A PLETHORA OF MULTI-PULSED SOLUTIONS FOR A BOUSSINESQ SYSTEM

MIN CHEN

Department of Mathematics, Penn State University
University Park, PA16802, USA.

Abstract. This paper studies traveling-wave solutions of the regularized Boussinesq system which model waves in a horizontal water channel traveling in both directions. An exact solution with a single trough is found analytically. Interesting new multi-pulsed traveling-wave solutions which consist of an arbitrary number of troughs are found numerically. The bifurcation diagrams for N-trough solutions with respect to phase speed are presented.

1. Regularized Boussinesq System

In this report, we study the regularized Boussinesq system

$$
\begin{aligned}
&\eta_t + u_x + (\eta u)_x - \frac{1}{6}\eta_{xxt} = 0, \\
&u_t + \eta_x + u u_x - \frac{1}{6}u_{xxt} = 0,
\end{aligned} \tag{1}
$$

which describes approximately the two-dimensional propagation of surface waves in a uniform horizontal channel of length L filled with an irrotational, incompressible, invisid fluid which in its undisturbed state has depth h. The non-dimensional variables $\eta(x,t)$ and $u(x,t)$ represent, respectively, the deviation of the water surface from its undisturbed position and the horizontal velocity at water level $\sqrt{\frac{2}{3}}h$. Among a class of formally equivalent models, system (1) is particular interesting because that the dispersion relation is stabilizing for all wave numbers and it is reasonably straightforward to implement the initial-boundary-value problems that arise in laboratory experiments. Moreover, the regularized Boussinesq system is much simpler than the full Euler equations and can be used on a wider class of problems than a model equation with the assumption of waves traveling in only one direction.

Since we are interested in modeling wave motion in a wave-tank experiment which is initiated by wave makers at both ends of the channel, the initial- and boundary-conditions are

$$
\begin{aligned}
&\eta(0,t) = h_1(t), \quad \eta(L,t) = h_2(t), \\
&u(0,t) = v_1(t), \quad u(L,t) = v_2(t), \\
&\eta(x,0) = f_1(x), \quad u(x,0) = f_2(x).
\end{aligned} \tag{2}
$$

The consistency requirements are the obvious ones dictated by continuity considerations, namely

$$
h_1(0) = f_1(0), \quad h_2(0) = f_1(L), \quad v_1(0) = f_2(0), \quad v_2(0) = f_2(L). \tag{3}
$$

Key words and phrases. Boussinesq system for water waves, traveling waves, multi-pulsed solutions, homoclinic orbits.

It is shown in [3] that this initial- and boundary-value problem (1)-(2)-(3) is well-posed.

Theorem 1. *Let* $\mathbf{f} = (f_1, f_2) \in C^2(0,L)^2$, $\mathbf{h} = (h_1, h_2), \mathbf{v} = (v_1, v_2) \in C^1(0,T)^2$ *for some* T, $L > 0$. *Suppose* $\mathbf{f}$, $\mathbf{h}$ *and* $\mathbf{v}$ *to satisfy the compatibility conditions (3). Define* $\|\mathbf{f}\| = \max\{\|f_1\|_{C(0,L)}, \|f_2\|_{C(0,L)}\}$, $\|\mathbf{h}\| = \max\{\|h_1\|_{C(0,T)}, \|h_2\|_{C(0,T)}\}$, *and* $\|\mathbf{v}\| = \max\{\|v_1\|_{C(0,T)}, \|v_2\|_{C(0,T)}\}$. *Then there is a* $T_0 = T_0(T, L, ||\mathbf{f}||, ||\mathbf{h}||, ||\mathbf{v}||) \leq T$ *and an unique solution pair* (η, u) *in* $C^1(0, T_0; C^2(0,L))^2$ *that satisfies (1).*

To our knowledge, this is the only existence and uniqueness result available for this physically relevant initial- and non-homogeneous Dirichlet-boundary-value problem for a system describing two-way propagation of water waves. We note that one of the difficulties with the KdV-equation posed on a bounded domain is that it requires three boundary conditions, which is physically unfavorable.

2. A Non-trivial Exact Solution

Letting $\xi = x - kt$ where k is the speed of the traveling-wave solution, one can write the solution in the form $\eta(x,t) = \eta(\xi)$ and $u(x,t) = u(\xi)$. Supposing the solution decays at large distance from its crests or troughs, it is natural to impose the boundary conditions

$$(\eta^{(n)}(\xi), u^{(n)}(\xi)) \to 0 \text{ as } \xi \to \pm\infty, \text{ for at least } n = 0, 1, 2. \tag{4}$$

In the moving frame, the functions η and u satisfy the ordinary differential equations

$$\begin{aligned} &\frac{1}{6}k\eta'' + u - k\eta + u\eta = 0, \\ &\frac{1}{6}ku'' - ku + \eta + \frac{1}{2}u^2 = 0, \end{aligned} \tag{5}$$

where the derivatives are with respect to ξ. It is clear that a homoclinic solution about the origin of (5), will lead to a traveling-wave solution of (1). Therefore, the problem of finding traveling-wave solutions becomes that of finding homoclinic orbits of (5). It is easy to check that system (5) can be written as a single equation

$$u'''' + \frac{12}{k}uu'' - 12u'' + \frac{6}{k}(u')^2 + \frac{18}{k^2}u^3 - \frac{54}{k}u^2 + 36u - \frac{36}{k^2}u = 0. \tag{6}$$

Instead of solving $u(\xi)$ from (6) directly, which is very difficult if it is not impossible, the technique used by Kichennassamy and Olver in [7] for a single 5th-order equation is adopted here. In the present case, the ordinary differential equation has coefficients depending on k which are part of the unknowns. Assuming that $u(\xi)$ can be reconstructed as the solution of a simple first-order ordinary differential equation

$$w(u) = (u')^2, \tag{7}$$

one finds that for $u' \neq 0$

$$(u')^2 = w, \quad u'' = \frac{1}{2}w', \quad u'''' = \frac{1}{2}ww''' + \frac{1}{4}w'w'', \tag{8}$$

where the primes on w indicate derivatives with respect to u. For a solution u of the form

$$u(\xi) = A\,\mathrm{sech}^2(\lambda\,\xi), \quad \lambda > 0, \tag{9}$$

the corresponding function $w(u)$ must be a cubic polynomial:

$$w(u) = 4\lambda^2(u^2 - \frac{1}{A}u^3).$$

Substituting $w(u)$ into (8) and using the resulting relationships in (6), one obtains a degree-three homogeneous polynomial in u which has to be zero. In order for u to be a nontrivial solution, all the coefficients have to be zero which yields an algebraic system of equations

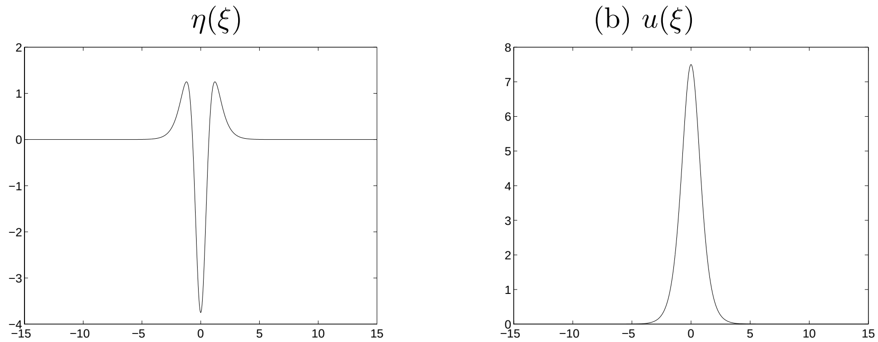

FIGURE 1. An exact solution of regularized Boussinesq system ($k = 2.5$)

on k, λ and A. Substituting the solution $A = \frac{15}{2}, \lambda^2 = \frac{9}{10}$ and $k^2 = \frac{25}{4}$ of the algebraic system into (9) and (5) one finds a pair of exact traveling-wave solutions

$$
\begin{aligned}
u(x,t) &= \pm\frac{15}{2}\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right), \\
\eta(x,t) &= \frac{15}{4}\left(2\mathrm{sech}^2\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right) - 3\mathrm{sech}^4\left(\frac{3}{\sqrt{10}}(x + x_0 \mp \frac{5}{2}t)\right)\right).
\end{aligned}
\tag{10}
$$

The right-moving exact solution is plotted in Figure 1 and it is clear that η has a trough with depth 3.75.

3. Existence of solitary-wave solutions for $k > 1$

Since the full Euler equations have solitary-wave solutions [2, 1] and the system (1) is its approximations, and as solitary waves play a central role in evolution of certain types of disturbances, it is important to show that the system (1) has solitary-wave solutions. The following theorem is proved in [5].

Theorem 2. *The regularized Boussinesq system possesses a homoclinic orbit about the origin with any phase speed $k > 1$ and it has the additional properties:*

(a) $(u(\xi), \eta(\xi))$ *is an even solution, namely* $(u(\xi), \eta(\xi)) = (u(-\xi), \eta(-\xi))$, $(u'(\xi), \eta'(\xi)) = -(u'(-\xi), \eta'(-\xi))$ *for any* $\xi \in I\!R$,

(b) $(u(\xi), \eta(\xi))$ *is positive and monotonically decreasing for* $\xi > 0$,

(c) $\dfrac{k^2 - 1}{k} < |u(\xi)|_{L^\infty} = u(0) < 2(k-1)$; $2(k-1) < |\eta(\xi)|_{L^\infty} = \eta(0) < k^2 - 1$,

(d) $u(0) = 2(k-1) + O((k-1)^2)$; $\eta(0) = 2(k-1) + O((k-1)^2)$ *as* $k \to 1$,

(e) $u(0) = \dfrac{\eta(0)(k + \sqrt{k^2 - 1 - \eta(0)})}{1 + \eta(0)}$,

(f) $(k - \sqrt{k^2-1})u(\xi) < \eta(\xi) < (k + \sqrt{k^2-1})u(\xi)$.

It is worth to note that in contrast to the KdV-type equations where a traveling-wave solution satisfying (4) is unique for a fixed k, such solutions are not unique for the regularized Boussinesq system. In particular, an analytical solution (10) and a solitary-wave solution exist for phase speed $k = 2.5$. For phase speed $k > 2.5$, the non-uniqueness is clearly seen in our numerical results presented in next section.

4. Even multi-pulsed solutions

To obtain a general picture of the multi-pulsed solutions of (1), it is helpful to study whole solution branches obtained by continuation from a given approximate solution. The

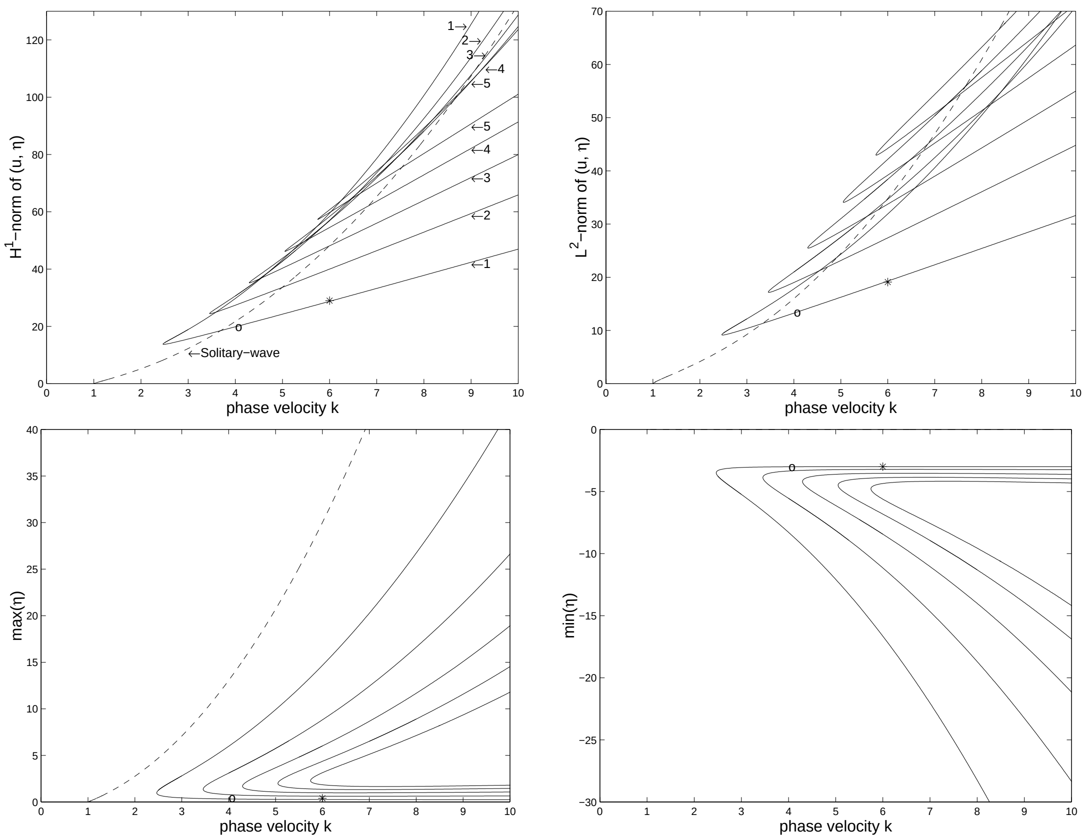

FIGURE 2. Bifurcation diagrams.

solution branches can be calculated numerically by using a toolbox HomCont in AUTO [6, 4] and the precise quantitative properties of these branches can be analyzed.

In Figure 2, the bifurcation diagrams of multi-pulsed solutions with 1 to 5 troughs (solid lines) and the solitary-wave solutions (dashed line) are presented. The H^1-norm and the L^2-norm of the solution (u, η), the height of the crests $\max \eta(\xi)$ and the depth of the troughs $\min \eta(\xi)$ with respect to k are plotted. The numbers marked in Figure 2 indicate the number of troughs.

From Figure 2, one can see that the only branch which continues through the entire range $(1, +\infty)$ is the solitary-wave branch. Other branches possess a turning point at a point $k_c > 1$ and the branches cease to exist below that. There is an one-parameter family of solitary waves with wave height ranging from 1 to $+\infty$ which confirms the result in Theorem 2. Note that the multi-pulsed solutions exist when the phase speed and amplitude are large, and the depth of the troughs is at least 2.5 measured from the still water level. Our numerical results show that the solutions with one trough exist for $k \geq 2.47$ and the waves with two troughs exist for $k \geq 3.46$. A solution from the "upper" branch has larger H^1-norm, and also has larger L^2-norm and larger wave height but smaller magnitude of velocity u, than the corresponding solution from the lower branch with the same k. The velocity u is always positive. At any given k, if there is a solution with N_0 troughs, then there is a solution with N troughs where N is any positive integer smaller than N_0.

In Figure 3, five three-trough solutions along the branch are presented. The locations of these solutions in Figure 2 can be determined by the values of the phase speed k.

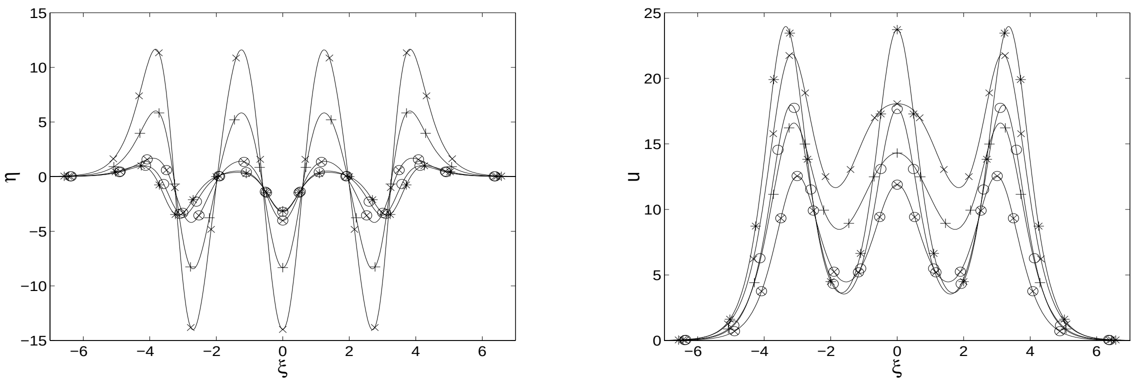

FIGURE 3. Solutions along the three-trough solution branch in Figure 2, "$\otimes$" is near the turning point with $k = 4.3$, "+" and "x" are from the upper branch with $k = 6$ and $k = 8$, "o" and "$*$" are from the lower branch with $k = 6$ and $k = 8$.

Since the KdV-type equations do not possess multi-pulsed solutions, our results show that there is a significant difference between the one-way model equations and the two-way model systems, although they are formally approximations of the same order to the Euler equation (cf. [3]). However, figure 2 reveals that the magnitude of all multi-trough solutions is quite large, so the difference happens in regimes where the solution is no longer small. Thus, our results do not call into question the validity of one or the other of KdV-type equations and Boussinesq systems, but does emphasize that when these models are used to simulate waves in a practical situation, they cannot both be accurate for large amplitude waves. Indeed, this interesting anomaly can be viewed as another cautionary lesson; with the conclusion that it is best to use these models well within the range they were derived to describe.

It is also interesting to note that even though the regularized Boussinesq system has the same formal accuracy as the uni-directional wave equations such as the KdV-equation and the regularized long wave equation, the solution set of traveling waves of this bi-directional wave system is much more complex.

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E-mail address: chen_m@math.psu.edu