomes the only tool available, as used in [9], and [10], depending on the dimension of the kernel. This small divisor problem also occurs when $b = c = d = 0$, $a < 0$. On the contrary, our method might apply on the model system (1) when (i) $abcd < 0$, (ii) $b = c = d = 0$, $a > 0$, (iii) $a = b = c = 0$, $d > 0$, and (iv) $a = 0$, $b = d > 0$, $c < 0$.

3. Bifurcation problem

In addition to the symmetry operator $\mathcal{S}$, let us introduce the following linear operators $\mathcal{T}_\tau$ for any real τ

$$\{\mathcal{T}_\tau(\eta, u)\}(x, t) = (\eta(x, t + \tau), u(x, t + \tau)).$$

The operators $\mathcal{T}_\tau$ and $\mathcal{S}$ commute with the system (4), (3) and we have $\mathcal{T}_\tau \mathcal{S} = \mathcal{S}\mathcal{T}_{-\tau}$. Due to the time periodicity, it results that the nonlinear system (4), (3) possesses a $O(2)$ symmetry associated with the above representation operators. The aim of this section is to use the Lyapunov–Schmidt method for obtaining the bifurcating standing waves. The role of the $O(2)$ equivariance of our system is to simplify the analysis. Moreover, in leaving free the origin of t, this avoids to restrict a priori the study to solutions (η, u) which have a fixed evenness in t. Let us consider (4), (3) for parameter values $(\alpha, \beta) = (\alpha_0 + \nu, \beta_0 + \mu)$, where (α_0, β_0) is as in the above Lemma 4

$$u_t + \beta_0 \eta_x - \alpha_0 u_{xxt} + \big((\beta_0 + \mu)u^2/2 + \mu\eta - \nu u_{xt}\big)_x = 0,$$
$$\eta_t + \beta_0 u_x - \alpha_0 \eta_{xxt} + \big((\beta_0 + \mu)u\eta + \mu u - \nu \eta_{xt}\big)_x = 0$$

with (ν, μ) close to 0, and let us look for nontrivial doubly periodic solutions (η, u) in $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$. We observe that for $k \geqslant 2$ (Sobolev imbedding theorem)

$$(u^2/2, u\eta) \in H^{k,e}_{\natural\natural} \times H^{k,o}_{\natural\natural}$$

hence, defining (g, f) by

$$g = -(\beta_0 + \mu)u^2/2 - \mu\eta + \nu u_{xt},$$
$$f = -(\beta_0 + \mu)u\eta - \mu u + \nu \eta_{xt}$$

the right-hand side of (5) has the properties required for (g, f) in Lemma 4, once the compatibility condition is satisfied. We can then apply the Lyapunov–Schmidt method for finding the bifurcation equation.

Remark. Here we have a big difference with the full water wave system, where it appears impossible to use such a method, because of the nonregularizing enough effect of the pseudo-inverse of the operator corresponding to $\mathcal{L}_0$, due to a small divisor problem (see [9] and [10]).

Let us define $U=(\eta,u)\in H^{k,e}_{\natural\natural,0}\times H^{k,o}_{\natural\natural}$ and write our system (2) as

$$\mathcal{L}_0U+\mu\mathcal{J}U-\nu\mathcal{K}U_{xt}+(\beta_0+\mu)\mathcal{N}(U,U)=0 \tag{18}$$

with

$$\mathcal{J}U=\big((\mathbb{I}-\pi_0)\eta,u\big),\quad \mathcal{K}U_{xt}=(u_{xt},\eta_{xt}),\quad \mathcal{N}(U,U)=\big((\mathbb{I}-\pi_0)u^2/2,u\eta\big),$$

which means that Eq. (18) already lies in $\ker(\pi_0\otimes 0)$. As already mentioned, Eq. (18) is equivariant under the $O(2)$ symmetry defined above:

$$\mathcal{T}_\tau\mathcal{L}_0=\mathcal{L}_0\mathcal{T}_\tau,\quad \mathcal{T}_\tau\mathcal{J}=\mathcal{J}\mathcal{T}_\tau,\quad \mathcal{T}_\tau\mathcal{N}=\mathcal{N}\circ\mathcal{T}_\tau, \tag{19}$$

$$\mathcal{S}\mathcal{L}_0=\mathcal{L}_0\mathcal{S},\quad \mathcal{S}\mathcal{J}=\mathcal{J}\mathcal{S},\quad \mathcal{S}\mathcal{N}=\mathcal{N}\circ\mathcal{S}. \tag{20}$$

Let us now decompose U as follows (we are looking for *real solutions*)

$$U=(\eta,u)=\Theta+\Upsilon$$

with

$$\Theta=A\xi_0+B\zeta_0+\overline{B\zeta_0},\qquad (\Upsilon,\zeta_0)=(\Upsilon,\overline{\zeta_0})=(\Upsilon,\xi_0)=0,$$

where the scalar product is the one of $(L^2_{\natural\natural})^2$, $A\in\mathbb{R}$ and $B\in\mathbb{C}$ are constants. We notice, as a consequence of the above decomposition ($(\Upsilon,\xi_0)=0$), that A *is the average of* $\eta(x,t)$:

$$A=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Let us define $F=(g,f)\in H^{k,e}_{\natural\natural}\times H^{k,o}_{\natural\natural}$, we also need a projection Q_0 expressing the compatibility condition (17)

$$Q_0F=F-\frac{1}{4\pi^2}(F,\zeta_0)\zeta_0-\frac{1}{4\pi^2}(F,\overline{\zeta_0})\overline{\zeta_0}$$

and we notice that the conditions $(F,\zeta_0)=(F,\overline{\zeta_0})=0$ are sufficient for F to belong to the range of $\mathcal{L}$, if already $\pi_0g=0$. We observe that

$$\mathcal{K}(\zeta_0)_{xt}=p_0q_0\zeta_0,\quad \mathcal{J}\zeta_0=\zeta_0,\quad \mathcal{J}\xi_0=0,$$

hence

$$Q_0\mathcal{K}\Theta_{xt}=0,\quad (\mathcal{K}\Upsilon_{xt},\zeta_0)=(\mathcal{K}\Upsilon_{xt},\overline{\zeta_0})=0,\quad Q_0\mathcal{J}\Theta=0.$$

It results that (18) projected on the range of $\mathcal{L}_0$ and on $\operatorname{span}\{\zeta_0,\overline{\zeta_0}\}$ (which is orthogonal to the range), may be written as the system

$$\mathcal{L}_0\Upsilon+\mu Q_0\mathcal{J}(\Upsilon)-\nu\mathcal{K}\Upsilon_{xt}+(\beta_0+\mu)Q_0\mathcal{N}(\Theta+\Upsilon,\Theta+\Upsilon)=0, \tag{21}$$

$$\big(\mu\mathcal{J}(\Theta+\Upsilon)-\nu\mathcal{K}\Theta_{xt}+(\beta_0+\mu)\mathcal{N}(\Theta+\Upsilon,\Theta+\Upsilon),\zeta_0\big)=0, \tag{22}$$

$$\big(\mu\mathcal{J}(\Theta+\Upsilon)-\nu\mathcal{K}\Theta_{xt}+(\beta_0+\mu)\mathcal{N}(\Theta+\Upsilon,\Theta+\Upsilon),\overline{\zeta_0}\big)=0. \tag{23}$$

Therefore, by first considering (21) we get

$$\Upsilon+\widetilde{\mathcal{L}}_0^{-1}\big\{\mu Q_0\mathcal{J}(\Upsilon)-\nu\mathcal{K}\Upsilon_{xt}+(\beta_0+\mu)Q_0\mathcal{N}(\Theta+\Upsilon,\Theta+\Upsilon)\big\}=0$$

which is of the form

$$\mathcal{F}(\Upsilon,A,B,\overline{B},\mu,\nu)=0 \tag{24}$$

and thanks to the boundedness properties of the operator $\widetilde{\mathcal{L}}_0^{-1}$ shown at Lemma 4, $\mathcal{F}$ is analytic:

$$\big\{(H^{k,e}_{\natural\natural,0}\times H^{k,o}_{\natural\natural})\cap\{\xi_0,\zeta_0,\overline{\zeta_0}\}^\perp\big\}\times\mathbb{R}\times\mathbb{C}^2\times\mathbb{R}^2\to\big\{(H^{k,e}_{\natural\natural,0}\times H^{k,o}_{\natural\natural})\cap\{\xi_0,\zeta_0,\overline{\zeta_0}\}^\perp\big\}$$

and satisfies

$$\mathcal{F}(0, A, 0, 0, \mu, \nu) = 0.$$

Moreover, because of the fact that

$$\mathcal{S}\zeta_0 = \overline{\zeta_0}, \quad \mathcal{T}_\tau \zeta_0 = \mathrm{e}^{\mathrm{i}q_0\tau}\zeta_0,$$
$$\mathcal{S}\xi_0 = \xi_0, \quad \mathcal{T}_\tau \xi_0 = \xi_0,$$

the equivariance properties (19), (20) of our system lead to

$$\mathcal{T}_\tau \mathcal{F}(\Upsilon, A, B, \overline{B}, \mu, \nu) = \mathcal{F}(\mathcal{T}_\tau \Upsilon, A, \mathrm{e}^{\mathrm{i}q_0\tau} B, \mathrm{e}^{-\mathrm{i}q_0\tau}\overline{B}, \mu, \nu),$$
$$\mathcal{S}\mathcal{F}(\Upsilon, A, B, \overline{B}, \mu, \nu) = \mathcal{F}(\mathcal{S}\Upsilon, A, \overline{B}, B, \mu, \nu).$$

The above Eq. (24) is solvable with respect to $\Upsilon \in H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$ by the *analytic implicit function theorem*, for A, B, μ, ν close enough to 0 in $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$. We then obtain a function

$$\Upsilon = \mathcal{Y}(A, B, \overline{B}, \mu, \nu),$$

where Υ is analytic in its arguments and its principal part is quadratic in $(A, B, \overline{B})$, given by (after a simple identification)

$$\Upsilon = -\beta_0 \widetilde{\mathcal{L}}^{-1} Q_0 \mathcal{N}(\Theta, \Theta) + \mathrm{O}\big\{\big(|\mu| + |\nu|\big)\|\Theta\|^2 + \|\Theta\|^3\big\}, \tag{25}$$

and, because of the uniqueness of $\mathcal{Y}$ (comes from the implicit function theorem), we have for any real τ

$$\mathcal{Y}(A, \overline{B}, B, \mu, \nu) = \mathcal{S}\mathcal{Y}(A, B, \overline{B}, \mu, \nu),$$
$$\mathcal{Y}(A, \mathrm{e}^{\mathrm{i}q_0\tau} B, \mathrm{e}^{-\mathrm{i}q_0\tau}\overline{B}, \mu, \nu) = \mathcal{T}_\tau \mathcal{Y}(A, B, \overline{B}, \mu, \nu).$$

In addition, we notice that, because of the existence of the family of trivial solutions of (18), $U = A\xi_0$ which correspond to $(\eta, u) = (A, 0)$, we have for any (A, μ, ν) close enough to 0, and thanks to the uniqueness of the solution

$$\mathcal{Y}(A, 0, 0, \mu, \nu) = 0.$$

By simple calculation, we have

$$\mathcal{N}(\Theta, \Theta) = \bigg(\frac{1}{4}\cos 2p_0x\big(B^2\mathrm{e}^{2\mathrm{i}q_0t} - 2|B|^2 + \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t}\big),$$
$$-\,\mathrm{i}\sin p_0x(AB\mathrm{e}^{\mathrm{i}q_0t} - A\overline{B}\mathrm{e}^{-\mathrm{i}q_0t}) - \frac{\mathrm{i}}{2}(B^2\mathrm{e}^{2\mathrm{i}q_0t} - \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t})\sin 2p_0x\bigg)$$

and

$$Q_0\mathcal{N}(\Theta, \Theta) = \bigg(-\frac{1}{2}\cos p_0x(AB\mathrm{e}^{\mathrm{i}q_0t} + A\overline{B}\mathrm{e}^{-\mathrm{i}q_0t}) + \frac{1}{4}\cos 2p_0x\big(B^2\mathrm{e}^{2\mathrm{i}q_0t} - 2|B|^2 + \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t}\big),$$
$$-\,\frac{\mathrm{i}}{2}\sin p_0x(AB\mathrm{e}^{\mathrm{i}q_0t} - A\overline{B}\mathrm{e}^{-\mathrm{i}q_0t}) - \frac{\mathrm{i}}{2}(B^2\mathrm{e}^{2\mathrm{i}q_0t} - \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t})\sin 2p_0x\bigg).$$

The principal part of $\mathcal{Y}$ is then given by

$$-\beta_0 \widetilde{\mathcal{L}}^{-1} Q_0 \mathcal{N}(\Theta, \Theta) := (y^{(1)}, y^{(2)})$$

with

$$y^{(1)} = \frac{1}{4}\cos p_0x(AB\mathrm{e}^{\mathrm{i}q_0t} + A\overline{B}\mathrm{e}^{-\mathrm{i}q_0t}) + \frac{1}{2}|B|^2\cos 2p_0x + \alpha_1\cos 2p_0x(B^2\mathrm{e}^{2\mathrm{i}q_0t} + \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t}),$$
$$y^{(2)} = \frac{\mathrm{i}}{4}\sin p_0x(AB\mathrm{e}^{\mathrm{i}q_0t} - A\overline{B}\mathrm{e}^{-\mathrm{i}q_0t}) + \mathrm{i}\beta_1\sin 2p_0x(B^2\mathrm{e}^{2\mathrm{i}q_0t} - \overline{B}^2\mathrm{e}^{-2\mathrm{i}q_0t}),$$

with

$$\alpha_1 = \frac{\beta_0(1 + 3\alpha_0p_0^2)}{4\alpha_0p_0q_0(2 + 5\alpha_0p_0^2)} = \frac{(1 + \alpha_0p_0^2)(1 + 3\alpha_0p_0^2)}{4\alpha_0p_0^2(2 + 5\alpha_0p_0^2)},$$
$$\beta_1 = \frac{-\beta_0(1 + 2\alpha_0p_0^2)}{4\alpha_0p_0q_0(2 + 5\alpha_0p_0^2)} = \frac{-(1 + \alpha_0p_0^2)(1 + 2\alpha_0p_0^2)}{4\alpha_0p_0^2(2 + 5\alpha_0p_0^2)}.$$

Now, substituting $\Upsilon = \mathcal{Y}(A, B, \overline{B}, \mu, \nu)$ into (22) we obtain an equation in $\mathbb{C}$ of the form

$$h(A, B, \overline{B}, \mu, \nu) = 0$$

while (23) gives its complex conjugate. Now, let us use the equivariance of our system. We then obtain the properties

$$h(A, B\mathrm{e}^{\mathrm{i}q_0\tau}, \overline{B}\mathrm{e}^{-\mathrm{i}q_0\tau}, \mu, \nu) = \mathrm{e}^{\mathrm{i}q_0\tau} h(A, B, \overline{B}, \mu, \nu),$$
$$h(A, \overline{B}, B, \mu, \nu) = \overline{h(A, B, \overline{B}, \mu, \nu)}$$

as it can be seen below. Indeed, we have thanks to (19)

$$\begin{aligned} h(A, B\mathrm{e}^{\mathrm{i}q_0\tau}, \overline{B}\mathrm{e}^{-\mathrm{i}q_0\tau}, \mu, \nu) &= \big(\mu\mathcal{J}\mathcal{T}_\tau(\Theta + \mathcal{Y}) - \nu\mathcal{T}_\tau\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}\big(\mathcal{T}_\tau(\Theta + \mathcal{Y}), \mathcal{T}_\tau(\Theta + \mathcal{Y})\big), \zeta_0\big) \\ &= \big(\mathcal{T}_\tau\big\{\mu\mathcal{J}(\Theta + \mathcal{Y}) - \nu\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}(\Theta + \mathcal{Y}, \Theta + \mathcal{Y})\big\}, \zeta_0\big) \\ &= \big(\mu\mathcal{J}(\Theta + \Upsilon) - \nu\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}(\Theta + \Upsilon, \Theta + \Upsilon), \mathcal{T}_{-\tau}\zeta_0\big) \\ &= \mathrm{e}^{\mathrm{i}q_0\tau}\big(\mu\mathcal{J}(\Theta + \Upsilon) - \nu\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}(\Theta + \Upsilon, \Theta + \Upsilon), \zeta_0\big) \\ &= \mathrm{e}^{\mathrm{i}q_0\tau} h(A, B, \overline{B}, \mu, \nu), \end{aligned}$$

and thanks to (20)

$$\begin{aligned} h(A, \overline{B}, B, \mu, \nu) &= \big(\mu\mathcal{J}S(\Theta + \mathcal{Y}) - \nu\mathcal{S}\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}\big(\mathcal{S}(\Theta + \mathcal{Y}), \mathcal{S}(\Theta + \mathcal{Y})\big), \zeta_0\big) \\ &= \big(\mathcal{S}\big\{\mu\mathcal{J}(\Theta + \mathcal{Y}) - \nu\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}(\Theta + \mathcal{Y}, \Theta + \mathcal{Y})\big\}, \zeta_0\big) \\ &= \big(\mu\mathcal{J}(\Theta + \mathcal{Y}) - \nu\Theta_{xt} + (\beta_0 + \mu)\mathcal{N}(\Theta + \mathcal{Y}, \Theta + \mathcal{Y}), \overline{\zeta_0}\big) \\ &= \overline{h(A, B, \overline{B}, \mu, \nu)}. \end{aligned}$$

It then results from its analyticity, that h takes the form

$$h(A, B, \overline{B}, \mu, \nu) = BH\big(A, |B|^2, \mu, \nu\big) \tag{26}$$

with an analytic function H taking only real values, and the complex equation $h = 0$ reduces to either $B = 0$ or the real equation $H = 0$. The same property holds for the complex conjugate equation (23). Now noticing that

$$(\mathcal{K}\Theta_{xt}, \zeta_0) = 4\pi^2 p_0 q_0 B,$$
$$\big(\mathcal{N}(\Theta, \Theta), \zeta_0\big) = 2\pi^2 AB,$$
$$(\mathcal{J}\Theta, \zeta_0) = 4\pi^2 B,$$

the bifurcation equation (22) reads

$$BH\big(A, |B|^2, \mu, \nu\big) = 0 \tag{27}$$

where

$$(4\pi^2)^{-1}H\big(A, |B|^2, \mu, \nu\big) = \mu - p_0 q_0 \nu + \frac{1}{2}\beta_0 A - \beta_2|B|^2 + \mathrm{O}\big(|B|^4 + \big(|\mu| + |\nu| + |A|\big)\big(|A| + |B|^2\big)\big) \tag{28}$$

and the term which is the most important to compute is the coefficient β_2 in (28). For this we need to introduce the symmetric bilinear operator associated with the quadratic operator $\mathcal{N}$ by

$$2\mathcal{N}(U_1, U_2) = \big((\mathbb{I} - \pi_0)u_1u_2, u_1\eta_2 + u_2\eta_1\big).$$

Since we have the principal part of $\mathcal{Y}$ under the form

$$y := (y^{(1)}, y^{(2)}) := ABy^{(1,1)} + A\overline{B}y^{(1,-1)} + |B|^2 y^{(2,0)} + B^2 y^{(2,2)} + \overline{B}^2 y^{(2,-2)}$$

we then obtain

$$\beta_2 = \frac{-\beta_0}{4\pi^2}\big(2\mathcal{N}(y^{(2,0)}, \zeta_0) + 2\mathcal{N}(y^{(2,2)}, \overline{\zeta_0}), \zeta_0\big),$$

therefore,

$$\beta_2 = \beta_0\left(\frac{1}{8} + \frac{\beta_1}{2} - \frac{\alpha_1}{4}\right) = \frac{3\beta_0[(\alpha_0 p_0^2 - 1)^2 - 2]}{16\alpha_0 p_0^2(2 + 5\alpha_0 p_0^2)}.$$

We can in addition give the exact term $H(A,0,\mu,\nu)$ independent of B. For this, let us look for all terms of degree one in B, and degree 0 in $\overline{B}$ in the expression $h(A,B,\overline{B},\mu,\nu)$. Due to the form of $\mathcal{Y}(A,B,\overline{B},\mu,\nu)$, these terms come from

$$\big(\mu\mathcal{J}(B\zeta_0+Y_B)-\nu B\mathcal{K}\zeta_{0,xt}+(\beta_0+\mu)\big\{2\mathcal{N}(A\xi_0,B\zeta_0)+2\mathcal{N}(A\xi_0,Y_B)\big\},\zeta_0\big), \tag{29}$$

where Y_B is the term in $\mathcal{Y}(A,B,\overline{B},\mu,\nu)$ of degree one in B, and degree 0 in $\overline{B}$. Now, Y_B is the solution of the affine equation

$$\mathcal{L}_0Y_B+\mu Q_0\mathcal{J}(Y_B)-\nu\mathcal{K}Y_{B,xt}+\beta Q_0\big\{2\mathcal{N}(A\xi_0,B\zeta_0)+2\mathcal{N}(A\xi_0,Y_B)\big\}=0, \tag{30}$$

and a careful examination shows that we can look for Y_B under the form

$$Y_B=B\gamma(A)(\mathrm{e}^{\mathrm{i}q_0t}\cos p_0x,\mathrm{i}\mathrm{e}^{\mathrm{i}q_0t}\sin p_0x).$$

A direct identification in (30) leads to

$$(2\beta_0+\mu+p_0q_0\nu)\gamma(A)-\frac{1}{2}\beta A\big\{1-\gamma(A)\big\}=0,$$

which gives $\gamma(A)$: (notice that $\beta_0+\mu=\beta$)

$$\gamma_{\mu,\nu}(A)=\frac{(\beta_0+\mu)A}{2(2\beta_0+\mu+p_0q_0\nu)+(\beta_0+\mu)A}=\frac{\beta A}{2(2\beta-\mu+p_0q_0\nu)+\beta A}, \tag{31}$$

which is coherent (taking the limit μ,ν,A tending towards 0) with the coefficient $y^{(1,1)}$ of AB in $(y^{(1)},y^{(2)})$. We then observe that

$$\big(\mathcal{J}(Y_B),\zeta_0\big)=0$$

which leads for the coefficient (29) to the following expression

$$4\pi^2B\left\{\mu-p_0q_0\nu+\frac{1}{2}(\beta_0+\mu)A\big(1-\gamma_{\mu,\nu}(A)\big)\right\}.$$

Now, from the form of $\gamma_{\mu,\nu}(A)$, and from the identity

$$\mu-p_0q_0\nu=\frac{1}{p_0}\big(p_0\beta-q_0(1+\alpha p_0^2)\big)$$

we obtain

$$\mu-p_0q_0\nu+\frac{1}{2}(\beta_0+\mu)A\big(1-\gamma_{\mu,\nu}(A)\big)=\frac{2\{p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2\}}{p_0\{2(p_0\beta+q_0(1+\alpha p_0^2))+p_0\beta A\}}.$$

We then obtain, in addition to the trivial family of solutions of (27) corresponding to $B=0$ (already seen), another bifurcating family given by the solutions of $H(A,|B|^2,\mu,\nu)=0$, i.e. thanks to the analyticity of H and the above computation, the solutions of the following improved form for (28)

$$\frac{2\{p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2\}}{p_0\{2(p_0\beta+q_0(1+\alpha p_0^2))+p_0\beta A\}}-\beta_2|B|^2+\mathrm{O}\big(|B|^4+\big(|\mu|+|\nu|+|A|\big)\big(|B|^2\big)\big)=0.$$

This provides standing waves, determined up to a phase shift in t, equivalent to an arbitrary choice of the phase of B. Moreover, for $\beta_2\neq 0$, we can obtain, via the implicit function theorem,

$$|B|^2=\frac{p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2}{2p_0^2\beta_0\beta_2}\big\{1+\mathrm{O}\big(|A|+|\mu|+|\nu|\big)\big\}$$

for arbitrary A,μ,ν close to 0, while the bifurcation only takes place either for $p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2>0$ or for $p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2<0$. We sum up our result in the following

Theorem 6. *Consider any positive (α_0,β_0) such that*

$$\Sigma_{(\alpha_0,\beta_0)}=\big\{(p,q)\mid(p,q)\in\mathbb{N}^2 \text{ and } q(1+\alpha_0p^2)-p\beta_0=0\big\}$$

has a unique element (p_0,q_0). Then, for μ,ν,A close enough to 0, where $\alpha=\alpha_0+\nu$, $\beta=\beta_0+\mu$, A is the average of $\eta(x,t)$, and for

$$\big\{p_0^2\beta^2(1+A)-q_0^2(1+\alpha p_0^2)^2\big\}\big(\alpha_0p_0^2-(1+\sqrt{2})\big)>0,$$

there is a three parameter (α, β, A) *family of bifurcating standing waves* $U = (\eta, u)$, *solution of the system* (3), (4) *in* $H^{k,e}_{\natural\natural,0} \times H^{k,o}_{\natural\natural}$, $k \geqslant 2$, *defined up to a time shift. More precisely, we have* $U = \mathcal{T}_\tau U_0$, $\tau \in \mathbb{R}$ (*time shifted solutions, with respect to* U_0), *and*

$$\begin{aligned}
&U_0(x,t) = (\eta_0, u_0)(x,t),\\
&\eta_0(x,t) = A + 2|B|\cos q_0 t \cos p_0 x + \mathrm{O}\big(|B|\big(|A| + |B|\big)\big),\\
&u_0(x,t) = 2|B|\sin q_0 t \sin p_0 x + \mathrm{O}\big(|B|\big(|A| + |B|\big)\big),\\
&|B|^2 = \frac{p_0^2\beta^2(1+A) - q_0^2(1+\alpha p_0^2)^2}{2p_0^2\beta_0\beta_2}\big\{1 + \mathrm{O}\big(|A| + |\mu| + |\nu|\big)\big\}.
\end{aligned}$$

Remark 1. In the cases when (α_0, β_0) is such that $\Sigma_{(\alpha_0,\beta_0)}$ contains more than one element, the kernel of $\mathcal{L}_0$ is still finite dimensional, and the bifurcation equation is a system of $2m$ equations for $A, B_1, \ldots, B_m$ where m is the number of elements in $\Sigma_{(\alpha_0,\beta_0)}$. Even though, the structure of this system is simplified by its $O(2)$ equivariance, the general study of such a nongeneric system is not studied here.

Remark 2. If we consider the linearization of the system (3), (4) at a point $(\eta, u) = (A, 0)$ instead of the origin, we obtain for the inverse operator, a new denominator replacing Δ in (6) by

$$q^2(1+\alpha p^2)^2 - p^2\beta^2(1+A).$$

This is the *quantity appearing in the expression of the square* $|B|^2$ *of the amplitude* of the bifurcating standing waves, as it is natural. Now, the set of (α, β, A) in the three-parameter space, where bifurcation takes place is when the above expression cancels, which is for fixed (p, q) a right conoid (axis: $\alpha = -1/p^2$, $\beta = 0$) called a *Whitney's umbrella*. The intersection of this surface with the plane $A = 0$ is the couple of straight lines already mentioned in the remark of Section 2. The above theorem shows that the bifurcation of standing waves takes place along a discrete set of such Whitney's umbrellas (do not forget that (p, q) is arbitrary in $\mathbb{N}^2$).

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