NONLINEAR GALERKIN METHOD WITH MULTILEVEL INCREMENTAL UNKNOWNS

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ABSTRACT. Multilevel methods are indispensable for the approximation of nonlinear evolution equations when complex physical phenomena involving the interaction of many scales are present (such as in, but without being limited to fluid turbulence). Incremental unknowns of different types have been proposed as a means to develop such numerical schemes in the context of finite difference discretizations.

In this article, we present several numerical schemes using the so-called multilevel wavelet-like incremental unknowns. The fully discretized explicit and semi-explicit schemes for reaction-diffusion equations are presented and analyzed. The stability conditions are improved when compared with the corresponding standard algorithms. Furthermore the complexity of the computation on each time step is comparable to the corresponding standard algorithm.

1. Introduction.

In the past, the approximation of nonlinear evolution equations was mostly restricted to short intervals of time or to long intervals of time when the solution converges to a stationary one as $t \to \infty$.

The new technologies and the increased power of the new computers offer to the numerical analysts new challenging problems, namely the approximation of nonlinear evolution equations on large intervals of time when complex physical phenomena appear. New numerical methods adapted to such problems need to be developed (see [9]); in particular multilevel methods are needed in order to treat appropriately the different scales appearing in a complex problem and to resolve at reasonable cost the smaller scales.

Incremental unknowns have been proposed as a means to address this new type of problems when finite difference discretizations are used. The idea is to treat differently the small and large scale components of a flow and in this way to avoid

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stiff systems; to save on computing time; and to obtain better CFL (Courant-Fredriche-Levy) stability conditions (see [9]).

After studying linear elliptic problems in [1] and [2], we consider here nonlinear evolution equations. As a first example, we apply the multilevel wavelet-like incremental unknowns to a Reaction-Diffusion equation:

$$\frac{\partial u}{\partial t} - \nu \Delta u + g(u) = 0 \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0 \text{ in } \Omega. \tag{1.3}$$

Here $\nu > 0, \Omega$ is an open bounded set in $\mathbb{R}^n$ with sufficient smooth boundary and

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0.$$

For the sake of simplicity, we shall consider only the one-dimensional case and $\Omega = [0,1]$ in the rest of the paper. The higher dimensional cases can be treated in the same way. The definition of the wavelet-like incremental unknowns in dimension two can be found in [3] and it is recalled below in dimension one.

The article is organized as follows. In Section 2 we recall the definition of the wavelet-like incremental unknowns (WIU) and describe their implementation in the *space discretization* of problem (1.1)-(1.3). Then in Section 3 we consider *space and time discretization.* Four different schemes are proposed which are of the nonlinear Galerkin type. Finally in Section 4 we develop the stability analysis of these schemes. The limitation of the time mesh $k = \Delta t$ are much better than those obtained with usual one-level spatial discretizations. Of course as usual for nonlinear problems, our stability analysis provides only sufficient stability conditions; however there are also numerical simulations performed for Burgers equation which confirm these improvements [5].

2. Multilevel Wavelet-like Incremental Unknowns (WIU).

In this section, we shall transform a spatial finite difference discretization in terms of U into a scheme involving Y and Z where U is the finite difference approximation of the solution u, while Y represents a coarse grid approximation and Z represents a fine grid correction (the incremental unknowns).

Considering spatial discretization by finite difference with mesh size $h_d = 1/(2^d N + 1)$, where $N \in \mathbb{N}$, we have

$$\frac{\partial U_{h_d}}{\partial t} + \nu A_{h_d} U_{h_d} + g(U_{h_d}) = 0, \tag{2.1}$$

where U_{h_d} is the vector of approximate values of u at the grid points, $U_{h_d} \in \mathbb{R}^{2^d N}$ and A_{h_d} is a regular matrix of order $2^d N$. For simplicity, we write $A_d = A_{h_d}, U_d =$

U_{h_d}. When a central difference scheme is used for the convection term, we have

$$A_d U_d(i) = \frac{1}{h_d^2}(2U_d(i) - U_d(i+1) - U_d(i-1)),$$

where $U_d(i)$ is the finite difference approximation of u at $x = ih_d$. Ordering $U_d(i), i = 1, 2, \ldots, 2^d N$ in its natural way, we see that A_d is a tri-diagonal matrix.

We now introduce the $(d+1)$ levels Wavelet Incremental Unknowns (WIU) into equation (2.1). We first separate evenly the unknowns into two parts, one part represents a coarser grid approximation, another represents a correction to the coarser grid approximation. We obtain 2-level wavelet-like incremental unknowns. After the first split, the unknowns which represent a coarser grid approximation can be separated again into two parts .... After d-time of separations, there are N unknowns (Y part) which represent the coarsest grid approximation and each of them is an average of 2^d unknowns from U_d. The other $(2^d - 1)N$ unknowns (Z part) are the correction of Y to bring the total accuracy of the approximation into the accuracy of U_d.

We now introduce the first separation. The incremental unknown $\overline{U}_d$ in this level as stated consists of two parts:

- the coarser grid approximation which is the average of two neighboring values of finer grid
$$y_{2i}^d = (U_d(2i-1) + U_d(2i))/2, \quad i = 1, \ldots, 2^{d-1}N, \tag{2.2}$$
- the increment on the fine grid approximation
$$z_{2i-1}^d = (U_d(2i-1) - U_d(2i))/2, \quad i = 1, \ldots, 2^{d-1}N. \tag{2.3}$$

The transformation from U_d to incremental unknowns $\overline{U}_d$ is the inverse of (2.2) and (2.3):

$$\begin{cases} U_d(2i) = y_{2i}^d - z_{2i-1}^d, \\ U_d(2i-1) = z_{2i-1}^d + y_{2i}^d, \end{cases} \quad \text{for } i = 1, \ldots, 2^{d-1}N. \tag{2.4}$$

We reordering U_d into $\widetilde{U}_d$ by letting

$$\widetilde{U}_d = (U_d(2), \ldots, U_d(2^d N), U_d(1), \ldots, U_d(2^d N - 1))^T,$$

and we see that

$$U_d = P_d \widetilde{U}_d,$$

where P_d is a permutation matrix. We then write (2.4) in the matrix form

$$\widetilde{U}_d = S_d \overline{U}_d,$$

where $\overline{U}_d = (Y_d, Z_d) = (y_2^d, \dots, y_{2^dN}^d, z_1^d, \dots, z_{2^dN-1}^d)$, and

$$S_d = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & -1 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} I_{d-1} & -I_{d-1} \\ I_{d-1} & I_{d-1} \end{pmatrix},$$

I_{d-1} being the identity matrix of order $2^{d-1}N$. We can easily see that $S^{-1} = \frac{1}{2}S^T$, and

$$U_d = P_d S_d \overline{U}_d. \tag{2.5}$$

Substituting (2.5) into the finite difference equation (2.1) and multiplying the equation by $(P_dS_d)^T$, we find

$$\frac{\partial (P_dS_d)^T P_d S_d \overline{U}_d}{\partial t} + \nu (P_dS_d)^T A_d P_d S_d \overline{U}_d + (P_dS_d)^T g(P_dS_d\overline{U}_d) = 0.$$

Noticing that $P_d^T P_d = I_d$, $S_d^T S_d = 2I_d$, P_d^T and g commute, we obtain

$$2\frac{\partial \overline{U}_d}{\partial t} + \nu S_d^T P_d^T A_d P_d S_d \overline{U}_d + S_d^T g(S_d \overline{U}_d) = 0, \tag{2.6}$$

which is the finite difference scheme obtained when 2-level wavelet-like incremental unknowns are used. Equation (2.6) is equivalent to (2.1) except that we have replaced U_d by $\overline{U}_d = (Y_d, Z_d)^T$.

We can again introduce the next level of WIU on Y_d by repeating exactly the same procedure. Let

$$\overline{Y}_d = (Y_{d-1}, Z_{d-1})^T = (y_4^{d-1}, y_8^{d-1}, \dots, y_{2^dN}^{d-1}, z_2^{d-1}, z_6^{d-1}, \dots, z_{2^dN-2}^{d-1})^T;$$

we replace U_d by Y_d and Z_d by Z_{d-1} and change the corresponding subscript in (2.4). Namely, we define

$$\begin{cases} y_{4i}^d = y_{4i}^{d-1} - z_{4i-2}^{d-1}, \\ y_{4i-2}^d = z_{4i-2}^{d-1} + y_{4i}^{d-1}, \end{cases} \quad \text{for } i = 1, \dots, 2^{d-2}N. \tag{2.7}$$

Therefore,

$$Y_d = P_{d-1} S_{d-1} \overline{Y}_d,$$

where P_{d-1} is a permutation matrix of order $2^{d-1}N$ and

$$S_{d-1} = \begin{pmatrix} I_{d-2} & -I_{d-2} \\ I_{d-2} & I_{d-2} \end{pmatrix}.$$

Noticing that the size of Y_d is only half that of U_d, we let

$$\overline{U}_{d-1} = (Y^{d-1}, Z^{d-1}, \frac{1}{\sqrt{2}} Z^d)^T,$$

$$\widetilde{P}_{d-1} = \begin{pmatrix} P_{d-1} & 0 \\ 0 & I_{d-1} \end{pmatrix}, \quad \widetilde{S}_{d-1} = \begin{pmatrix} S_{d-1} & 0 \\ 0 & \sqrt{2} I_{d-1} \end{pmatrix}.$$

Here $\widetilde{P}_{d-1}$ and $\widetilde{S}_{d-1}$ are matrices of order $2^d N$. We see that

$$\widetilde{P}_d \widetilde{P}_d^T = I_d, \qquad \widetilde{S}_d \widetilde{S}_d^T = 2I_d,$$

$$\overline{U}_d = \widetilde{P}_{d-1} \widetilde{S}_{d-1} \overline{U}_{d-1}.$$

Generally for $l = d-1, d-2, \ldots, 1$, we define $\overline{Y}_{l+1} = (Y_l, Z_l)^T$ and

$$\begin{cases} y^{l+1}_{2^{d-l+1}i} = y^l_{2^{d-l+1}i} - z^l_{2^{d-l+1}i-2^{d-l}}, \\ y^{l+1}_{2^{d-l+1}i-2^{d-l}} = z^l_{2^{d-l+1}i-2^{d-l}} + y^l_{2^{d-l+1}i}, \end{cases} \quad \text{for } i = 1, \ldots, 2^{l-1}N. \tag{2.8}$$

We can easily see as previously that

$$Y_{l+1} = P_l S_l \overline{Y}_l, \quad l = d-1, d-2, \ldots, 1, \tag{2.9}$$

where P_l, S_l have similar structures as P_d and S_d but with different sizes. We can include (2.5) into formula (2.9) with $l = d$ by denoting $Y_{d+1} = U_d$ and $\overline{Y}_d = \overline{U}_d$. Setting $\overline{U}_l = (Y_l, Z_l, \frac{1}{\sqrt{2}} Z_{l-1}, \ldots, \frac{1}{\sqrt{2}^{d-l}} Z_d)^T$, we obtain

$$\overline{U}_l = \widetilde{P}_l \widetilde{S}_l \overline{U}_{l-1} \tag{2.10}$$

where

$$\widetilde{P}_l = \begin{pmatrix} P_l & 0 \\ 0 & I_{k\times k} \end{pmatrix}, \quad \text{where } k = (2^d - 2^l)N.$$

$$\widetilde{S}_l = \begin{pmatrix} I_{l-1} & -I_{l-1} & 0 \\ I_{l-1} & I_{l-1} & 0 \\ 0 & 0 & \sqrt{2} I_{k\times k} \end{pmatrix} = \begin{pmatrix} S_l & 0 \\ 0 & I_{k\times k} \end{pmatrix},$$

$$\widetilde{S}_l \widetilde{S}_l^T = 2I_d, \quad \widetilde{P}_l \widetilde{P}_l^T = I_d.$$

Substituting (2.10) into equation (2.6) successively with $l = d, \ldots, 1$, we obtain the evolution equation expressed in terms of Y and Z where $Y = Y_0$, $Z = (Z_0, \frac{1}{\sqrt{2}} Z_1, \ldots, \frac{1}{\sqrt{2}^d} Z_d)^T$:

$$2^d \frac{\partial \overline{U}_0}{\partial t} + \nu S^T A_d S \overline{U}_0 + S^T g(S\overline{U}_0) = 0, \tag{2.11}$$

with $S = \widetilde{P}_d \widetilde{S}_d \ldots \widetilde{P}_1 \widetilde{S}_1$.

3. Nonlinear Galerkin method.

In this section, we propose some new schemes based on the utilization of the incremental unknowns introduced in the last section. The new schemes will not only simplify the formulas which make them easier to implement, but also improve the stability conditions comparing to the corresponding schemes in the last section while maintaining the same complexity of the computation (see Section 4). The schemes we shall propose are obtained by neglecting some small terms involving Z. A partial justification of these schemes can also be seen through the dynamical system theory (c.f. [6], [7], [8]). In this section, we shall first propose the new treatment of the spatial discretization. We then propose several fully discretized schemes. The stability conditions for the fully discretized schemes will be presented in the next section. The convergence of these schemes can be proved by using the stability results in the next section and then proceeding as in the proof of convergence of the nonlinear galerkin method for Navier-stokes type equations in [3].

We now analyze (2.11) and start with $d = 1$; we write

$$2\frac{\partial \overline{U}_d}{\partial t} + \nu S_d^T P_d^T A_d P_d S_d \overline{U}_d + S_d^T g(S_d \overline{U}_d) = 0.$$

But

$$S_d\overline{U}_d = \begin{pmatrix} I_{d-1} & -I_{d-1} \\ I_{d-1} & I_{d-1} \end{pmatrix} \begin{pmatrix} Y_d \\ Z_d \end{pmatrix} = \begin{pmatrix} Y_d - Z_d \\ Y_d + Z_d \end{pmatrix},$$

$$\begin{aligned} S_d^T g(S_d\overline{U}_d) &= \begin{pmatrix} I_{d-1} & I_{d-1} \\ -I_{d-1} & I_{d-1} \end{pmatrix} \begin{pmatrix} g(Y_d - Z_d) \\ g(Y_d + Z_d) \end{pmatrix} \\ &= \begin{pmatrix} g(Y_d - Z_d) + g(Y_d + Z_d) \\ g(Y_d + Z_d) - g(Y_d - Z_d) \end{pmatrix} = \begin{pmatrix} 2g(Y_d) + O(|Z_d|^2) \\ O(|Z_d|) \end{pmatrix}. \end{aligned} \tag{3.1}$$

We therefore obtain the 2-level nonlinear galerkin method

$$2\frac{\partial}{\partial t}\begin{pmatrix} Y_d \\ Z_d \end{pmatrix} + \nu S_d^T P_d^T A_d P_d S_d \overline{U}_d + 2\begin{pmatrix} g(Y_d) \\ 0 \end{pmatrix} = 0,$$

by neglecting a $O(|Z_d|^2)$ term in the evolution equation for Y_d and a $O(|Z_d|)$ term in the equation for the evolution of Z_d.

Now, when $d = 2$, we have

$$4\frac{\partial \overline{U}_{d-1}}{\partial t} + \nu S^T A_d S \overline{U}_{d-1} + \widetilde{S}_{d-1}^T \widetilde{P}_{d-1}^T S_d^T g(S_d \overline{U}_d) = 0.$$

Using the approximation for (3.1), P_{d-1} and g commute, we have

$$\begin{aligned} \widetilde{S}_{d-1}^T \widetilde{P}_{d-1}^T S_d^T g(S_d\overline{U}_d) &\approx 2\widetilde{S}_{d-1}^T \widetilde{P}_{d-1}^T \begin{pmatrix} g(Y_d) \\ 0 \end{pmatrix} \\ &= 2\begin{pmatrix} S_{d-1}^T & 0 \\ 0 & I_{k\times k} \end{pmatrix} \begin{pmatrix} \widetilde{P}_{d-1}^T g(\widetilde{P}_{d-1} S_{d-1} \overline{Y}_d) \\ 0 \end{pmatrix} = 2\begin{pmatrix} S_{d-1}^T g(S_{d-1}^T \overline{Y}_d) \\ 0 \end{pmatrix}. \end{aligned}$$

We can again use the same approximation technique as for (3.1) and obtain

$$\widetilde{S}_{d-1}^T \widetilde{P}_{d-1}^T S_d^T g(S_d \overline{U}_d) \approx 4 \begin{pmatrix} g(Y_{d-1}) \\ 0 \end{pmatrix}.$$

Therefore, we can easily see that the nonlinear galerkin method with the use of $(d+1)$-level incremental unknowns leads to equation

$$2^d \frac{\partial}{\partial t} \begin{pmatrix} Y_0 \\ Z \end{pmatrix} + \nu S^T A_d S \overline{U}_0 + 2^d \begin{pmatrix} g(Y_0) \\ 0 \end{pmatrix} = 0 \tag{3.2}$$

where $Y_0 \in I\!R^N$ and Z is a vector of dimension $(2^d - 1)N$.

From the theory of inertial manifolds, we sometime prefer to neglect also the $\frac{\partial Z}{\partial t}$ term. Therefore, another similar scheme can be proposed

$$2^d \frac{\partial}{\partial t} \begin{pmatrix} Y_0 \\ 0 \end{pmatrix} + \nu S^T A_d S \overline{U}_0 + 2^d \begin{pmatrix} g(Y_0) \\ 0 \end{pmatrix} = 0. \tag{3.3}$$

Now we consider time discretization. We can easily obtain an explicit scheme for (3.2) by using the explicit Euler scheme.

Scheme I. *Explicit scheme*

$$\frac{2^d}{k} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \overline{U}_0^n + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

Based on (3.3), we can also obtain a similar scheme by omitting the discretized time derivative of Z:

Scheme I'.

$$\frac{2^d}{k} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ 0 \end{pmatrix} + \nu S^T A_d S \overline{U}_0^n + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

Alternatively, taking a backward Euler scheme for the time discretization of the linear terms, we obtain semi-implicit schemes:

Scheme II. *Semi-implicit scheme*

$$\frac{2^d}{k} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \overline{U}_0^{n+1} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

Scheme II'.

$$\frac{2^d}{k} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ 0 \end{pmatrix} + \nu S^T A_d S \overline{U}_0^{n+1} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

The effective implementation of the above schemes is very similar to using incremental unknowns for solving linear problems (c.f. [1], [2]). The product of $S^T A_d S^T$ with a vector can be obtained without writing out the explicit form of S; $O(2^d N)$ flops are required which is the order of flops required for the product of A_d with a vector.

4. Stability Analysis for the Fully Discretized schemes.

Let $\mathcal{V}_{h_d}$ be the function space spanned by the basis functions $w_{h_d,M}, M = ih_d, i = 1, 2, \ldots, 2^d N$; w_{h_d,ih_d} is equal to 1 on the interval $[ih_d, (i+1)h_d)$ and vanishes outside this interval; let $u_{h_d}(x)$ be a step function in $\mathcal{V}_{h_d}$ and $u_{h_d}(x) = U_d(i)$, for $ih_d \leq x < (i+1)h_d, i = 1, 2, \ldots, 2^d N$. Hence

$$u_{h_d}(x) = \sum_{i=1}^{2^d N} U_d(i) w_{h_d,ih_d}, \qquad x \in \Omega.$$

We introduce the finite difference operator

$$\nabla_{h_d}\phi(x) = \frac{1}{h_d}\{\phi(x+h_d) - \phi(x)\},$$

and endow $\mathcal{V}_{h_d}$ with the scalar product

$$((u_{h_d}, v_{h_d}))_{h_d} = (\nabla_{h_d} u_{h_d}, \nabla_{h_d} v_{h_d}),$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$. We set $||\cdot||_{h_d} = \{((\cdot,\cdot))_{h_d}\}^{1/2}$ and observe that $||\cdot||_{h_d}$ and $|\cdot|$ are Hilbert norms on $\mathcal{V}_{h_d}$.

Using the space $\mathcal{V}_{h_d}$, we can write the finite difference discretization scheme (2.1) in variational form as

$$(\frac{\partial u_{h_d}}{\partial t}, \tilde{u}) + \nu((u_{h_d}, \tilde{u}))_{h_d} + (g(u_{h_d}), \tilde{u}) = 0, \quad \forall \tilde{u} \in \mathcal{V}_{h_d}. \tag{4.1}$$

We can recover (2.1) by choosing $\tilde{u} = w_{h_d,ih_d}$. It is not hard to see that we can recover the definition of wavelet-like incremental unknowns by a suitable decomposition of the space $\mathcal{V}_{h_d}$ (c.f. [3]). We define $\mathcal{Y}_{h_d}$ (or simply $\mathcal{Y}_d$) as the space spanned by the basis functions $\psi_{2h_d,M}$, where $M = 2ih_d, i = 1, 2, \ldots, 2^{d-1}N$; here $\psi_{2h_d,2ih_d}$ is equal to 1 on the interval $[2ih_d - h_d, 2ih_d + h_d)$ and vanishes outside this interval. Thus

$$y_d(x) = \sum_{i=1}^{2^{d-1}N} y_d(2ih_d)\psi_{2h_d,2ih_d}, \qquad x \in \Omega, \quad \forall y_d \in \mathcal{Y}_d.$$

We then define $\mathcal{Z}_d$ as the space spanned by $\chi_{h_d,M} = w_{h_d,M} - w_{h_d,M+h_d}$, where $M = (2i-1)h_d, i = 1, 2, \ldots, 2^{d-1}N$. We have

$$z_d(x) = \sum_{i=1}^{2^{d-1}N} z_d((2i-1)h_d)\chi_{2h_d,2ih_d-h_d}, \qquad x \in \Omega, \quad \forall z_d \in \mathcal{Z}_d.$$

Therefore,

$$\mathcal{V}_{h_d} = \mathcal{Y}_d \oplus \mathcal{Z}_d. \tag{4.2}$$

We now decompose the approximate solution $u_{h_d} \in \mathcal{V}_{h_d}$ into:

$$u_{h_d} = y_d + z_d, \qquad y_d \in \mathcal{Y}_d, \quad z_d \in \mathcal{Z}_d.$$

By identifying y_d and z_d on each interval $[2ih_d - h_d, 2ih_d + h_d), i = 1, \dots, 2^{d-1}N$ and writing $y_d(2ih_d) = y_{2i}^d,\ z_d(ih_d) = z_i^d$, we obtain exactly (2.4).

With decomposition (4.2), (2.6) is identical to

$$\begin{aligned}
&(\frac{\partial y_d}{\partial t}, \tilde{y}) + \nu((y_d + z_d, \tilde{y}))_{h_d} + (g(y_d + z_d), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}_d \\
&(\frac{\partial z_d}{\partial t}, \tilde{z}) + \nu((y_d + z_d, \tilde{z}))_{h_d} + (g(y_d + z_d), \tilde{z}) = 0, \quad \forall \tilde{z}_d \in \mathcal{Z}_d
\end{aligned}$$

Multilevel incremental unknowns can be recovered in a similar fashion. We decompose $\mathcal{Y}_l$, $l = d, \dots, 1$, into

$$\mathcal{Y}_l = \mathcal{Y}_{l-1} \oplus \mathcal{Z}_{l-1},$$

and we recover (2.8) by defining $\mathcal{Y}_{l-1}$ and $\mathcal{Z}_{l-1}$ accordingly. We therefore see that for any function $u_{h_d} \in \mathcal{V}_{h_d}$, we can write it as

$$u_{h_d} = y + z,$$

where $y = y_{h_0} \in \mathcal{Y} = \mathcal{Y}_0$ and $z \in \mathcal{Z} = \mathcal{Z}_0 \oplus \mathcal{Z}_1 \oplus \cdots \oplus \mathcal{Z}_d$; $\mathcal{Y}_0$ is a function space spanned by the step functions with step size $h_0 = 2^d h_d$ and

$$(y, z) = 0, \quad \forall y \in \mathcal{Y}_0, \forall z \in \mathcal{Z}.$$

Equation (2.11) is therefore identical to

$$\begin{aligned}
(\frac{\partial y_{h_0}}{\partial t}, \tilde{y}) + \nu((y_{h_0} + z, \tilde{y}))_{h_d} + (g(y_{h_0} + z), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}_0, \\
(\frac{\partial z}{\partial t}, \tilde{z}) + \nu((y_{h_0} + z, \tilde{z}))_{h_d} + (g(y_{h_0} + z), \tilde{z}) = 0, \quad \forall \tilde{z} \in \mathcal{Z},
\end{aligned}$$

and (3.3) is identical to

$$\begin{aligned}
(\frac{\partial y_{h_0}}{\partial t}, \tilde{y}) + \nu((y_{h_0} + z, \tilde{y}))_{h_d} + (g(y_{h_0}), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}_0, \\
(\frac{\partial z}{\partial t}, \tilde{z}) + \nu((y_{h_0} + z, \tilde{z}))_{h_d} = 0, \quad \forall \tilde{z} \in \mathcal{Z}.
\end{aligned} \tag{4.3}$$

Before presenting the stability theory, let us introduce some easy lemmas. Their proof can be found for example in [4].

Lemma 2.1. *There exist constants c_1 and c_2 such that the function g above satisfies*

$$g(s)s \geq \frac{1}{2} b_{2p-1} s^{2p} - c_1, \tag{4.4}$$

$$g(s)^2 \leq 2b_{2p-1}^2 s^{4p-2} + c_2, \quad \forall s. \tag{4.5}$$

Lemma 2.2. *For every function $u_h \in \mathcal{V}_h$,*

$$\sqrt{2}|u_h| \leq \|u_h\|_h \leq \frac{1}{S_1(h)}|u_h|,$$

where $S_1(h) = h/2$.

Lemma 2.3. *For every function $y_{h_0} \in \mathcal{Y}_0$,*

$$S_2(h_0)|y_{h_0}|_\infty^2 \leq |y_{h_0}|^2, \text{ with } S_2(h_0) = h_0.$$

$$\overline{S}_1(h_0, h_d)\|y_{h_0}\|_{h_d} \leq |y_{h_0}|, \text{ with } \overline{S}_1(h_0, h_d) = \frac{1}{2}\sqrt{h_0 h_d}.$$

Here $|y_{h_0}|_\infty$ is the maximum (L^∞) norm of y_{h_0}.

Theorem 2.1. *Stability condition for Scheme I*

We assume that $k \leq K_0$ for some K_0 fixed and let

$$M_0 = |u_{h_d}^0|^2 + \frac{1}{\nu}(c_1 + c_2 K_0)|\Omega|.$$

If

$$\frac{k}{h_d^2} \leq \frac{1}{4\nu}\left(\frac{2^d}{2+2^d}\right) \tag{4.6}$$

and

$$\frac{k}{(h_d)^{p-1}} \leq \frac{2^{d(p-1)}}{4b_{2p-1}M_0^{p-1}}, \tag{4.7}$$

we have for Scheme I the following estimate:

$$|u_{h_d}^n|^2 = |y_{h_0}^n|^2 + |z^n|^2 \leq M_0 \text{ for any } n \geq 0. \tag{4.8}$$

Proof. Using (4.3), we can write Scheme I in its variational form:

$$\left(\frac{y_{h_0}^{n+1} - y_{h_0}^n}{k}, \tilde{y}\right) + \nu((y_{h_0}^n + z^n, \tilde{y}))_{h_d} + (g(y_{h_0}^n), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}_0, \tag{4.9}$$

$$\left(\frac{z^{n+1} - z^n}{k}, \tilde{z}\right) + \nu((y_{h_0}^n + z^n, \tilde{z}))_{h_d} = 0, \quad \forall \tilde{z} \in \mathcal{Z}. \tag{4.10}$$

We let $\tilde{y} = 2ky^n_{h_0}$ in (4.9) and $\tilde{z} = 2kz^n$ in (4.10) and add these relations, since $2(a-b,b) = |a|^2 - |b|^2 - |a-b|^2$, we obtain

$$|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 - |y^{n+1}_{h_0} - y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 - |z^{n+1} - z^n|^2 + 2k\nu \|y^n_{h_0} + z^n\|^2_{h_d} + 2k(g(y^n_{h_0}), y^n_{h_0}) = 0.$$

By using (4.4) in Lemma 2.1, we find

$$|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 - |y^{n+1}_{h_0} - y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 - |z^{n+1} - z^n|^2 + 2k\nu \|y^n_{h_0} + z^n\|^2_{h_d} + kb_{2p-1}\int_\Omega (y^n_{h_0})^{2p}dx \leq 2kc_1|\Omega|.$$

Now, let $\tilde{y} = k(y^{n+1}_{h_0} - y^n_{h_0})$ in (4.9):

$$|y^{n+1}_{h_0} - y^n_{h_0}|^2 + k\nu((y^n_{h_0} + z^n, y^{n+1}_{h_0} - y^n_{h_0}))_{h_d} + k(g(y^n_{h_0}), y^{n+1}_{h_0} - y^n_{h_0}) = 0.$$

We obtain by using Cauchy-Schwarz inequality and Lemma 2.3

$$\begin{aligned}|y^{n+1}_{h_0} - y^n_{h_0}|^2 &\leq k\nu \|y^n_{h_0} + z^n\|_{h_d} \, \|y^{n+1}_{h_0} - y^n_{h_0}\|_{h_d} + k|g(y^n_{h_0})| \, |y^{n+1}_{h_0} - y^n_{h_0}| \\ &\leq k\nu \frac{1}{\overline{S}_1(h_0,h_d)} \|y^n_{h_0} + z^n\|_{h_d} \, |y^{n+1}_{h_0} - y^n_{h_0}| + k^2|g(y^n_{h_0})|^2 + \frac{1}{4}|y^{n+1}_{h_0} - y^n_{h_0}|^2 \\ &\leq \frac{1}{2}|y^{n+1}_{h_0} - y^n_{h_0}|^2 + k^2|g(y^n_{h_0})|^2 + \frac{k^2\nu^2}{\overline{S}_1(h_0,h_d)^2}\|y^n_{h_0} + z^n\|^2_{h_d}.\end{aligned}$$

Therefore

$$|y^{n+1}_{h_0} - y^n_{h_0}|^2 \leq 2k^2|g(y^n_{h_0})|^2 + \frac{2k^2\nu^2}{\overline{S}_1(h_0,h_d)^2}\|y^n_{h_0} + z^n\|^2_{h_d}.$$

We can bound $|z^{n+1} - z^n|^2$ by the same method. Let $\tilde{z} = k(z^{n+1} - z^n)$ in (4.10):

$$|z^{n+1} - z^n|^2 + k\nu((y^n_{h_0} + z^n, z^{n+1} - z^n))_{h_d} = 0.$$

We therefore obtain using Lemma 2.2 with $h = h_d$,

$$\begin{aligned}|z^{n+1} - z^n|^2 &\leq k\nu \|y^n_{h_0} + z^n\|_{h_d} \, \|z^{n+1} - z^n\|_{h_d} \\ &\leq k\nu \frac{1}{S_1(h_d)} \|y^n_{h_0} + z^n\|_{h_d} \, |z^{n+1} - z^n|.\end{aligned}$$

Therefore

$$|z^{n+1} - z^n|^2 \leq \frac{k^2\nu^2}{S_1(h_d)^2}\|y^n_{h_0} + z^n\|^2_{h_d}.$$

Combining these relations, we see that

$$|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 + (2k\nu - \frac{k^2\nu^2}{S_1(h_d)^2} - \frac{2k^2\nu^2}{\overline{S}_1(h_0,h_d)^2})\|y^n_{h_0} + z^n\|^2_{h_d} + kb_{2p-1}\int_\Omega (y^n_{h_0})^{2p}dx \leq 2kc_1|\Omega| + 2k^2|g(y^n_{h_0})|^2.$$

Using (4.5) pointwise and Lemma 2.3, we obtain

$$
\begin{aligned}
k^2|g(y^n_{h_0})|^2 &\le 2k^2 b^2_{2p-1}\int_\Omega (y^n_{h_0})^{4p-2}dx + k^2c_2|\Omega| \\
&\le 2k^2 b^2_{2p-1}|y^n_{h_0}|^{2p-2}_\infty \int_\Omega (y^n_{h_0})^{2p}dx + k^2c_2|\Omega| \qquad (4.11)\\
&\le \frac{2k^2 b^2_{2p-1}}{S_2(h_0)^{p-1}}|y^n_{h_0}|^{2p-2}\int_\Omega (y^n_{h_0})^{2p}dx + k^2c_2|\Omega|.
\end{aligned}
$$

This yields

$$
\begin{aligned}
|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 + (2k\nu - \frac{k^2\nu^2}{S_1(h_d)^2} - \frac{2k^2\nu^2}{\overline{S}_1(h_0,h_d)^2})\|y^n_{h_0}+z^n\|^2_{h_d} \\
+ (kb_{2p-1} - \frac{4k^2b^2_{2p-1}}{S_2(h_0)^{p-1}}|y^n_{h_0}|^{2p-2})\int_\Omega (y^n_{h_0})^{2p}dx \le 2kc_1|\Omega| + 2k^2c_2|\Omega|.
\end{aligned}
$$

Since $S_1(h_d) = h_d/2$, $\overline{S}_1(h_0,h_d) = \frac{1}{2}\sqrt{h_0h_d}$, $S_2(h_0) = h_0$,

$$
\begin{aligned}
|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 + 2k\nu(1 - \frac{2k\nu}{h_d^2} - \frac{4k\nu}{2^d h_d^2})\|y^n_{h_0}+z^n\|^2_{h_d} \\
+ (kb_{2p-1} - \frac{4k^2b^2_{2p-1}}{h_0^{p-1}}|y^n_{h_0}|^{2p-2})\int_\Omega (y^n_{h_0})^{2p}dx \le 2kc_1|\Omega| + 2k^2c_2|\Omega|.
\end{aligned}
$$

We are now ready to prove Theorem 2.1 by induction:

- $q = 0$ is obvious since $|y^0_{h_0}|^2 + |z^0|^2 \le M_0$.
- Assuming (4.8) is correct up to $q = n$, we then have $|y^n_{h_0}|^2 + |z^n|^2 \le M_0$.
- For $q = n+1$, using condition (4.6) and (4.7), we write

$$
|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 + k\nu\|y^n_{h_0}+z^n\|^2_{h_d} \le 2kc_1|\Omega| + 2k^2c_2|\Omega|.
$$

With the use of Lemma 2.2 and $|y^n_{h_0}+z^n|^2 = |y^n_{h_0}|^2 + |z^n|^2$, we obtain

$$
|y^{n+1}_{h_0}|^2 + |z^{n+1}|^2 \le (1-2k\nu)(|y^n_{h_0}|^2 + |z^n|^2) + 2kc_1|\Omega| + 2k^2c_2|\Omega|.
$$

Using above inequality corresponding to $n, n-1, \dots, 1, 0$, we obtain

$$
\begin{aligned}
|y^{n+1}_{h_0}|^2 + |z^{n+1}|^2 &\le (1-2k\nu)^{n+1}(|y^0_{h_0}|^2 + |z^0|^2) \\
&\quad + (1 + (1-2k\nu) + (1-2k\nu)^2 + \cdots + (1-2k\nu)^n)(2kc_1|\Omega| + 2k^2c_2|\Omega|) \\
&\le (1-2k\nu)^{n+1}(|y^0_{h_0}|^2 + |z^0|^2) + \frac{1}{1-(1-2k\nu)}(2kc_1|\Omega| + 2k^2c_2|\Omega|).
\end{aligned}
$$

Therefore

$$
|y^{n+1}_{h_0}|^2 + |z^{n+1}|^2 \le (1-2k\nu)^{n+1}(|y^0_{h_0}|^2 + |z^0|^2) + \frac{|\Omega|}{\nu}(c_1 + K_0c_2) \le M_0. \qquad (4.12)
$$

Hence (4.8).

Remark 1. By using the same method, we observe that the stability condition for the classical explicit approximation scheme of (4.1) (i.e. two levels in time, one level in space) reads

$$\frac{k}{h_d^2} \le \frac{1}{8\nu}$$

and

$$\frac{k}{h_d^{p-1}} \le \frac{1}{4b_{2p-1}M_0^{p-1}}. \tag{4.13}$$

Since for $d \ge 1$, $\frac{2^d}{2+2^d} > \frac{1}{2}$, we observe an improved stability for any $d \ge 1$. When the nonlinear effect is strong, that is when (4.7) and (4.13) are dominant, the stability condition of the nonlinear galerkin method is better. The time step can be taken about $2^{d(p-1)}$ larger than the step size if we deal with u_{h_d} directly.

Remark 2. From (4.12), we see that there is an absorbing set in the L^2–norm for the approximate solution. That is there exists an $R_0 > 0$ ($R_0 = \frac{|\Omega|}{\nu}(c_1 + K_0 c_2)$), which depends only on g (and not on k and h), such that the ball centered at origin with radius R, $B_R(0)$, for any $R > R_0$ absorbs the solutions: namely for any initial data u_0, we have $|y_{h_0}^n|^2 + |z^n|^2 \le R$ when n is large enough.

Remark 3. We have better stability results for Scheme I'. But the equation involving z becomes implicit which might add the complexity of computation.

In order to present the stability condition for Scheme II, we need the following well-known result:

Lemma. *(Discrete Gronwall Lemma)*

Let a^n, b^n be two nonnegative sequences satisfying

$$\frac{a^{n+1} - a^n}{k} + \lambda a^{n+1} \le b^n, \quad b^n \le b, \quad \forall n \ge 0.$$

Then,

$$a^n \le \frac{1}{(1+k\lambda)^n} a^0 + \frac{1+k\lambda}{\lambda}(1 - \frac{1}{(1+k\lambda)^{n+1}})b, \quad \forall n \ge 0,$$

provided $k > 0$ and $1 + k\lambda > 0$.

Theorem 2.2.

Assuming $k \le K_0$ for some K_0 fixed, we set

$$M_1 = |y_{h_0}|^2 + |z_0|^2 + \frac{1+4K_0\nu}{4\nu}(2c_1 + c_2 K_0)|\Omega|.$$

Then if

$$\frac{k}{(h_d)^{p-1}} \le \frac{2^{d(p-1)}}{2b_{2p-1}M_1^{p-1}}, \tag{4.14}$$

we have for Scheme II the following estimate:

$$|y_{h_0}^n|^2 + |z^n|^2 \leq M_1 \text{ for all } n \geq 0.$$

Proof. Again, we first write Scheme II in its variational form

$$(\frac{y_{h_0}^{n+1} - y_{h_0}^n}{k}, \tilde{y}) + \nu((y_{h_0}^{n+1} + z^{n+1}, \tilde{y}))_{h_d} + (g(y_{h_0}^n), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}_0, \tag{4.15}$$

$$(\frac{z^{n+1} - z^n}{k}, \tilde{z}) + \nu((y_{h_0}^{n+1} + z^{n+1}, \tilde{z}))_{h_d} = 0, \quad \forall \tilde{z} \in \mathcal{Z}. \tag{4.16}$$

We let $\tilde{y} = 2ky_{h_0}^{n+1}$ in (4.15) and $\tilde{z} = 2kz^{n+1}$ in (4.16) and use $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ and inequality (4.4). We obtain after adding these relations

$$\begin{aligned} &|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |y_{h_0}^{n+1} - y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2 + |z^{n+1} - z^n|^2 + 2k\nu\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}^2 \\ &\quad = -2k(g(y_{h_0}^n), y_{h_0}^{n+1}) = -2k(g(y_{h_0}^n), y_{h_0}^{n+1} - y_{h_0}^n) - 2k(g(y_{h_0}^n), y_{h_0}^n) \\ &\quad\quad \leq -2k(g(y_{h_0}^n), y_{h_0}^{n+1} - y_{h_0}^n) - kb_{2p-1}\int_\Omega (y_{h_0}^n)^{2p} dx + 2kc_1|\Omega|, \end{aligned}$$

$$\begin{aligned} &|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |y_{h_0}^{n+1} - y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2 + |z^{n+1} - z^n|^2 \\ &+ 2k\nu\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}^2 + kb_{2p-1}\int_\Omega (y_{h_0}^n)^{2p} dx \leq -2k(g(y_{h_0}^n), y_{h_0}^{n+1} - y_{h_0}^n) + 2kc_1|\Omega|. \end{aligned}$$

Now using formula (4.11)

$$\begin{aligned} &-2k(g(y_{h_0}^n), y_{h_0}^{n+1} - y_{h_0}^n) \leq 2k|g(y_{h_0}^n)|\,|y_{h_0}^{n+1} - y_{h_0}^n| \leq |y_{h_0}^{n+1} - y_{h_0}^n|^2 + k^2|g(y_{h_0}^n)|^2 \\ &\quad \leq |y_{h_0}^{n+1} - y_{h_0}^n|^2 + \frac{2k^2 b_{2p-1}^2}{S_2(h_0)^{p-1}}|y_{h_0}^n|^{2p-2}\int_\Omega (y_{h_0}^n)^{2p} dx + k^2 c_2|\Omega|. \end{aligned}$$

We therefore obtain

$$\begin{aligned} &|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2 + |z^{n+1} - z^n|^2 + 2k\nu\|y_{h_0}^n + z^n\|_{h_d}^2 \\ &\quad + kb_{2p-1}(1 - \frac{2kb_{2p-1}}{S_2(h_0)^{p-1}}|y_{h_0}^n|^{2p-2})\int_\Omega (y_{h_0}^n)^{2p} dx \leq 2kc_1|\Omega| + k^2 c_2|\Omega|. \end{aligned}$$

Lemma 2.2 yields then

$$|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2 + 4k\nu|y_{h_0}^n + z^n|^2 \leq 2kc_1|\Omega| + k^2 c_2|\Omega|.$$

Since

$$|y_{h_0}^n + z^n|^2 = |y_{h_0}^n|^2 + |z^n|^2,$$

we have

$$\frac{|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2}{k} + 4\nu(|y_{h_0}^{n+1}|^2 + |z^{n+1}|^2) \leq 2c_1|\Omega| + kc_2|\Omega|.$$

Now the discrete Gronwall lemma implies

$$|y_{h_0}^n|^2 + |z^n|^2 \leq \frac{1}{(1+4k\nu)^n}(|y_{h_0}^0|^2 + |z^0|^2) + \frac{1+4k\nu}{4\nu}(1 - \frac{1}{(1+4k\nu)^n})|\Omega|(2c_1 + K_0c_2).$$

Therefore

$$|y_{h_0}^n|^2 + |z^n|^2 \leq \frac{1}{(1+4k\nu)^n}(|y_{h_0}^0|^2 + |z^0|^2) + \frac{1+4k\nu}{4\nu}|\Omega|(2c_1 + K_0c_2),$$

$$|y_{h_0}^n|^2 + |z^n|^2 \leq M_1.$$

Remark 4. Comparing the stability condition of Scheme II with that of the standard semi-implicit approximation scheme of (4.1), we see that the size of the time step can be taken $2^{d(p-1)}$ times larger.

Remark 5. As in Remark 2, any ball $B_R(0)$ with $R > R_0$ is an absorbing ball, where $R_0 = \frac{1+4k\nu}{4\nu}|\Omega|(2c_1 + K_0c_2)$.

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