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Boussinesq Equations and Other Systems for Small-Amplitude Long Waves in Nonlinear Dispersive Media. I: Derivation and Linear Theory

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Summary. Considered herein are a number of variants of the classical Boussinesq system and their higher-order generalizations. Such equations were first derived by Boussinesq to describe the two-way propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal. These systems arise also when modeling the propagation of long-crested waves on large lakes or the ocean and in other contexts. Depending on the modeling of dispersion, the resulting system may or may not have a linearization about the rest state which is well posed. Even when well posed, the linearized system may exhibit a lack of conservation of energy that is at odds with its status as an approximation to the Euler equations. In the present script, we derive a four-parameter family of Boussinesq systems from the two-dimensional Euler equations for free-surface flow and formulate criteria to help decide which of these equations one might choose in a given modeling situation. The analysis of the systems according to these criteria is initiated.

Key words. water waves, two-way propagation, Boussinesq systems, local well-posedness, global well-posedness

1. Introduction

In a continuum approximation, waves on the surface of an ideal fluid under the force of gravity are governed by the Euler equations. These are expected to provide a good model of irrotational waves on the surface of water, say, in situations where dissipative and surface tension effects may be safely ignored. In many field and laboratory studies and in engineering applications, the full Euler equations appear more complex than is necessary for the modeling situation at hand, and consequently there have appeared many approximate models applying to restricted physical regimes.

A regime that arises in practical situations is that of waves in a channel of approximately constant depth h that are uniform across the channel, and which are of small amplitude and long wavelength, and such that the associated nonlinear and dispersive effects are balanced. Similar waves appear as long-crested disturbances on larger bodies of water. If A connotes a typical wave amplitude and ℓ a typical wavelength, the conditions just mentioned amount to

$$\alpha = \frac{A}{h} \ll 1, \qquad \beta = \frac{h^2}{\ell^2} \ll 1, \qquad S = \frac{\alpha}{\beta} = \frac{A\ell^2}{h^3} \approx 1. \tag{1.1}$$

In the 1870s, Boussinesq derived some model evolution equations which are applicable in principle to describe motions that are sensibly two-dimensional and which have the form of a perturbation of the one-dimensional wave equation. Perhaps the best known is the equation

$$w_{tt} = w_{xx} + (w^2)_{xx} + w_{xxxx}, \tag{1.2}$$

or its regularized version

$$w_{tt} = w_{xx} + (w^2)_{xx} + w_{xxtt} \tag{1.3}$$

(see Boussinesq [29], Keulegan & Patterson [44], Ursell [59], Benjamin et al. [9], Kano & Nishida [42], Bona & Sachs [21]). The variables appearing in (1.2) and (1.3) are dimensionless but unscaled, so w itself is of order α; w_x, w_t are of order $\alpha\beta^{\frac{1}{2}}$; w_{xx}, w_{xt}, and w_{tt} are of order $\alpha\beta$, and so on. Contrary to what one might guess, these equations are derived directly from the Eulerian formulation of the water wave problem using an assumption, among others, that the waves travel only in one direction. (It is worth noting that the assumption of unidirectionality is not needed in the derivation for the Lagrangian formulation, as Craig showed in [36, p. 799]. This configuration is not under consideration here, however.) In consequence, (1.2) and (1.3) are formally comparable to the well-known model put forth by Korteweg and de Vries [45], but also written down by Boussinesq (see [29]). Boussinesq [28] also derived from the Euler equations a system of two coupled equations,

$$\begin{aligned} \eta_t + w_x + (w\eta)_x &= 0, \\ w_t + \eta_x + ww_x + \frac{1}{3}\eta_{xtt} &= 0, \end{aligned} \tag{1.4}$$

or its regularization (cf. Whitham [61]),

$$\begin{aligned} \eta_t + w_x + (w\eta)_x &= 0, \\ w_t + \eta_x + ww_x - \frac{1}{3}w_{xxt} &= 0, \end{aligned} \tag{1.5}$$

which are free of the presumption of unidirectionality that is the hallmark of (1.2), (1.3), and Korteweg–de Vries–type equations as models of surface-wave propagation. One therefore expects that these Boussinesq systems will have more intrinsic interest than the one-way models (an appellation we append to any model derived under the assumption of unidirectionality of propagation) on account of their considerably wider range of potential applicability. Because of the very rich mathematical theory that obtains for unidirectional models like the Korteweg–de Vries equation, and which for the most part seems to have no counterpart for systems of equations, much of the existing mathematical discussion has been centered around one-way models. However, it is the Boussinesq systems to which the present paper is devoted.

As with one-way models, there are potentially many different but formally equivalent Boussinesq systems. As explained by Bona and Smith [26] (and see also Section 2 of the present script), the plethora of possibilities is owed in the main to the fact that the lower-order relations can be used systematically to alter the higher-order terms without disturbing the formal level of approximation, and to the considerable choice of dependent variables available for the description of the motion. Despite their formal equivalence as models for small-amplitude long waves, these systems may have rather different mathematical properties.

It is our principal purpose to examine some of the properties of a family of Boussinesq systems of the form

$$\begin{aligned} \eta_t + w_x + (w\eta)_x + aw_{xxx} - b\eta_{xxt} &= 0, \\ w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{xxt} &= 0, \end{aligned} \tag{1.6}$$

which are all first-order approximations (in the small parameters α and β introduced in (1.1)) to the Euler equations. We also aim to derive Boussinesq systems of the form

$$\begin{aligned} \eta_t - b\eta_{xxt} + b_1\eta_{xxxxt} =& - w_x - (\eta w)_x - aw_{xxx} \\ &+ b(\eta w)_{xxx} - \left(a + b - \frac{1}{3}\right)(\eta w_{xx})_x - a_1 w_{xxxxx}, \\ w_t - dw_{xxt} + d_1 w_{xxxxt} =& - \eta_x - c\eta_{xxx} - ww_x - c(ww_x)_{xx} - (\eta\eta_{xx})_x \\ &+ (c + d - 1)w_x w_{xx} + (c + d)ww_{xxx} - c_1\eta_{xxxxx}, \end{aligned} \tag{1.7}$$

which are second-order approximations. The derivation of these model systems from the full Euler equations is addressed in Section 2.

The parameters $a, b, c, \ldots$ appearing in (1.6) and (1.7) are not independently specified. As will appear in Section 2, the constants in (1.6) obey the relations

$$\begin{aligned} a + b &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \\ c + d &= \frac{1}{2}(1 - \theta^2) \geq 0, \\ a + b + c + d &= \frac{1}{3}, \end{aligned} \tag{1.8}$$

the last of which follows from the first two, where $\theta \in [0, 1]$ specifies which horizontal velocity variables w represents ($w = w_\theta$ is the nondimensional horizontal velocity in the flow corresponding to the physical velocity at height θh where h, as above, is the undisturbed depth of the liquid). As will appear presently, the constants in (1.6) arise naturally in the form

$$\begin{aligned} a &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \qquad & b &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda), \\ c &= \frac{1}{2}(1-\theta^2)\mu, \qquad & d &= \frac{1}{2}(1-\theta^2)(1-\mu), \end{aligned} \tag{1.9}$$

where $\lambda, \mu \in \mathbb{R}$ are modeling parameters (parameters that do not possess a direct physical interpretation as does θ). Of course (1.8) follows from (1.9), but the significance of λ and μ will become apparent in Section 2. Similar but more elaborate restrictions apply to the parameters in (1.7).

A few specializations of (1.6) have appeared already in the literature. In addition to the regularized version (1.5) of the classical Boussinesq system, Kaup [43], Bona and Smith [26], and Bona and Chen [14] have put forward models that have attracted further attention (see e.g., [46], [53], [35] for the Kaup system; [62], [57], [58] for the Bona-Smith system, and [33] for the system studied in [14]). These and some other interesting specializations are listed now to give concrete form to the discussion:

- Classical Boussinesq system ($\theta^2 = \frac{1}{3},\ \lambda$ arbitrary, $\mu = 0$)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x &= 0, \\ w_t + \eta_x + ww_x - \frac{1}{3}w_{xxt} &= 0; \end{aligned} \tag{1.10}$$

- Kaup system ($\theta^2 = 1,\ \lambda = 1,\ \mu$ arbitrary)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x + \frac{1}{3}w_{xxx} &= 0, \\ w_t + \eta_x + ww_x &= 0; \end{aligned} \tag{1.11}$$

- Bona–Smith system ($\theta^2 = (\frac{4}{3} - \mu)/(2-\mu),\ \lambda = 0, \mu < 0$ arbitrary)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x - b\eta_{xxt} &= 0, \\ w_t + \eta_x + ww_x + c\eta_{xxx} - bw_{xxt} &= 0, \end{aligned} \tag{1.12}$$

  where, in the notation of (1.6),

$$b = d = \frac{1-\mu}{3(2-\mu)} > 0 \quad \text{and} \quad c = \frac{\mu}{3(2-\mu)} < 0;$$

- Coupled BBM-system ($\theta^2 = \frac{2}{3},\ \lambda = 0,\ \mu = 0$)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x - \frac{1}{6}\eta_{xxt} &= 0, \\ w_t + \eta_x + ww_x - \frac{1}{6}w_{xxt} &= 0; \end{aligned} \tag{1.13}$$

- Coupled K-dV system ($\theta^2 = \frac{2}{3},\ \lambda = 1,\ \mu = 1$)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x + \frac{1}{6} w_{xxx} &= 0, \\ w_t + \eta_x + w w_x + \frac{1}{6} \eta_{xxx} &= 0; \end{aligned} \tag{1.14}$$

- Coupled K-dV–BBM system ($\theta^2 = \frac{2}{3},\ \lambda = 1,\ \mu = 0$)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x + \frac{1}{6} w_{xxx} &= 0, \\ w_t + \eta_x + w w_x - \frac{1}{6} w_{xxt} &= 0; \end{aligned} \tag{1.15}$$

- Coupled BBM –K-dV system ($\theta^2 = \frac{2}{3},\ \lambda = 0,\ \mu = 1$)

$$\begin{aligned} \eta_t + w_x + (w\eta)_x - \frac{1}{6} \eta_{xxt} &= 0, \\ w_t + \eta_x + w w_x + \frac{1}{6} \eta_{xxx} &= 0. \end{aligned} \tag{1.16}$$

Some preliminary commentary is warranted concerning some of the preceding models. Among other things, the Kaup system was featured in Craig's comparison [36] with the full Euler equations for two-dimensional water waves. On the other hand, as will appear in Section 3, Kaup's version of Boussinesq's system is linearly ill-posed for the initial-value problem

$$\eta(x, 0) = \varphi(x), \qquad w(x, 0) = \psi(x), \tag{1.17}$$

for $x \in \mathbb{R}$. The system (1.14), while referred to as a coupled K-dV system, is not at all the same as a pair of K-dV equations for the two dependent variables η and w coupled through mixed nonlinear effects. The operator $\partial_t \pm \partial_x^3$ which is characteristic of the K-dV equation does not appear in its pure form. Nevertheless, because of the appearance of third-order spatial derivatives in the dispersive term, this appellation seems useful. A further change of dependent variables does render (1.14) into the form of a pair of K-dV equations coupled through nonlinearity. System (1.13), on the other hand, is exactly a pair of nonlinear BBM equations or regularized long-wave equations (see Benjamin et al. [9]) coupled through nonlinear effects. The special case of the system (1.12) in which the parameters take their limiting values $\theta^2 = 1$, $b = d = \frac{1}{3}$ and $c = -\frac{1}{3}$, as $\mu \to -\infty$ was considered by Bona and Smith [26]. Notice that if reference is made to the definition of c and d in (1.9), the value $\theta = 1$ would imply $c = d = 0$. However, if one takes the limit as $\mu \to -\infty$, the combination $-\frac{1}{3}(\eta_{xxx} + w_{xxt})$ remains in the formal limit. When referred to the size of dependent variables η and w, this quantity has relative size of order β^2, as will be apparent in Section 2. Consequently, its appearance plays no role at a formal level, though Bona and Smith appended it to gain a useful mathematical advantage. It will be seen later that their analysis of (1.12) with $\mu = -\infty$ may be adapted to the system (1.12) for any value of $\mu \leq 0$. Of course, when $\mu = 0$, the coupled BBM-system (1.13) is recovered.

One might wonder if there is a need to have available more than one of the three-parameter family displayed in (1.6). Experience with the unidirectional models K-dV, BBM, the equation

$$u_t + u_x + uu_x + u_{xtt} = 0$$

arising readily from Boussinesq's original system (1.4), and the model

$$u_t + u_x + uu_x - u_{ttt} = 0$$

shows that, while they are known rigorously to be equivalent models for initial data obeying (1.1), (see [19], [2], [25], [13]), they nevertheless possess different mathematical properties. These differences mean that in both theoretical and practical studies, one can and does choose the model in accordance with the particular questions under investigation. The same flexibility is desirable for two-way models.

This latter remark suggests another point that needs explication. Namely, what general properties must a model such as one of those posited in (1.6) or (1.7) possess to be considered a desirable prospect for approximating water waves? Here is a list of aspects that seem likely to be wanted in both theoretical and practical situations.

(i) Well-posedness of initial-value problems; for the linear equation, for the nonlinear equation, locally in time, globally in time. Well-posedness for initial-boundary-value problems appropriate to generation of waves by a wavemaker or as a model for waves impinging on a quiescent stretch of the medium (e.g., waves approaching the shore from deep water).

(ii) Energy preservation; in each Fourier mode for the linearized equations, of the total energy for the nonlinear problem.

(iii) Existence of solitary-wave solutions at least for small amplitudes.

(iv) Relative ease of constructing accurate, efficient numerical schemes for approximating solutions of interesting initial-boundary-value problems.

(v) Accurate comparison with laboratory and field data in the range of parameters where the model is formally deemed to be valid.

These criteria deserve some discussion. As is well known, and will be seen again in our Section 2, all these models are derived from the full Euler equations for two-dimensional water waves under the force of gravity by way of truncating a formal expansion of one sort or another (e.g., a Taylor expansion of the velocity potential as here, an operator expansion of the Hamiltonian [37], [51] or the like). Underlying such formalisms is existence and regularity of the Euler flow corresponding at least to physically relevant data. Indeed, in the work of Craig [36], Kano and Nishida [42], and the more recent discussion of Schneider and Wayne [54], theoretical justification has been provided for some aspects of the passage from the Euler equations to some of the one- and two-way models considered here. Related comparisons between coupled BBM and the one-way BBM model are given in the forthcoming work [1]. In spaces of functions analytic in a strip, such comparisons also follow from the ideas in [42], albeit over limited time scales. As the overlying equations are linearly globally well-posed, and at least nonlinearly locally well-posed, requiring the same of the models is not asking too much. Whether or not the models need to be globally well-posed is perhaps not so clear. They should be well-posed at least on a time scale of order $\frac{1}{\alpha^2}$. Moreover, they are only expected to be valid where they are smooth, so global well posedness is certainly convenient if not necessary. In any event, a globally well-posed model is less likely to give trouble when forming an associated numerical scheme to simulate solutions as in point (iv) above.

As far as preservation of energy is concerned, the linear Euler equations preserve energy mode by mode. Thus the total energy in each Fourier mode remains unchanged as time evolves. While this is no longer true for the nonlinear problem, the total energy is preserved under the passage of time for various initial-boundary-value problems for the Euler equations. Maintaining this property in the model seems a very good idea both as regards theoretical issues and the potential accuracy of an associated numerical scheme in approximating Euler flows.

Solitary waves are known to play a distinguished role in the evolution of general initial disturbances for the K-dV equation. This is a rigorous consequence of the inverse scattering theory for the K-dV equation. Numerical simulations show similar results for the BBM equation and indeed for a whole range of unidirectional, nonlinear, dispersive wave equations (see, e.g., [20], [17], [41], [16]). Laboratory experiments (see [63], [38], [18], [50]) show the same results for real water waves, in both unidirectional and bidirectional situations. Moreover, the full Euler equations admit solitary-wave solutions (see [47], [48], [3], [8]). This preponderance of evidence suggests that in choosing a model from the class in (1.6), attention might reasonably be restricted to those having solitary waves of their own.

Of course the final point is the ultimate test of any mathematical model. However, such comparisons can be subtle to make because in laboratory or field situations, effects not accounted for by the models often prove to be significant. For example, viscous effects cannot be ignored except perhaps on planetary spatial scales (see [18], [38], [39], [31], [49]).

All the foregoing commentary centered around (1.6) applies mutatis mutandis to the equations appearing in (1.7).

After deriving the systems (1.6) and (1.7) in Section 2, attention is turned to Criterion (i). Such evolution equations are unlikely to be well-posed unless they are linearly well posed. Section 3 focuses on the latter question, namely which of the models in (1.6) are well posed when the quadratic terms are dropped from consideration.

Of course, the fact of linear well-posedness does not by itself imply that the associated nonlinear initial-value problem will be well posed, nor does it assure that interesting initial-boundary-value problems arising in applications will be properly posed. Nevertheless, the successful study of linear well-posedness is a step in the direction of this understanding. Somewhat surprisingly, though the evolution equations appearing in (1.6) are all formally equivalent, they possess strikingly varied linear well-posedness properties. Indeed, some are not linearly well-posed even in the Hilbert-space setting of L_2–based function classes. Among choices that are well posed in Hadamard's classical sense, there are more delicate aspects such as spurious growth in energy that make some unsuitable as models of the underlying physical phenomena, even though well posed.

Section 4 is concerned with the more delicate issue of well-posedness of these linear systems in L_p–spaces and some of their interesting dispersive blow-up properties. These aspects have to do with whether or not the solution maintains regularity as t increases (see [23] for related work on unidirectional models in this context). Also, L_p-theory $p \neq 2$, is sometimes very helpful when analyzing nonlinear problems.

The higher-order correct systems in (1.7) are considered in Section 5 and a preliminary foray made into their analysis. This project has independent interest, but our main impetus was the need for a better model for the two-way propagation of waves in connection with some surface-wave experiments in a laboratory setting and in field studies of wave-generated sediment transport. The second-order correct models both have increased accuracy, and, just as important, are formally valid on time and spatial scales an order of magnitude longer than first-order correct models. These points are explained in more detail in Section 2.

While the second-order correct systems are more involved than the first-order systems, the level of complexity that arises is still well below that exhibited by the full, two-dimensional Euler equations with a free boundary, as regards mathematical analysis, imposition of boundary conditions, and especially as regards the prospect for associated algorithms for the numerical approximation of waves. Especially with an eye to practical implementation, discussion of the systems in (1.7) seems amply justified.

The paper concludes with a summary and a short discussion of related lines of inquiry.

In Part II, which will appear separately, we deal with the nonlinear initial-value problems corresponding to those systems which are linearly well-posed. A theory of local well-posedness will be developed for all the linearly well-posed systems, though the function classes in which well-posedness obtains vary from equation to equation. Some of the relevant systems possess a Hamiltonian structure, and this together with local well-posedness sometimes implies global well-posedness of the initial-value problem at least for physically relevant initial data.

Finally, it is worth note that the models derived in this paper depend upon small parameters. For instance, in (1.11), one has $(w\eta)_x << \eta_t$, etc. Thus the equations have not been made independent of all the small parameters. Indeed, this is not an important point in the present perspective, as our models are intended for the description of realistic waves with finite rather than asymptotically small amplitudes.

2. Model Systems

The origin of the systems (1.6) and (1.7) of partial differential equations is explained in this section. The methods are standard, but some of these equations are derived for the first time. They are usefully set forth to provide a context for the further modeling considerations to follow.

Let Ω_t be the domain in $\mathbb{R}^3$ which is occupied by an inviscid, incompressible fluid at time t. The system describing the motion of such a fluid is the classical Euler equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} + \nabla p = -g\vec{k}, \quad \text{in } \Omega_t, \tag{2.1}$$

$$\nabla \cdot \vec{v} = 0, \quad \text{in } \Omega_t, \tag{2.2}$$

where g denotes the acceleration of gravity, $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$ denotes the velocity field, $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors along the $x-$, $y-$, and $z-$axis, respectively, in $\mathbb{R}^3$, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$, and p denotes the pressure field. Assuming the initial velocity field is irrotational so that $\nabla \times \vec{v} = 0$, Helmholtz's vorticity principle implies that the velocity field remains irrotational. So long as a regular solution obtains, and hence that

$$\vec{v} = \nabla\phi \tag{2.3}$$

for some potential function $\phi = \phi((x, y, z), t)$. It follows from (2.2) that ϕ satisfies Laplace's equation

$$\Delta\phi = 0 \tag{2.4}$$

in Ω_t, for each t.

View the boundary of Ω_t as consisting of two parts: the fixed surface located at $z = -h(x, y)$, and the free surface $z = \eta(x, y, t)$. The domain is taken to be unbounded in the horizontal directions so that lateral boundaries do not intrude at this stage of the analysis. Note that $\eta(x, y, t)$ is a fundamental unknown of the problem. On the fixed portion of the boundary, the condition of impermeability (no flow through the solid boundary) $\vec{v} \cdot \vec{n} = 0$ is satisfied with $\vec{n}$ being the normal direction of the surface, which is to say

$$\phi_x h_x + \phi_y h_y + \phi_z = 0, \quad \text{on } z = -h(x, y). \tag{2.5}$$

Since the free surface is a material surface, it satisfies the kinematic condition $\frac{D(\eta - z)}{Dt} = 0$, where $\frac{D}{Dt}$ is the usual material derivative $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. In consequence, we have

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0, \quad \text{on } z = \eta(x, y, t). \tag{2.6}$$

Assuming the pressure on the free surface is equal to the ambient air pressure, it follows from (2.1) and (2.3) that the Bernoulli condition (cf. Whitham [61], pp. 432)

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz = 0, \quad \text{on } z = \eta(x, y, t), \tag{2.7}$$

is satisfied on the free surface as well.

Summarizing the equations for the unknown functions η and ϕ, they consist of (2.4), (2.5), (2.6), and (2.7). This system is challenging to treat either numerically or analytically because Ω_t is changing with time through the evolution of η and the boundary conditions (2.6) and (2.7) on the free surface are nonlinear. Consider now the simpler case of an open channel in which the fluid motion is irrotational, inviscid and uniform in the cross-channel direction. Suppose the bottom of the channel to be flat and horizontal and let h denote the depth of the liquid in its undisturbed state. Then the preceding formulation reduces to

$$\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0, \quad \text{in } -h < z < \eta(x, t),\\
\phi_z &= 0, \quad \text{on } z = -h,\\
\eta_t + \phi_x \eta_x - \phi_z &= 0, \quad \text{on } z = \eta(x, t),\\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz &= 0, \quad \text{on } z = \eta(x, t),
\end{aligned}$$

where the undisturbed free surface is located at $z = 0, -\infty < x < +\infty$, for all $t \geq 0$. This system of equations is posed together with suitable boundary conditions as $x \to \pm\infty$ and an initial condition at $t = 0$.

Consider now the regime mentioned before where the free surface has small amplitude, long wavelength and the classical Stokes number $S = \frac{\alpha}{\beta}$ is of order one. In this circumstance, the two small parameters α and β may be treated on an equal footing. Choosing β to be the primary parameter, we seek to write solutions of (2.4)–(2.7) in a Taylor series with respect to β, and thereby to obtain approximate equations corresponding to orders of accuracy characterized by β^n for $n = 1, 2, \ldots$.

The procedure is most transparent when working with the variables scaled in such a way that the dependent quantities and the initial data appearing in the initial-value problem are all of order one, while the assumptions about small amplitude and long wavelength appear explicitly connected with small parameters in the equations of motion. Such consideration leads to the scaled, dimensionless variables

$$x = \ell\tilde{x}, \qquad z = h(\tilde{z} - 1), \qquad \eta = A\tilde{\eta}, \qquad t = \ell\tilde{t}/c_0, \qquad \phi = gA\ell\tilde{\phi}/c_0, \quad (2.8)$$

where $c_0 = \sqrt{gh}$. In these variables, the last set of equations becomes the system

$$\beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0, \quad \text{in } 0 < \tilde{z} < 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{2.9}$$

$$\tilde{\phi}_{\tilde{z}} = 0, \quad \text{on } \tilde{z} = 0, \tag{2.10}$$

$$\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\phi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \frac{1}{\beta}\tilde{\phi}_{\tilde{z}} = 0, \quad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{2.11}$$

$$\tilde{\eta} + \tilde{\phi}_{\tilde{t}} + \frac{1}{2}\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{1}{2}\frac{\alpha}{\beta}\tilde{\phi}_{\tilde{z}}^2 = 0, \quad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x}, \tilde{t}), \tag{2.12}$$

for $-\infty < \tilde{x} < \infty$, $\tilde{t} > 0$. For clarity, we drop the tilde over the new variables in our further machinations.

The next procedure, which is a standard one (cf. Peregrine [52], Benjamin [7], and Whitham [61], Ch. 13), begins by representing ϕ as a formal expansion,

$$\phi(x, z, t) = \sum_{m=0}^{\infty} f_m(x, t) z^m.$$

Demanding that ϕ formally satisfy Laplace's equation (2.9) leads to the recurrence relation

$$(m + 2)(m + 1) f_{m+2}(x, t) = -\beta(f_m(x, t))_{xx}, \tag{2.13}$$

for $m = 0, 1, 2, \ldots$. Let $F = f_0$ denote the velocity potential at the bottom $z = 0$ and use (2.13) repeatedly to obtain

$$f_{2k}(x, t) = \frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}F(x, t)}{\partial x^{2k}}, \qquad k = 0, 1, 2, \ldots.$$

Equation (2.10) implies that $f_1(x, t) = 0$, so

$$f_{2k+1}(x, t) = 0, \qquad k = 0, 1, 2, \ldots,$$

and therefore

$$\phi(x, z, t) = \sum_{k=0}^{\infty} \frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}F(x, t)}{\partial x^{2k}} z^{2k}.$$

Substitute the latter representation into (2.11) and (2.12) to obtain a system of equations for $\eta(x,t)$ and $F(x,t)$, namely

$$\eta_t + \alpha\eta_x \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} z^{2k} - \sum_{k=1}^{\infty} \frac{(-1)^k \beta^{k-1} 2k}{(2k)!} \frac{\partial^{2k} F}{\partial x^{2k}} z^{2k-1} = 0,$$

$$\eta + \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k} \partial t} z^{2k} + \frac{1}{2}\alpha \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} z^{2k} \right\}^2 + \frac{1}{2}\frac{\alpha}{\beta} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^k \beta^k 2k}{(2k)!} \frac{\partial^{2k} F}{\partial x^{2k}} z^{2k-1} \right\}^2 = 0,$$

on

$$z = 1 + \alpha\eta(x,t).$$

Substituting the value of z into the last two equations leads to

$$\eta_t + \alpha\eta_x \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} (1+\alpha\eta)^{2k} \right\} \beta^k + \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k+1)!} \frac{\partial^{2k+2} F}{\partial x^{2k+2}} (1+\alpha\eta)^{2k+1} \right\} \beta^k = 0, \tag{2.14}$$

and

$$\eta + \sum_{k=0}^{\infty} \left\{ \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k} \partial t} (1+\alpha\eta)^{2k} \right\} \beta^k + \frac{1}{2}\alpha \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\partial^{2k+1} F}{\partial x^{2k+1}} (1+\alpha\eta)^{2k} \beta^k \right\}^2 + \frac{1}{2}\alpha\beta \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\partial^{2k+2} F}{\partial x^{2k+2}} (1+\alpha\eta)^{2k+1} \beta^k \right\}^2 = 0. \tag{2.15}$$

Account is now taken of the formal order of the various terms appearing in (2.14) and (2.15). The parameters α and β have the same small order, while F and η have been scaled so that they and their partial derivatives are of order one. Keeping only the terms in (2.14) and (2.15) which are of lowest order, there obtains the system

$$\eta_t + \frac{\partial^2 F}{\partial x^2} = \text{terms linear in } \alpha, \beta,$$

$$\eta + \frac{\partial F}{\partial t} = \text{terms linear in } \alpha, \beta.$$

Differentiate the second equation with respect to x and let $\frac{\partial F(x,t)}{\partial x} = u(x,t)$, the scaled horizontal velocity at the bottom of the channel. With this new dependent variable at hand, the last equation becomes

$$\eta_t + u_x \ = \ \text{terms linear in } \alpha, \beta, \tag{2.16}$$

$$\eta_x + u_t \ = \ \text{terms linear in } \alpha, \beta, \tag{2.17}$$

which is simply the linear wave equation if the terms of formal order α and β are ignored.

The next order of approximation keeps all the terms in (2.14)–(2.15) which are at most linear in α or β. This leads to the system

$$\begin{aligned} \eta_t + \frac{\partial^2 F}{\partial x^2} + \alpha\eta_x \frac{\partial F}{\partial x} + \alpha\eta \frac{\partial^2 F}{\partial x^2} - \frac{1}{6}\beta \frac{\partial^4 F}{\partial x^4} \ &= \ \text{terms quadratic in } \alpha, \beta, \\ \eta + \frac{\partial F}{\partial t} - \frac{1}{2}\beta \frac{\partial^3 F}{\partial x^2 \partial t} + \frac{1}{2}\alpha \left(\frac{\partial F}{\partial x}\right)^2 \ &= \ \text{terms quadratic in } \alpha, \beta. \end{aligned}$$

Differentiate the second equation with respect to x and substitute u for $\frac{dF}{dx}$ as before to recover the first-order Boussinesq system (cf. [27]),

$$\begin{aligned} \eta_t + u_x + \alpha\eta_x u + \alpha\eta u_x - \frac{1}{6}\beta u_{xxx} &= \text{terms quadratic in } \alpha, \beta, \\ \eta_x + u_t + \alpha u u_x - \frac{1}{2}\beta u_{xxt} &= \text{terms quadratic in } \alpha, \beta. \end{aligned} \tag{2.18}$$

By the same process, one may derive higher-order approximations. For the second-order case, (2.14)–(2.15) yield

$$\begin{aligned} &\eta_t + \alpha\eta_x \left(\frac{\partial F}{\partial x} - \frac{1}{2}\frac{\partial^3 F}{\partial x^3}(1+\alpha\eta)^2\beta\right) + \frac{\partial^2 F}{\partial x^2}(1+\alpha\eta) \\ &\qquad - \frac{1}{6}\frac{\partial^4 F}{\partial x^4}(1+\alpha\eta)^3\beta + \frac{1}{5!}\frac{\partial^6 F}{\partial x^6}(1+\alpha\eta)^5\beta^2 \ = \ \text{terms cubic in } \alpha, \beta, \\ &\eta + \frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^3 F}{\partial x^2 \partial t}(1+\alpha\eta)^2\beta + \frac{1}{4!}\frac{\partial^5 F}{\partial x^4 \partial t}(1+\alpha\eta)^4\beta^2 \\ &\quad + \frac{1}{2}\alpha \left\{\frac{\partial F}{\partial x} - \frac{1}{2}\frac{\partial^3 F}{\partial x^3}(1+\alpha\eta)^2\beta\right\}^2 + \frac{1}{2}\alpha\beta\Big(\frac{\partial^2 F}{\partial x^2}\,(1+\alpha\eta)\Big)^2 \ = \ \text{terms cubic in } \alpha, \beta. \end{aligned}$$

Expand the powers, rewrite the system, and combine the resulting higher-order terms with the error terms to obtain

$$\begin{aligned} &\eta_t + \alpha\eta_x \left(\frac{\partial F}{\partial x} - \frac{1}{2}\frac{\partial^3 F}{\partial x^3}\beta\right) + \frac{\partial^2 F}{x^2} + \alpha\eta \frac{\partial^2 F}{\partial x^2} \\ &\qquad - \frac{1}{6}\frac{\partial^4 F}{\partial x^4}(1+3\alpha\eta)\beta + \frac{1}{120}\frac{\partial^6 F}{\partial x^6}\beta^2 \ = \ \text{terms cubic in } \alpha, \beta, \\ &\eta + \frac{\partial F}{\partial t} - \frac{1}{2}\frac{\partial^3 F}{\partial x^2 \partial t}(1+2\alpha\eta)\beta + \frac{1}{24}\frac{\partial^5 F}{\partial x^4 \partial t}\beta^2 \\ &\quad + \frac{1}{2}\alpha \left(\frac{\partial F}{\partial x}\right)^2 - \frac{1}{2}\alpha\beta \frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3} + \frac{1}{2}\alpha\beta\Big(\frac{\partial^2 F}{\partial x^2}\Big)^2 \ = \ \text{terms cubic in } \alpha, \beta. \end{aligned}$$

Differentiate the second equation with respect to x and substitute u as before to reach a second-order correct Boussinesq-system of equations

$$\begin{aligned} &\eta_t + u_x + \alpha\eta_x u + \alpha\eta u_x - \frac{1}{6}\beta u_{xxx} - \frac{1}{2}\alpha\beta\eta_x u_{xx} \\ &\quad -\frac{1}{2}\alpha\beta\eta u_{xxx} + \frac{1}{120}\beta^2 u_{xxxxx} = \text{terms cubic in } \alpha, \beta, \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\eta_x + u_t - \frac{1}{2}\beta u_{xxt} + \alpha u u_x - \alpha\beta\eta u_{xxt} - \alpha\beta\eta_x u_{xt} + \frac{1}{2}\alpha\beta u_x u_{xx} \\ &\quad -\frac{1}{2}\alpha\beta u u_{xxx} + \frac{1}{24}\beta^2 u_{xxxxt} = \text{terms cubic in } \alpha, \beta. \end{aligned} \tag{2.20}$$

Our purpose now is to derive a class of systems all of which are formally equivalent to the system displayed in (2.19)–(2.20). This will be accomplished by considering changes in the dependent variables and by making use of lower-order relations in higher-order terms. (As mentioned in the Introduction, different members of this putative class are likely to be useful in different circumstances.) Toward this goal, begin by letting w be the scaled horizontal velocity corresponding to the depth $(1-\theta)h$ below the undisturbed surface. Of course, $0 \le \theta \le 1$ with $\theta = 0$ leading to $w = u$, the horizontal velocity at the bottom. One anticipates on the basis of prior experience that, when the system model is based upon η and w rather than η and u, the structure of the associated differential equations may change markedly. A formal use of Taylor's formula with remainder shows that

$$\begin{aligned} w = \phi_x|_{z=\theta} &= F_x - \frac{1}{2}\beta\theta^2 F_{xxx} + \frac{1}{4!}\beta^2\theta^4 F_{xxxxx} + O(\beta^3) \\ &= u - \frac{1}{2}\beta\theta^2 u_{xx} + \frac{1}{4!}\beta^2\theta^4 u_{xxxx} + O(\beta^3), \end{aligned}$$

as $\beta \to 0$. By using the Fourier transform, the latter relationship may be written as

$$\hat{w} = (1 + \frac{1}{2}\beta\theta^2 k^2 + \frac{1}{4!}\beta^2\theta^4 k^4)\hat{u} + O(\beta^3),$$

where the Fourier transform of a function f of the spatial variable x is

$$\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x)\, dx.$$

Inverting the positive Fourier multiplier yields

$$\begin{aligned} \hat{u} &= \left(1 + \frac{1}{2}\beta\theta^2 k^2 + \frac{1}{4!}\beta^2\theta^4 k^4\right)^{-1} \hat{w} + O(\beta^3) \\ &= \left(1 - \frac{1}{2}\beta\theta^2 k^2 - \frac{1}{4!}\beta^2\theta^4 k^4 + \frac{1}{4}\beta^2\theta^4 k^4\right)\hat{w} + O\left(\beta^3\right) \\ &= \left(1 - \frac{1}{2}\beta\theta^2 k^2 + \frac{5}{24}\beta^2\theta^4 k^4\right)\hat{w} + O(\beta^3), \end{aligned}$$

as $\beta \to 0$. Thus there appears the relationship

$$u = w + \frac{1}{2}\beta\theta^2 w_{xx} + \frac{5}{24}\beta^2\theta^4 w_{xxxx} + O(\beta^3), \tag{2.21}$$

as $\beta \to 0$. Substitute this relation into (2.19) and (2.20) and combine higher-order terms with the formal error to obtain, after some straightforward calculation,

$$\begin{aligned}
\eta_t + w_x + \alpha(\eta w)_x + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta w_{xxx} + \frac{1}{2}(\theta^2 - 1)\alpha\beta(\eta w_{xx})_x \\
+ \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2 \beta^2 w_{xxxxx} = \text{terms cubic in } \alpha, \beta,
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\eta_x + w_t + \frac{1}{2}(\theta^2 - 1)\beta w_{xxt} + \alpha w w_x - \alpha\beta\eta w_{xxt} \\
-\alpha\beta\eta_x w_{xt} + \frac{1}{2}(\theta^2 + 1)\alpha\beta w_x w_{xx} + \frac{1}{2}(\theta^2 - 1)\alpha\beta w w_{xxx} \\
+ \frac{5}{24}(\theta^2 - 1)\left(\theta^2 - \frac{1}{5}\right)\beta^2 w_{xxxxt} = \text{terms cubic in } \alpha, \beta.
\end{aligned} \tag{2.23}$$

Other systems of equations correct to second order in α, β can be obtained by altering the higher-order terms using the lower-order approximations. Moving all the terms in (2.22) and (2.23) quadratic in α and β to the right-hand side, it is seen that

$$\eta_t + w_x + \alpha(\eta w)_x + \frac{1}{2}(\theta^2 - \frac{1}{3})\beta w_{xxx} \; = \; \text{terms quadratic in } \alpha, \beta, \tag{2.24}$$

$$\eta_x + w_t + \frac{1}{2}(\theta^2 - 1)\beta w_{xxt} + \alpha w w_x \; = \; \text{terms quadratic in } \alpha, \beta. \tag{2.25}$$

The idea is to use the relationships (2.24) and (2.25) to alter the evolution equations without changing the formal order of approximation. As an example, consider the fourth and sixth terms

$$\frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta w_{xxx} + \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2 \beta^2 w_{xxxxx}$$

on the left-hand side of (2.22). Differentiate (2.24) twice with respect to x and solve for w_{xxx} to obtain

$$w_{xxx} = -\eta_{xxt} - \alpha(\eta w)_{xxx} - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta w_{xxxxx} + \text{terms quadratic in } \alpha, \beta.$$

Let $\lambda \in \mathbb{R}$ and write

$$\begin{aligned}
\beta w_{xxx} \; &= \; \lambda\beta w_{xxx} + (1-\lambda)\beta w_{xxx} \\
&= \; \lambda\beta w_{xxx} + (1-\lambda)\beta\left(-\eta_{xxt} - \alpha(\eta w)_{xxx} - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta w_{xxxxx}\right) \\
&\quad + \text{terms cubic in } \alpha, \beta.
\end{aligned}$$

Similarly, it is clear because of (2.16)–(2.17) that, formally,

$$\beta^2 w_{xxxxx} = -\beta^2 \eta_{xxxxt} + \text{terms cubic in } \alpha, \beta.$$

Thus we may write

$$\beta^2 w_{xxxxx} = \beta^2 \lambda_1 w_{xxxxx} - \beta^2 (1-\lambda_1)\eta_{xxxxt} + \text{terms cubic in } \alpha, \beta.$$

Proceeding systematically to apply the same type of representation to other terms (use (2.25) on the fifth, sixth, and parts of the third and ninth terms on the left-hand side of (2.23)), we obtain the system

$$\begin{aligned}
\eta_t \;&+\; w_x + \alpha(\eta w)_x + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda\beta w_{xxx} \\
&+\; \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda)\beta\left(-\eta_t - \alpha(\eta w)_x - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta w_{xxx}\right)_{xx} \\
&+\; \frac{1}{2}(\theta^2 - 1)\alpha\beta(\eta w_{xx})_x + \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2 \lambda_1 \beta^2 w_{xxxxx} \\
&-\; \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2 (1-\lambda_1)\beta^2 \eta_{xxxxt} = \text{terms cubic in } \alpha, \beta,
\end{aligned}$$

and

$$\begin{aligned}
\eta_x \;&+\; w_t + \frac{1}{2}(\theta^2 - 1)(1-\mu)\beta w_{xxt} \\
&+\; \frac{1}{2}(\theta^2 - 1)\mu\beta\left(-\eta_x - \frac{1}{2}(\theta^2 - 1)\beta w_{xxt} - \alpha w w_x\right)_{xx} \\
&+\; \alpha w w_x + \alpha\beta(\eta\eta_{xx})_x + \frac{1}{2}(\theta^2 + 1)\alpha\beta w_x w_{xx} \\
&+\; \frac{1}{2}(\theta^2 - 1)\alpha\beta w w_{xxx} + \frac{5}{24}(\theta^2 - 1)\left(\theta^2 - \frac{1}{5}\right)\mu_1 \beta^2 w_{xxxxt} \\
&-\; \frac{5}{24}(\theta^2 - 1)\left(\theta^2 - \frac{1}{5}\right)(1-\mu_1)\beta^2 \eta_{xxxxx} = \text{terms cubic in } \alpha, \beta.
\end{aligned}$$

By neglecting the terms cubic in α and β on the right-hand side, we derive the systems of equations

$$\begin{aligned}
\eta_t - b\beta\eta_{xxt} + b_1\beta^2\eta_{xxxxt} &= -w_x - \alpha(\eta w)_x - a\beta w_{xxx} + b\alpha\beta(\eta w)_{xxx} \\
&\quad - \left(a + b - \frac{1}{3}\right)\alpha\beta(\eta w_{xx})_{\tilde{x}} - a_1\beta^2 w_{xxxxx}, \\
w_t - d\beta w_{xxt} + d_1\beta^2 w_{xxxxt} &= -\eta_x - c\beta\eta_{xxx} - \alpha w w_x - c\alpha\beta(w w_x)_{xx} \qquad (2.26) \\
&\quad - \alpha\beta(\eta\eta_{xx})_x + (c + d - 1)\alpha\beta w_x w_{xx} \\
&\quad + (c + d)\alpha\beta w w_{xxx} - c_1\beta^2\eta_{xxxxx},
\end{aligned}$$

written in the scaled variables (η, w), which are correct renditions through terms of formal order $\beta^2, \alpha\beta, \alpha^2$, where the constants a, b, c, d are given by (1.9) and

$$\begin{aligned} a_1 &= -\frac{1}{4}\left(\theta^2 - \frac{1}{3}\right)^2(1-\lambda) + \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2\lambda_1, \\ b_1 &= -\frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2(1-\lambda_1), \\ c_1 &= \frac{5}{24}(1-\theta^2)\left(\theta^2 - \frac{1}{5}\right)(1-\mu_1), \\ d_1 &= -\frac{1}{4}(1-\theta^2)^2\mu - \frac{5}{24}(1-\theta^2)\left(\theta^2 - \frac{1}{5}\right)\mu_1. \end{aligned} \tag{2.27}$$

An appraisal of the scales of the various terms above suggests making the following change of variables:

$$x = \beta^{\frac{1}{2}}\hat{x}, \qquad t = \beta^{\frac{1}{2}}\hat{t}, \qquad \eta = \alpha^{-1}\hat{\eta}, \qquad w = \alpha^{-1}\hat{w}.$$

Recall that the tilde over the variables introduced in (2.8) has been suppressed in (2.26). In the new variables, the second-order correct system has the form

$$\begin{aligned} \hat{\eta}_{\hat{t}} - b\hat{\eta}_{\hat{x}\hat{x}\hat{t}} + b_1\hat{\eta}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} = &-\hat{w}_{\hat{x}} - (\hat{\eta}\hat{w})_{\hat{x}} - a\hat{w}_{\hat{x}\hat{x}\hat{x}} + b(\hat{\eta}\hat{w})_{\hat{x}\hat{x}\hat{x}} \\ &- \left(a + b - \frac{1}{3}\right)(\hat{\eta}\hat{w}_{\hat{x}\hat{x}})_{\hat{x}} - a_1\hat{w}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} \hat{w}_{\hat{t}} - d\hat{w}_{\hat{x}\hat{x}\hat{t}} + d_1\hat{w}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} = &-\hat{\eta}_{\hat{x}} - c\hat{\eta}_{\hat{x}\hat{x}\hat{x}} - \hat{w}\hat{w}_{\hat{x}} - c(\hat{w}\hat{w}_{\hat{x}})_{\hat{x}\hat{x}} \\ &-(\hat{\eta}\hat{\eta}_{\hat{x}\hat{x}})_{\hat{x}} + (c + d - 1)\hat{w}_{\hat{x}}\hat{w}_{\hat{x}\hat{x}} \\ &+(c+d)\hat{w}\hat{w}_{\hat{x}\hat{x}\hat{x}} - c_1\hat{\eta}_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}}. \end{aligned} \tag{2.29}$$

These are the systems (1.7), though the circumflexes surmounting the variables were there omitted in the interest of visual clarity. By dropping the second-order terms in (2.26) and using the scaling introduced above, one obtains the first-order correct systems displayed in (1.6).

The constants $a, b, c, d, a_1, b_1, c_1, d_1$ form a restricted eight-parameter family which depends on θ in the range $0 \le \theta \le 1$ and λ, μ, λ_1 and μ_1 in $\mathbb{R}$. To the second-order with respect to the small parameters α and β, all the systems in (2.28)–(2.29) are formally equivalent. The relationship between the original physical variables x, t, η, w and the new variables $\hat{x}, \hat{t}, \hat{\eta}, \hat{w}$ is

$$x = h\hat{x}, \qquad t = h\hat{t}/c_0, \qquad \eta = h\hat{\eta}, \qquad w = c_0\hat{w}.$$

Thus, the variables $\hat{x}, \hat{t}, \hat{\eta}$, and $\hat{w}$ are the standard nondimensionalization of x, t, η, and w wherein the length scale is taken to be h and the time scale to be h/c_0.

A check of the above calculations is to determine the dispersion relation of the system (2.28)–(2.29) linearized around the rest state. By transforming the linearized system into a single equation and positing a solution of the form $e^{i(kx-\omega t)}$, the frequency ω is determined as a function of wave number k, *viz.*

$$\omega^2(k) = k^2\frac{(1 - ak^2 + a_1k^4)(1 - ck^2 + c_1k^4)}{(1 + bk^2 + b_1k^4)(1 + dk^2 + d_1k^4)}.$$

Expanding the right-hand side of the last equation for small values of k (long wavelength) gives an expression for the phase speed c, namely

$$c^2(k) = \frac{\omega^2(k)}{k^2} = 1 - \frac{1}{3}k^2 + \frac{2}{15}k^4 \\ + g(\theta, \lambda, \lambda_1, \mu, \mu_1)k^6 + O(k^8), \tag{2.30}$$

as $k \to 0$, where

$$\begin{aligned} g(\theta, \lambda, \lambda_1, \mu, \mu_1) \; = \; & \frac{1}{720}\{(-66 - \lambda - \lambda_1 + 15\mu + 15\mu_1) \\ & + (172 + 13\lambda + 13\lambda_1 - 105\mu - 105\mu_1)\theta^2 \\ & + 5(-34 - 11\lambda - 11\lambda_1 + 33\mu + 33\mu_1)\theta^4 \\ & + 75(\lambda + \lambda_1 - \mu - \mu_1)\theta^6\}. \end{aligned}$$

The first three terms, which are independent of θ, λ, λ_1, μ, and μ_1, correspond to the expansion of the full linearized dispersion relation

$$c^2(k) = \frac{\tanh(k)}{k}$$

of the Euler equations written in the variables displayed in (2.8) (cf. Whitham [61]).

It is interesting to note that while the linear dispersion relation for (2.28)–(2.29) always matches that of the linearized Euler equations through terms of order k^4 in the Taylor expansion around the origin, by an adroit choice of the constants, the linearized dispersion relation can be made to match through terms of sixth-order.

It is worthwhile to review what we expect from the models introduced above. If one takes seriously the power series representation of the velocity potential ϕ, then the formal error in the free-surface deviation made in using the second-order correct model versus using the Euler equations has the form

$$\sup_{x\in\mathbb{R}} |\tilde{\eta}_{Euler}(x,t) - \tilde{\eta}_{Bouss}(x,t)| \le C\beta^3 t \tag{2.31}$$

in the variables appearing in (2.26). The constant C depends on a norm of the initial data and the Stokes number $S = \frac{\alpha}{\beta}$. In consequence, it is an order 1 quantity. The two renditions of the free surface are likewise of order 1. Hence, so long as the error is of order β or less, the Boussinesq approximation would be judged a good one. Thus, for time scales t of order $\frac{1}{\beta^2}$, the expectation is that the higher-order Boussinesq model will provide a satisfactory description of the wave motion.

In the variables appearing in (2.28)–(2.29) or (1.7), this estimate of the divergence of the model from the Euler equations becomes

$$\sup_{x\in\mathbb{R}} |\hat{\eta}_{Euler}(x,t) - \hat{\eta}_{Bouss}(x,t)| \le C_1\alpha\beta^3 t. \tag{2.32}$$

Of course, if one poses an initial-value problem for (2.28)–(2.29), then for the solution to be physically relevant, the initial data

$$\hat{\eta}(\hat{x}, 0) = \phi(\hat{x}), \qquad \hat{w}(\hat{x}, 0) = \psi(\hat{x}), \tag{2.33}$$

should satisfy the small-amplitude, long-wavelength assumptions inherent in the derivation of the models. That is, in principle, $\phi(\hat{x})$ and $\psi(\hat{x})$ should be of the form

$$\phi(\hat{x}) = \alpha f(\beta \hat{x}), \qquad \psi(\hat{x}) = \alpha g(\beta \hat{x}),$$

where $\alpha = \frac{A}{h}$ and $\beta = \frac{h^2}{\ell^2}$ are small and f, g and their first few derivatives are all of order one. (It is worth remark here that, depending on the norm used to measure functions, it appears that one must assume f and g possess five or six derivatives to justify the model equation; see Benjamin et al. [9] and Craig [36]). Thus, in this case, both $\hat{\eta}_{Euler}$ and $\hat{\eta}_{Bouss}$ are of order α. The inequality (2.32) thus shows that the difference between the two profiles is at the neglected relative order error at least over the time scale $\hat{t}$ of order $\frac{1}{\beta^{\frac{5}{2}}}$. Similar remarks apply to the first-order correct system (1.6). However, the analog of (2.31) is

$$\sup_{x \in \mathbb{R}} |\eta_{Euler}(x,t) - \eta_{Bouss}(x,t)| \le C\beta^2 t, \tag{2.34}$$

and similarly for (2.32).

Remark 2.1. It deserves comment that no attempt has been made to alter the nonlinearities in the Boussinesq systems. Such a program where one looks more deeply into the nonlinear structure is being pursued separately (see [5], [6], [4]).

Remark 2.2. The water channel in the Penn State Mathematics Department is about 10 meters long and 30 cm wide. Experiments are planned in a water depth of 10 cm and the wavelength, which is determined by the frequency of a piston-type wavemaker, is about a meter. The wave height can be controlled by varying the throw of the wavemaker. Wave amplitudes from 10^{-2} cm to 1 cm are in view, and thus α will vary between 10^{-3} and 10^{-1}. The Stokes number will therefore lie in the range $10^{-1} \lessapprox S \lessapprox 10$.

3. First-Order Correct Linear Boussinesq Systems

The initial-value problem for the linear model obtained from the system (1.6) by ignoring the quadratic terms is analyzed here. The system in question is

$$\begin{aligned} \eta_t + w_x + a w_{xxx} - b\eta_{xxt} &= 0, \\ w_t + \eta_x + c\eta_{xxx} - d w_{xxt} &= 0, \end{aligned} \tag{3.1}$$

where a, b, c, d are as defined in (1.9).

Because the nonlinearity has been dropped, the system (3.1) is straightforwardly understood using Fourier analysis.

Notation. The standard norm in $L_p(\mathbb{R})$ will be written $|\cdot|_p$ for $1 \le p \le \infty$. If $f \in H^s = H^s(\mathbb{R})$, where $s \ge 0$, the Sobolev class of L_2-functions whose first s derivatives also lie in L_2, then its norm is written $\|f\|_s$. If s is not an integer, the notion is extended via the Fourier transform in the usual way. For $f \in H^s(\mathbb{R})$, its Fourier transform is $\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x)\,dx$ and

$$\|f\|_s = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 (1+k^2)^s dk \right)^{\frac{1}{2}} \tag{3.2}$$

is a norm on $H^s(\mathbb{R})$ which is equivalent to the usual norm

$$\left(\sum_{j=0}^{s} \left\| f^{(j)}(x) \right\|_0^2 \right)^{\frac{1}{2}}$$

when s is a positive integer (cf. [40]). When $s = 0$, Parseval's formula implies

$$\|f\|_0 = |f|_2 = \left(\int_{-\infty}^{+\infty} |f(x)|^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} |\hat{f}|_2.$$

If X is any other Banach space, its norm will be denoted, unabbreviated, as $\|\cdot\|_X$. The product space $X \times X$ will be abbreviated by X^2 and it carries the norm

$$\|\mathbf{f}\|_{X^2} = \left(\|f_1\|_X^2 + \|f_2\|_X^2 \right)^{\frac{1}{2}}$$

for $\mathbf{f} = (f_1, f_2)$. We denote by $B(X, Y)$ the set of all bounded linear operators from X to Y. The associated norm is denoted by $\|\cdot\|_{X,Y}$. The domain of an operator T is written $D(T)$. If X is a Banach space, the continuous mappings $w\colon\ [a, b] \to X$, equipped with the maximum norm

$$\max_{a \le t \le b} \|w(t)\|_X,$$

is again a Banach space denoted by $C(a, b; X)$.

Attention is given to the pure initial-value problem (3.1)–(1.17) where φ, ψ are selected from a class of functions evanescent at ∞, say $L_2(\mathbb{R})$ or $H^s(\mathbb{R})$ for some $s > 0$. Taking the Fourier transform with respect to x, the system (3.1) may be written in the form

$$\frac{d}{dt} \begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} + ik\mathcal{A}(k) \begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} = 0, \tag{3.3}$$

where

$$\mathcal{A}(k) = \begin{pmatrix} 0 & \omega_1(k) \\ \omega_2(k) & 0 \end{pmatrix} \tag{3.4}$$

and

$$\omega_1(k) = \frac{1 - ak^2}{1 + bk^2}, \qquad \omega_2(k) = \frac{1 - ck^2}{1 + dk^2}. \tag{3.5}$$

Interest is first turned to determining when the initial-value problem for the system (3.1) is well posed. We begin by investigating when the initial-value problem for (3.1) is ill posed in the sense to be explained below. In the complementary class, where the system (3.1) is well posed, it will appear nevertheless that not all choices preserve energy. Because these models are meant to serve in place of the Euler equations, the versions that preserve energy are more natural candidates as good approximations. Indeed, we shall demand that the initial-value problem for (3.1) defines a group in appropriate Cartesian products of L_2-based Sobolev classes H^s.

To understand the initial-value problem (3.1)–(1.17), it suffices to solve formally the Fourier-transformed system (3.3) and investigate the resulting representation of solutions. Let the quantity $\sigma(k)$ be defined to be

$$\sigma(k) = |\omega_1(k)\omega_2(k)|^{\frac{1}{2}}.$$

The eigenvalues of the matrix $\mathcal{A}(k)$ are $\pm\lambda(k)$ where

$$\lambda(k) = \sigma(k), \quad \text{when} \quad \omega_1(k)\omega_2(k) \geq 0,$$

and

$$\lambda(k) = i\sigma(k), \quad \text{when} \quad \omega_1(k)\omega_2(k) < 0.$$

The formal solution of (3.1) with initial data $(\hat{\varphi}, \hat{\psi})$ is

$$\begin{pmatrix} \hat{\eta}(k,t) \\ \hat{w}(k,t) \end{pmatrix} = e^{-ik\mathcal{A}(k)t} \begin{pmatrix} \hat{\varphi} \\ \hat{\psi} \end{pmatrix}, \tag{3.6}$$

and a straightforward computation shows that when $\omega_1(k)\omega_2(k) \geq 0$,

$$e^{-ik\mathcal{A}(k)t} = \begin{pmatrix} \cos(k\sigma(k)t) & -i\sin(k\sigma(k)t)\frac{\omega_1(k)}{\sigma(k)} \\ -i\sin(k\sigma(k)t)\frac{\omega_2(k)}{\sigma(k)} & \cos(k\sigma(k)t) \end{pmatrix}, \tag{3.7}$$

whereas, if $\omega_1(k)\omega_2(k) < 0$, then

$$e^{-ik\mathcal{A}(k)t} = \begin{pmatrix} \cosh(k\sigma(k)t) & i\sinh(k\sigma(k)t)\frac{\omega_1(k)}{\sigma(k)} \\ i\sinh(k\sigma(k)t)\frac{\omega_2(k)}{\sigma(k)} & \cosh(k\sigma(k)t) \end{pmatrix}. \tag{3.8}$$

The first point to appreciate is that the Fourier multiplier

$$m(k,t) = e^{ik\mathcal{A}(k)t}$$

need not be bounded for bounded values of k. If m is unbounded at finite values of k, the linear initial-value problem is certainly not well-posed in any of the Sobolev classes H^s because the operators

$$T_m(f) \equiv \mathcal{F}^{-1}(m(k)\hat{f}(k))(x),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform, are not bounded maps from H^s to L_2 for any value of s. This is because the bounded Fourier multipliers on L_2 are exactly the L_∞-functions. Insisting on boundedness of m on bounded sets and reference to the restrictions in (1.8) on the coefficients a, b, c, d lead to the following result.

Proposition 3.1. *For a, b, c, d satisfying (1.8), the matrix $e^{-ik\mathcal{A}(k)t}$ is bounded on finite intervals of wavenumbers k if and only if one of the following sets of conditions hold:*

(C1) $b \geq 0, \quad d \geq 0, \quad a \leq 0, \quad c \leq 0$;
(C2) $b \geq 0, \quad d \geq 0, \quad a = c > 0$;
(C3) $b = d < 0, \quad a = c > 0$.

This follows by noting that m is bounded on bounded intervals if and only if the ratio $\frac{\omega_1(k)}{\omega_2(k)}$, which is a rational function of order 2 or 4, has neither zeroes nor poles on the real axis. Exhaustive reference to the restriction (1.8) on the coefficients a, b, c, and d then leads to the stated conclusions.

Notice that in all three cases (C1), (C2), and (C3), the product $\omega_1(k)\omega_2(k) \geq 0$ for all k. In consequence, it is clear that the coefficients of the symbol $m(k, t)$ are bounded on bounded sets and grow at most quadratically as $k \to \pm\infty$. Thus, as will appear presently, the initial-value problem (3.1), when a, b, c, d satisfy one of (C1) to (C3), is well posed in L_2-based Sobolev classes. Indeed, a consequence of (3.6)–(3.7) is that

$$|\hat{\eta}|^2 + \left(\frac{\omega_1}{\omega_2}\right)|\hat{w}|^2 = |\hat{\varphi}|^2 + \left(\frac{\omega_1}{\omega_2}\right)|\hat{\psi}|^2, \tag{3.9}$$

for all k and t. If we define the Fourier multiplier operator $\mathcal{H}$ by

$$\hat{\mathcal{H}g}(k) = h(k)\hat{g}(k), \quad \text{where } h(k) = h(k,t) = \left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}}, \tag{3.10}$$

then the "energy" of the solution is conserved in the sense that for any value of the Sobolev index s,

$$\|\eta\|_s^2 + \|\mathcal{H}w\|_s^2 = \|\varphi\|_s^2 + \|\mathcal{H}\psi\|_s^2. \tag{3.11}$$

Because of (3.9), and depending on the behavior of $\frac{\omega_1}{\omega_2}$ at infinity, a precise result about global well-posedness for the initial-value problem for (3.1) can be stated as follows.

Theorem 3.2. *Let a, b, c, d satisfy one of the conditions (C1)–(C3) in Proposition 3.1. Define the order l pseudodifferential operator $\mathcal{H}$ as in (3.10), and set $m_1 = \max(0, -l)$, $m_2 = \max(0, l)$. Then the corresponding linear initial-value problem (3.1) is well posed in $H^{s+m_1} \times H^{s+m_2}$ for any $s \geq 0$.*

Taking $s = 0$, Theorem 3.2 implies the following helpful result.

Corollary 3.3. *Let $\mathcal{H}$ be as defined in (3.10).*

- *If $\mathcal{H}$ is of order 2, then (3.1) is well posed in $H^0 \times H^2$;*
- *If $\mathcal{H}$ is of order 1, then (3.1) is well posed in $H^0 \times H^1$;*
- *If $\mathcal{H}$ is of order 0, then (3.1) is well posed in $H^0 \times H^0$;*
- *If $\mathcal{H}$ is of order -1, then (3.1) is well posed in $H^1 \times H^0$;*
- *If $\mathcal{H}$ is of order -2, then (3.1) is well posed in $H^2 \times H^0$.*

Remark 3.4. The systems with $\mathcal{H}$ having order -2 are not admissible as models of the underlying physical situation. For if $\mathcal{H}$ has order -2, then $a = d = 0$ and $b \neq 0$, $c \neq 0$. This is incompatible with (1.8) and any of the conditions C1, C2, or C3.

4. Well-posedness of the Linear Cauchy Problem in L_∞: Dispersive Singularity Formation

Thus far, consideration has been given to the well-posedness of the Cauchy problem for the linearized Boussinesq systems in L_2 or in Sobolev spaces based on L_2. In this section, we will investigate to what extent the linear systems well-posed in L_2 are well-posed in L_p for $p \neq 2$ in the usual range $1 \leq p \leq \infty$. There are several reasons for this study. First, as discussed in [22], [23], ill-posedness in L_p for some $p \neq 2$ is linked to the short-wave behavior of the phase velocity. As these equations are meant to be long-wave models, the short-wave behavior should not introduce spurious difficulties, and especially not instabilities or singularity formation. Another point is that the linear Euler system from which all these models derive is hyperbolic, and so locally well-posed in L_∞. It is not unnatural, therefore, to require the same qualitative property of the model system as a means of insuring a more robust approximation of solutions of the full problem. A final point is that L_p-instability, and especially L_∞-instability due to short-wave modeling, could very easily lead to instability in an approximating numerical code; these instabilities would be an artifact of the modeling and have nothing to do with the full Euler equations.

Consider a general, constant coefficient system

$$\begin{aligned} &\frac{\partial \vec{v}}{\partial t} + P(\partial_x)\vec{v} = 0, \qquad x \in \mathbb{R}, \qquad t \geq 0, \\ &\vec{v}(x,0) = \phi(x), \end{aligned} \tag{4.1}$$

where $\vec{v} = (v_1(x,t), \dots, v_d(x,t))$ and $P(\partial_x)$ is a matrix of constant coefficient pseudodifferential operators (Fourier multiplier operators). The precise definition of well-posedness in force in this context is the following.

Definition 4.1. *Let p lie in the range $1 \leq p \leq \infty$. The system (4.1) is called L_p-well-posed if for any $\phi \in L_p^d$ (the d-fold product of L_p with itself), there is a unique solution $\vec{v}$ in the sense of distributions of (4.1) such that, for any $T > 0$, $\vec{v} \in C(0,T;L_p^d)$ and there is a constant C_T such that*

$$\sup_{t\in[0,T]} |\vec{v}(\cdot,t)|_p \leq C_T |\phi|_p. \tag{4.2}$$

Some classical facts about Fourier multipliers (see [40], [56]) are recalled here for convenience. A function $m \in L_\infty$ is an L_p-multiplier for some $p \in [1,\infty]$ if there exists a constant C_p such that for every $f \in L_p$, $T_m(f) \equiv \mathcal{F}^{-1}\Big(m(k)\hat{f}(k)\Big) \in L_p$ and satisfies

$$|T_m(f)|_p \leq C_p |f|_p. \tag{4.3}$$

The multiplier norm $\|m\|_{M_p}$ is the smallest value of C_p for which (4.3) holds for all $f \in L_p$. Denote by M_p the Banach algebra of L_p-multipliers equipped with this norm.

Lemma 4.2. (i) *Fix p with* $1 \le p \le \infty$. *Then* $M_p = M_{p'}$ *for* p' *satisfying* $\frac{1}{p} + \frac{1}{p'} = 1$, *and if* $m \in M_p$, *then* $\|m\|_{M_p} = \|m\|_{M_{p'}}$.

(ii) $M_1 = M_\infty = \mathcal{F}(\mathcal{M})$, *where* $\mathcal{M}$ *is the space of bounded Borel measures.*

(iii) $M_2 = L_\infty$ *with equality of norms.*

(iv) *If* $a, b \in M_p$, *then* $ab \in M_p$ *and* $\|ab\|_{M_p} \le \|a\|_{M_p}\|b\|_{M_p}$.

Lemma 4.3. *For* $k \in \mathbb{R}$, *let* $s(k) = P(ik)$ *be the symbol of the operator* $P(\partial_x)$ *appearing in (4.1). Fix* $p \in [1, \infty]$. *Then (4.1) is* L_p*–well-posed if and only if* $exp(-ts(k)) \in M_p$ *for all* $t \in \mathbb{R}$ *and, for any* $T > 0$, *there is a constant* C_T *such that*

$$\|exp(-ts(k))\|_{M_p} \le C_T, \quad \text{for all } t \in [0, T].$$

Remark 4.4. If $s(k)$ is an L_p-multiplier, so is $exp(-ts(k))$. This helpful fact follows because when $s \in M_p$, the system (4.1) may be viewed as a linear ordinary differential equation posed in L_p.

We prepare the way for investigating the L_p-well-posedness of the linear Boussinesq systems by studying scalar initial-value problems of the form

$$\begin{aligned} &\tfrac{\partial u}{\partial t} + (I - \partial_x^2)^{-\frac{\lambda}{2}} \tfrac{\partial u}{\partial x} = 0, \qquad x \in \mathbb{R}, \qquad t \ge 0, \\ &u(x, 0) = \phi(x), \end{aligned} \tag{4.4}$$

where $\lambda > 0$ and $u = u(x, t)$ is a real-valued function of x and t.

Proposition 4.5. *For the system (4.4), the following two results are valid.*

(i) *If there is a* $p \in (1, \infty)$, $p \ne 2$, *such that (4.4) is well posed in* L_p, *then* $\lambda \ge 1$. *Conversely, if* $\lambda \ge 1$, *then (4.4) is well posed in* L_p *for all* $p \in (1, \infty)$.

(ii) *The initial-value problem (4.4) is* L_∞*- and* L_1*-well-posed if and only if* $\lambda > 1$.

Proof. Since $P(\partial_x) = (I - \partial_x^2)^{-\frac{\lambda}{2}} \partial_x$, in this case the symbol of the operator is $s(k) = ik(1 + |k|^2)^{-\frac{\lambda}{2}}$. Note that s is bounded if and only if $\lambda \ge 1$.

If $\lambda \ge 1$, then by the Hörmander-Mikhlin theorem (see [56]), s is a multiplier in L_p for any $p \in (1, \infty)$. On account of Remark 4.4, the initial-value problem (4.4) is seen to be well posed in L_p for all $p \in (1, \infty)$ (Lemma 4.2 is not needed here).

On the other hand, suppose $0 < \lambda < 1$. Then $P(\partial_x) = (I - \partial_x^2)^{-\frac{\lambda}{2}} \partial_x$ has a positive order $1 - \lambda > 0$ and Brenner's theorem [30] implies (4.4) to be ill posed in L_p for all $p \in (1, \infty)$, $p \ne 2$.

Now presume $\lambda > 1$. To show that (4.4) is L_∞- and L_1-well-posed, it suffices by Remark 4.4 to show that in this case, $s \in M_1 = M_\infty$ is a multiplier in L_∞, say. On account of Lemma 4.2, it suffices in this endeavor to show that the inverse Fourier transform of s is an L_1-function.

It is well known (see, e.g., [55]) that the inverse Fourier transform of s is the derivative of the function

$$f_\lambda(x) = \frac{2\pi^{\frac{\lambda}{2}}}{\Gamma(\frac{\lambda}{2})} |x|^{\frac{\lambda-1}{2}} \mathcal{K}_{\frac{1}{2}-\frac{\lambda}{2}}(2\pi|x|),$$

where $\mathcal{K}_\mu$ connotes the Bessel function of the third kind of order μ. Recall that $\mathcal{K}_\mu = \mathcal{K}_{-\mu}$,

$$\mathcal{K}_\mu(z) \sim \left(\frac{\pi}{2|z|}\right)^{\frac{1}{2}} e^{-|z|}, \quad \text{as } |z| \to +\infty,$$

$$\mathcal{K}_\mu(z) \sim \frac{1}{2}\Gamma(\mu)\left(\frac{1}{2}z\right)^{-\mu}, \quad \text{for } \mu > 0, \quad \text{as } |z| \to 0,$$

and

$$\mathcal{K}_0(z) \sim -\log(z), \quad \text{as } |z| \to 0.$$

Using the classical formulas (see [60]),

$$\frac{d}{dz}(z^\mu \mathcal{K}_\mu(z)) = z^\mu \mathcal{K}_{\mu-1}(z),$$

and noticing that $\mu = \frac{\lambda-1}{2} > 0$, one determines that f'_λ lies in $L_1(Q)$ where $Q = (-\infty, -m) \cup (m, \infty)$ for any $m > 0$. Concerning the behavior of f'_λ at 0, it is convenient to distinguish two cases:

(i) $\mu - 1 < 0$, i.e., $1 < \lambda < 3$, so that

$$f'_\lambda(x) \sim c|x|^{2\mu-1}, \quad \text{as } x \to 0;$$

in this case, f'_λ is integrable in a neighborhood of 0 since $1 - 2\mu < 1$ is equivalent to $\lambda > 1$; and

(ii) $\mu - 1 \geq 0$, i.e., $\lambda \geq 3$, in which case $f'_\lambda \sim c|x|$ as $x \to 0$ and thus there is no singularity at the origin.

In any event, it is now transparent that $\mathcal{F}^{-1}(s) \in L_1$.

Now, suppose the system (4.4) is L_∞- and L_1-well-posed. We claim that $\lambda > 1$ and argue by contradiction. If $0 < \lambda < 1$, the L_∞-ill-posedness of (4.4) results from a theorem of Brenner [30]. When $\lambda = 1$, $s(k) = ik(1+k^2)^{-\frac{1}{2}}$. The above computations show that $s \notin M_\infty$. We need to prove also that $m(k) = e^{-s(k)} \notin M_\infty$. By Lemma 4.2, it suffices to prove that $M = \mathcal{F}^{-1}(m)$ is not a bounded measure. We will show in fact that

$$M = G + c\delta_0 + d\Psi(x) PV\left(\frac{1}{x}\right), \tag{4.5}$$

where c, d are constants with $d \neq 0$, δ_0 is the Dirac delta-function at the origin, $G \in L_1$, $\Psi \in C_0^\infty$, and $\Psi \equiv 1$ in a neighborhood of 0. Denote by g the distribution xM and compute its Fourier transform, *viz.*

$$\widehat{g} = \widehat{xM} = -i\frac{d}{dk}\hat{M} = -i\frac{dm}{dk} = (1+k^2)^{-\frac{3}{2}} e^{-ik(1+k^2)^{-\frac{1}{2}}}. \tag{4.6}$$

The right-hand side of (4.6) is an H^1-function and $(1+k^2)^\alpha \frac{dm}{dk}$ belongs to L_2 provided $\alpha < \frac{5}{4}$. Hence $g \in L_1 \bigcap H^\beta$ for any $\beta < \frac{5}{2}$. In particular, $g \in C^{1,\gamma}$ for any γ with $0 < \gamma < 1$. Let Ψ be a nonnegative function in C_0^∞ with $\Psi(x) = 1$, for $x \in [-\frac{1}{2}, \frac{1}{2}]$, and $\Psi(x) = 0$, for $|x| \geq 1$. In terms of Ψ, write

$$\begin{aligned} xM &= g(x) - g(0)\Psi(x) + g(0)\Psi(x) \\ &= x\left[\frac{g(x) - g(0)\Psi(x)}{x} + g(0)\Psi(x)PV\left(\frac{1}{x}\right)\right], \end{aligned}$$

so

$$x\left[M - G(x) - g(0)\Psi(x)PV\left(\frac{1}{x}\right)\right] = 0, \tag{4.7}$$

where $G(x) = \frac{g(x)-g(0)\Psi(x)}{x}$. The identity (4.7) implies there is a constant c such that

$$M = G(x) + g(0)\Psi(x)PV\left(\frac{1}{x}\right) + c\delta_0.$$

Since g is smooth and integrable, $G \in L_1$,

$$g(0) = \int_{-\infty}^{\infty} \hat{g}(k)dk = i(e^i - e^{-i}) \neq 0,$$

and $\Psi(x)PV(\frac{1}{x})$ is not a bounded measure. Thus M is not a bounded measure.

The stage is now set to consider the well-posedness of the linear Boussinesq systems (3.1) in L_p with $1 \leq p \leq +\infty$, $p \neq 2$. It will be assumed that (1.8) and one of the conditions (C1)–(C3) is fulfilled, so the linear Cauchy problem associated to (3.1) is well-posed in an L_2-setting. It will be convenient to consider the system after the change of variables

$$\eta = \mathcal{H}(v_1 + v_2), \qquad w = v_1 - v_2, \tag{4.8}$$

where $\mathcal{H}$ is the Fourier multiplier defined in (3.10). This leads to the equivalent linear system

$$\begin{aligned} &\frac{\partial v_1}{\partial t} + \Sigma\frac{\partial v_1}{\partial x} = 0, \qquad v_1(x,0) = v_0 \equiv \frac{1}{2}(\mathcal{H}^{-1}\phi + \psi), \\ &\frac{\partial v_2}{\partial t} - \Sigma\frac{\partial v_2}{\partial x} = 0, \qquad v_2(x,0) = w_0 \equiv \frac{1}{2}(\mathcal{H}^{-1}\phi - \psi), \end{aligned} \tag{4.9}$$

where Σ is the pseudodifferential operator with symbol $\sigma(k)$. This has the salutary effect of decoupling the system, and thus reduces the issue to qualitative properties of a single equation. The L_p-well-posedness of the system is thereby reduced to the L_p-well-posedness for the scalar initial-value problem

$$\begin{aligned} &\frac{\partial v}{\partial t} + \Sigma\frac{\partial v}{\partial x} = 0, \\ &v(x,0) = v_0(x). \end{aligned} \tag{4.10}$$

Theorem 4.6. *Suppose the system (3.1) to have coefficients satisfying (1.8) and one of conditions (C1)–(C3). The Cauchy problem for (4.10) is well-posed in L_p for $1 < p < +\infty$, $p \neq 2$, if and only if $\sigma(k)$ has order 0, -1, or -2. It is well posed in L_1 and L_∞ if and only if $\sigma(k)$ has order 0 or -2.*

Proof. The theorem is proved by considering separately the cases where Σ has order $2, 1, 0, -1$, and -2, respectively. As stated, consideration is given only to systems which satisfy (1.8) and one of the conditions in (C1)–(C3). (These are well posed in L_2-based spaces and so are candidates to be physically relevant.)

If Σ has order 2, then

$$b = d = 0, \qquad a = c = \frac{1}{6} > 0, \qquad \text{and so } \sigma(k) = \left(1 - \frac{1}{6}k^2\right)$$

is the only admissible case. In this case, (4.10) is the linear K-dV equation which is ill posed in L_p for $1 \leq p \leq +\infty$, $p \neq 2$, and which displays the "dispersive blow-up" phenomena studied in [22], [23].

If Σ has order 1, the admissible sets of parameters are

(1) $a < 0, c < 0, b = 0, d > 0$, with

$$\sigma(k) = \left(\frac{(1 - ak^2)(1 - ck^2)}{(1 + dk^2)}\right)^{\frac{1}{2}}, \tag{4.11}$$

which gives a linear "Benjamin-Ono" type Boussinesq system;

(2) $b = 0, a = c > 0, d > 0, \sigma(k) = \left(\frac{(1-ak^2)^2}{(1+dk^2)}\right)^{\frac{1}{2}}$,

(3) $d = 0, a = c > 0, b > 0, \sigma(k) = \left(\frac{(1-ak^2)^2}{(1+bk^2)}\right)^{\frac{1}{2}}$.

In the first case, (4.10) has the form

$$w_t + (\tilde{a} H \partial_x^2 - \tilde{b} H - \mathcal{K}) w = 0, \tag{4.12}$$

where H is the Hilbert transform and $\mathcal{K}$ is an operator skew-adjoint in L_2 which is bounded on all L_p spaces, $1 \leq p \leq +\infty$, and is of order -2 (that is, $\mathcal{K} \in B(H^s, H^{s+2})$, for $s \in \mathbb{R}$). This is a perturbed linear Benjamin-Ono equation for which the L_p-ill-posedness (and the dispersive blow-up phenomena) is established in [24].

If Σ has order 0, there are six possibilities:

(1) $a < 0, c < 0, b > 0, d > 0,\ \sigma(k) = \left(\frac{(1 - ak^2)(1 - ck^2)}{(1 + bk^2)(1 + dk^2)}\right)^{\frac{1}{2}}$,

(2) $a = c > 0, b > 0, d > 0,\ \sigma(k) = \left(\frac{(1 - ak^2)^2}{(1 + bk^2)(1 + dk^2)}\right)^{\frac{1}{2}}$,

(3) $b = c = 0, a < 0, d > 0,\ \sigma(k) = \left(\frac{(1 - ak^2)}{(1 + dk^2)}\right)^{\frac{1}{2}}$,

(4) $a = b = 0, c < 0, d > 0,\ \sigma(k) = \left(\frac{(1 - ck^2)}{(1 + dk^2)}\right)^{\frac{1}{2}}$,

(5) $c = d = 0, a < 0, b > 0,\ \sigma(k) = \left(\dfrac{(1-ak^2)}{(1+bk^2)}\right)^{\frac{1}{2}}$,
(6) $a = c > 0, b = d < 0,\ \sigma(k) = \left(\dfrac{(1-ak^2)^2}{(1+bk^2)^2}\right)^{\frac{1}{2}}$.

Note that the condition $a+b+c+d = \frac{1}{3}$ prevents the condition $\sigma(k) \equiv 1$, so (4.10) is never the advection equation $v_t + v_x = 0$ (which is obviously well-posed in all the L_p spaces!).

We consider in detail only the first case, the other cases being similar. Write σ in the form

$$\sigma(k) = \left(\frac{ac}{bd}\right)^{\frac{1}{2}} + r(k),$$

where

$$r(k) = \frac{r_1(k)}{r_2(k)}$$

with

$$r_1(k) = bd - ac - (abc + adc + abd + bdc)k^2$$

and

$$\begin{aligned} r_2(k) \ &= \ bd[(1-ak^2)(1-ck^2)(1+bk^2)(1+dk^2)]^{\frac{1}{2}} \\ &\quad + (abcd)^{\frac{1}{2}}(1+bk^2)(1+dk^2). \end{aligned}$$

By arguments similar to those appearing in the proof of Proposition 4.5, one checks that $kr(k)$ is a multiplier in L_p for $1 \le p \le +\infty$, and, consequently, so is $\exp(itkr(k))$ for $t \ge 0$. It follows that

$$e^{itk\sigma(k)} = e^{itk(\frac{ac}{bd})^{\frac{1}{2}}} e^{itk\, r(k)}$$

is also a Fourier multiplier in L_p, for $1 \le p \le \infty$.

If Σ has order -1, the admissible cases are the following:

(1) $a = c = d = 0, b > 0,\ \sigma(k) = (1+bk^2)^{-\frac{1}{2}}$,
(2) $a = b = c = 0, d > 0,\ \sigma(k) = (1+dk^2)^{-\frac{1}{2}}$,
(3) $a = 0, b > 0, c < 0, d > 0,\ \sigma(k) = \left(\frac{(1-ck^2)}{(1+bk^2)(1+dk^2)}\right)^{\frac{1}{2}}$,
(4) $c = 0, a < 0, b > 0, d > 0\ \sigma(k) = \left(\frac{(1-ak^2)}{(1+bk^2)(1+dk^2)}\right)^{\frac{1}{2}}$.

In all these cases, (4.10) fits exactly into the framework of Proposition 4.5, part (ii). The results in the case of order -1 follow at once.

If σ has order -2, the admissible cases correspond to the coupled BBM-system (1.13) when $a = c = 0, b > 0, d > 0$. In this case, (4.10) reduces to two uncoupled linear BBM equations, and these are certainly well-posed in L_p for $1 \le p \le \infty$.

The study of the systems obtained via the change of variables (4.8) is complete. We now consider returning to the original variables (η, u) and the system (3.1). This also involves looking at several cases. The results are conveniently classified by the order of $\mathcal{H}$ and, for a given order, according to the order of σ, of course, keeping only the admissible cases. Here is the outcome of such an analysis.

Theorem 4.7. *A. Assume that $\mathcal{H}$ has order 0.*

(1) If σ has order 2 (the "K-dV case"), then the linear system (3.1) is ill posed in L_p for any $p \in [1, \infty]$ with $p \neq 2$. Moreover, the equation displays the dispersive blow-up phenomena.

(2) If σ has order 0 (which includes the "generic" case), the linear system (3.1) is well posed in L_p for all $p \in [1, \infty]$.

(3) If σ has order -2 (the "BBM case"), then (3.1) is well posed in L_p for all $p \in [1, \infty]$.

B. Assume that $\mathcal{H}$ has order -1.

(1) If σ has order -1 and $a = c = d = 0$, $b > 0$ or $a = 0$, $b > 0, c < 0, d > 0$, then (3.1) is well posed in $W_p^{s+1}(\mathbb{R}) \times W_p^s(\mathbb{R})$ for any $s \geq 0$ and $1 < p < \infty$, and ill posed in the same space if $s \geq 0$ and $p = 1$ or $p = \infty$.

(2) If σ has order 1 and $a = c > 0, b > 0, d = 0$, then (3.1) is ill posed in $W_p^{s+1}(\mathbb{R}) \times W_p^s(\mathbb{R})$ for any $s \geq 0$ and $p \in [1, \infty]$, but $p \neq 2$.

C. Assume that $\mathcal{H}$ has order 1.

(1) If σ has order -1 and $a = b = c = 0$, or $a, c < 0, d > 0$ or, alternatively, $b = 0, a = c > 0, d > 0$, then (3.1) is well posed in $W_p^s(\mathbb{R}) \times W_p^{s+1}(\mathbb{R})$ provided that $s \geq 0$ and $p \in [1, \infty]$.

(2) If σ has order 1 and $a = b = c = 0$, $d > 0$, or $c = 0$, $a < 0$, $b, c > 0$, then (3.1) is well posed in $W_p^s(\mathbb{R}) \times W_p^{s+1}(\mathbb{R})$ for $s \geq 0$ and $1 < p < \infty$. It is ill posed for the same range of s in the same spaces if $p = 1, \infty$.

D. Assume that $\mathcal{H}$ has order 2.

(1) The only admissible case is when σ has order 0, $b = c = 0, a < 0$ and $d > 0$. In this case, (3.1) is well posed for all values of $p \in [1, \infty]$ in $W_p^s(\mathbb{R}) \times W_p^{s+1}(\mathbb{R})$ provided that $s \geq 0$

To establish this result, we use the following lemma.

Lemma 4.8. *If $\mathcal{H}$ has order 0, then it is bounded as a mapping of $L_p(\mathbb{R})$ for $1 \leq p \leq \infty$.*

Proof. It suffices to consider the case when $\mathcal{H}$ has symbol $h(k) = (\frac{1-ck^2}{1-ak^2})^{\frac{1}{2}}$, $a < 0, c < 0$. The full result follows by composing two such mappings.

The L_2-boundedness is trivial since $h \in L_\infty(\mathbb{R})$. The L_p-boundedness, $1 < p < +\infty$, results from Mikhlin's multiplier theorem. The remaining cases $p = 1, +\infty$ follow from the fact that the Fourier transform of h is a bounded measure. In fact, write h in the form

$$
\begin{aligned}
h(k) &= \left(\frac{c}{a}\right)^{\frac{1}{2}} + \frac{a-c}{a}\frac{1}{[(1-ak^2)(1-ck^2)]^{\frac{1}{2}} + (\frac{c}{a})^{\frac{1}{2}}(1-ak^2)} \\
&\equiv \left(\frac{c}{a}\right)^{\frac{1}{2}} + f(k),
\end{aligned}
$$

where $f \in H^1(\mathbb{R})$. A consequence of this representation is that

$$\hat{h} = \left(\frac{c}{a}\right)^{\frac{1}{2}} \delta + \hat{f}, \quad \text{with } \hat{f} \in L_1(\mathbb{R}).$$

The lemma is thus established. □

Proof of Theorem 4.7. Part A. For (1), the only admissible situation corresponds to $a = c > 0$ and $b = d = 0$, in which case $\mathcal{H}$ is the identity mapping and the diagonalized variables are linked to the physical variables via the relations

$$\eta = v_1 + v_2 \quad \text{and} \quad w = v_1 - v_2,$$

so that

$$v_1 = \frac{1}{2}(\eta + w) \quad \text{and} \quad v_2 = \frac{1}{2}(\eta - w).$$

The proposition then follows directly from the results in [23].

For cases (2) and (3), the result follows on account of Lemma 4.8 in light of the results in Theorem 4.6 for the (v_1, v_2) variables.

Part B. (1) The well-posedness in $W_p^{s+1}(\mathbb{R}) \times W_p^s(\mathbb{R})$ for $1 < p < \infty$ results from (4.8) and Proposition 4.5. To establish the ill-posedness for $p = 1, \infty$ take $\phi = 0$ so that, by (4.8), $v_1^0 = \frac{1}{2}\psi$ and $v_2^0 = -\frac{1}{2}\psi$ with $\psi \in L_p$. This leads to an L_p-instability in (4.10).

The further cases noted in the theorem follow similarly from the foregoing results; we pass over the details. □

5. Linearized Higher-Order Systems and Some Interesting Sample Systems

In this section, consideration is given to the linear part of the systems in (1.7). The initial-value problem in question is

$$\begin{aligned} &\eta_t - b\eta_{xxt} + b_1\eta_{xxxxt} = -w_x - aw_{xxx} - a_1 w_{xxxxx}, \\ &w_t - dw_{xxt} + d_1 w_{xxxxt} = -\eta_x - c\eta_{xxx} - c_1\eta_{xxxxx}, \\ &\eta(x,0) = \phi(x), \qquad w(x,0) = \psi(x), \end{aligned} \tag{5.1}$$

for $x \in \mathbb{R}$, $t \geq 0$, where ϕ and ψ are to be selected from appropriate function classes.

As before when the nonlinear terms have been dropped, the system may be analyzed via Fourier analysis. Taking the Fourier transform of (5.1) with respect to the spatial variable x, it is seen that

$$\frac{d}{dt}\begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} + ik\mathcal{A}(k)\begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix} = 0, \quad \text{where} \quad \mathcal{A}(k) = \begin{pmatrix} 0 & \omega_1(k) \\ \omega_2(k) & 0 \end{pmatrix},$$

and

$$\omega_1(k) = \frac{1 - ak^2 + a_1k^4}{1 + bk^2 + b_1k^4}, \qquad \omega_2(k) = \frac{1 - ck^2 + c_1k^4}{1 + dk^2 + d_1k^4}.$$

Just as in Section 3, the solution of this system is

$$\begin{pmatrix} \hat{\eta} \\ \hat{w} \end{pmatrix}(k,t) = e^{-ik\mathcal{A}(k)t} \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix}(k). \tag{5.2}$$

The first point is to recognize when the matrix of Fourier multipliers $m(k,t) = e^{ik\mathcal{A}(k)t}$ has entries that are bounded on bounded intervals in k-space. Letting

$$\sigma(k) = |\omega_1(k)\omega_2(k)|^{\frac{1}{2}},$$

the next proposition follows immediately from the formulas (3.7)–(3.8) for $m(k,t)$.

Proposition 5.1. *The entries of the matrix $e^{-ik\mathcal{A}(k)t}$ are bounded on bounded sets of wavenumbers k if*

$$\left|\frac{\omega_1(k)}{\omega_2(k)}\right|, \qquad \left|\frac{\omega_2(k)}{\omega_1(k)}\right| \quad \textit{and} \; -\omega_1(k)\omega_2(k) \tag{5.3}$$

are bounded above.

The following theorem then applies.

Theorem 5.2. *Suppose the constants $a, b, c, d, a_1, b_1, c_1, d_1$ to satisfy the relations (1.9), (2.27) and to be such that (5.3) is valid. Define the pseudodifferential operator $\mathcal{H}$ by*

$$\widehat{\mathcal{H}g}(k) = h(k)\hat{g}(k), \quad \textit{where} \quad h(k) = \left(\frac{\omega_1(k)}{\omega_2(k)}\right)^{\frac{1}{2}}. \tag{5.4}$$

Suppose $\mathcal{H}$ to have order l, and set $m_1 = \max(0, -l)$ and $m_2 = \max(0, l)$. Then the corresponding linear initial-value problem (5.2) is well posed in $H^{s+m_1} \times H^{s+m_2}$ for any $s \geq 0$.

This theorem applies to some interesting specializations of the linear versions of the systems displayed in (1.7)–(1.9)–(2.27).

K-dV–type systems: Let $b = b_1 = d = d_1 = 0$ in (1.7), a situation obtained by choosing

$$\lambda = \lambda_1 = 1, \qquad \mu = 1, \qquad \mu_1 = \frac{6}{5}\frac{(\theta^2 - 1)}{(\theta^2 - \frac{1}{5})} \quad \left(\mu_1 \text{ is arbitrary if } \theta^2 = \frac{1}{5}\right),$$

in (1.9). Then, it transpires that

$$\begin{aligned} a &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad a_1 = \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2, \\ c &= \frac{1}{2}(1 - \theta^2), \qquad c_1 = \frac{1}{24}(1 - \theta^2)(5 - \theta^2), \end{aligned} \tag{5.5}$$

and the system of equations becomes

$$
\begin{aligned}
\eta_t &= -w_x - (\eta w)_x - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right) w_{xxx} \\
&\quad + \frac{1}{2}(1-\theta^2)(\eta w_{xx})_x - \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2 w_{xxxxx}, \\
w_t &= -\eta_x + \frac{1}{2}(\theta^2 - 1)\eta_{xxx} - w w_x - (\eta\eta_{xx})_x \\
&\quad - (2-\theta^2) w_x w_{xx} - \frac{1}{24}(1-\theta^2)(5-\theta^2)\eta_{xxxxx}.
\end{aligned}
\tag{5.6}
$$

Corollary 5.3. *For any θ in $[0, 1]$, the initial-value problem for the linear K-dV–type systems (5.6) is well posed in $H^{s+m_1} \times H^{s+m_2}$ for any $s \geq 0$, where m_1 and m_2 are defined as above.*

Proof. For this case,

$$
\omega_1(k) = 1 - ak^2 + a_1 k^4 \quad \text{and} \quad \omega_2(k) = 1 - ck^2 + c_1 k^4.
$$

It is easy to check that $\omega_1 > 0$ and $\omega_2 > 0$ and that condition (5.3) is satisfied. □

Lower-degree systems: Let $a_1 = b_1 = c_1 = d_1 = 0$, a situation obtained for $\theta^2 \neq \frac{1}{3}$ by letting

$$
\lambda_1 = \mu_1 = 1, \qquad \lambda = 1 - \frac{5}{6}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})^2}, \qquad \mu = \frac{(5\theta^2 - 1)}{6(\theta^2 - 1)}
$$

(μ is arbitrary if $\theta^2 = 1$).

For these values of the constants, one finds that

$$
\begin{aligned}
a &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right) - \frac{5}{12}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})}, \qquad b = \frac{5}{12}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})}, \\
c &= -\frac{5}{12}\left(\theta^2 - \frac{1}{5}\right), \qquad d = \frac{1}{12}(5 - \theta^2).
\end{aligned}
\tag{5.7}
$$

The corresponding system of equations, which has order three, is

$$
\begin{aligned}
\eta_t - b\eta_{xxt} &= -w_x - (\eta w)_x \\
&\quad - a w_{xxx} + b(\eta w)_{xxx} + \frac{1}{2}(1-\theta^2)(\eta w_{xx})_x, \\
w_t - d w_{xxt} &= -\eta_x - c\eta_{xxx} - w w_x - (\eta\eta_{xx})_x \\
&\quad - \frac{3}{4}(1-\theta^2) w_x w_{xx} + \frac{1}{12}(5-\theta^2) w w_{xxx}.
\end{aligned}
\tag{5.8}
$$

In this case, the linearized phase speed is

$$
c^2(k) = 1 - \frac{1}{3}k^2 + \frac{2}{15}k^4 + \frac{-201 + 655\theta^2 - 195\theta^4 + 125\theta^6}{3600(1 - 3\theta^2)}k^6 + O(k^8),
$$

according to (2.30). If the coefficient of k^6 equals $-17/315$, then the linearized phase speed for the system agrees with that of the linearized Euler equations to the sixth order. In the present case, this occurs for values of θ^2 about 0.1438 and 0.3504. With these choices of θ, there obtains a third-order system of equations whose linearized phase speed agrees with the linearized Euler equations to sixth order.

Corollary 5.4. *For the lower-degree systems (5.8) with a, b, c, and d satisfying (5.7) for some $\theta^2 \in (\frac{1}{3}, 1]$, if the pseudodifferential operator $\mathcal{H}$ has order l and $m_1 = \max(0, -l)$, $m_2 = \max(0, l)$, then the corresponding linear initial-value problem (5.8) is well-posed in $H^{s+m_1} \times H^{s+m_2}$ for any $s \geq 0$ for any value of θ in $(\frac{1}{3}, 1]$.*

Proof. Because these systems are third-order, Theorem 3.2 can be applied. For $\frac{1}{3} < \theta^2 \leq 1$, one of Conditions (C1)–(C3) holds and the result follows. □

BBM-type systems: There are many options for obtaining systems of equations which correspond to the regularized long-wave equation in the modeling of unidirectional waves. To make the linearized equation well posed, it suffices to require

$$\begin{aligned} &b \geq 0, \qquad b_1 > 0, \qquad a < 0, \qquad a_1 = 0,\\ &d \geq 0, \qquad d_1 > 0, \qquad c < 0, \quad c_1 = 0, \end{aligned}$$

which is equivalent to asking that

$$\begin{aligned} &\frac{1}{3} < \theta^2 < 1, \qquad \lambda = 1 - \frac{5}{6}\left(\frac{\theta^2 - 1/5}{\theta^2 - 1/3}\right)^2 \lambda_1,\\ &\lambda_1 > 1, \qquad \mu \leq 1, \qquad \mu_1 = 1. \end{aligned} \tag{5.9}$$

Corollary 5.5. *For the regularized long-wave-type systems, namely θ^2, λ, λ_1, μ, and μ_1 satisfying (5.9), the pseudodifferential operator $\mathcal{H}$ has order 0 and the corresponding linear initial-value problem (5.1) is well posed in $H^s \times H^s$ for any $s \geq 0$ for any value of θ in $(\frac{1}{3}, 1)$.*

Proof. This is a direct consequence of Theorem 5.2. □

6. Conclusion

Put forward here is a class of Boussinesq systems for the two-way propagation of surface water waves. An analysis of the associated linearized systems shows only a subclass to be linearly well-posed and to not feature spurious growth or dissipation.

This line of research needs further development both theoretically and with respect to applications. The foregoing linear theory, while helpful and suggestive, needs to be supplemented with associated nonlinear theories of well-posedness. In particular, one would like to know which of the class (1.6) and (1.7) are globally well-posed at least for small-amplitude, long-wavelength initial data. A start on this program will appear in Part

II of the present paper [15]. Another natural theoretical point is whether or not various of the systems possess solitary waves, and if they have such traveling-wave solutions, how they interact and what their stability properties are. Work in this direction was initiated in [32], [33].

We hope to make quantitative comparisons between model predictions and careful, laboratory measurement, along the lines of those made for unidirectional models by Zabusky and Galvin [63], Hammack and Segur [39], and Bona, Pritchard, and Scott [18]. In this regard, it will be necessary to incorporate damping into the models and analyze appropriate two-point boundary-value problems as was done in [14] for one of the first-order correct models. A longer-term outlook is the incorporation of variable bottom boundary structure with an eye toward improving models for wave-generated, sediment transport (cf. [10], [11], [12], [34]).

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