SOLITARY-WAVE AND MULTI-PULSED TRAVELING-WAVE SOLUTIONS OF BOUSSINESQ SYSTEMS

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ABSTRACT. This paper studies traveling-wave solutions of the partial differential equations which model waves in a horizontal water channel traveling in both directions. The existence of solitary-wave solutions for a number of systems is proved. Interesting new multi-pulsed traveling-wave solutions which consist of an arbitrary number of troughs are found numerically. The bifurcation diagrams of N-trough solutions with respect to phase speed and system parameters are presented.

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1. Introduction

This paper studies solitary-wave and multi-pulsed solutions of the Boussinesq systems

$$\begin{aligned} \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} &= 0. \end{aligned} \tag{1}$$

The variable $\eta(x,t)$ is the non-dimensional deviation of water surface from its undisturbed position and $u(x,t)$ is the non-dimensional horizontal velocity at a height above the bottom of the channel corresponding to θh_0, where h_0 is the undisturbed water depth and θ lies in $[0,1]$. The real constants a, b, c and d satisfy

$$\begin{aligned} a &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \qquad & b &= \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda), \\ c &= \frac{1}{2}\left(1-\theta^2\right)\mu, \qquad & d &= \frac{1}{2}\left(1-\theta^2\right)(1-\mu), \end{aligned} \tag{2}$$

where λ and μ are modeling parameters which can be any real numbers. As explained in Bona, Chen and Saut [8], the systems in (1)-(2) are formally equivalent and have the same formal justification as the classical Boussinesq system (cf. [10]) for the two-way propagation of small-amplitude long waves. Several members of the systems (1)-(2) have been studied in the past, including the classical Boussinesq system [10, 11, 12, 29, 31, 1, 32, 28], the integrable version of the Boussinesq system or Kaup system [26, 27, 30, 20], the Bona-Smith system [9, 38, 33, 34] and the regularized Boussinesq system [7]. An overview of all the systems in (1)-(2) was presented in [8]. Previous research was focused on the local and global wellposedness, the regularity of the solutions, and the exact solutions of these systems. But to the best of my knowledge, there is no result regarding the multi-pulsed solutions for any of the systems in (1)-(2).

In this paper, we attempt to study the traveling-wave solutions of the systems with general a, b, c and d. Special attention will be given to the *regularized Boussinesq system* which is the member of (1)-(2) with $a = c = 0$ and $b = d = \frac{1}{6}$. This system is particularly interesting because its dispersion relation is stabilizing for all wave numbers and the natural initial-boundary-value problems that arise in laboratory experiments are well-posed [7].

Letting $\xi = x - kt$ where k is the speed of the traveling-wave solution, one can write the solution in the form $\eta(x,t) = \eta(\xi)$ and $u(x,t) = u(\xi)$. Supposing the solution decays at large distance from its crests or troughs, it is natural to impose the boundary conditions

$$\left(\eta^{(n)}(\xi), u^{(n)}(\xi)\right) \to 0 \text{ as } \xi \to \pm\infty, \text{ for at least } n = 0, 1, 2. \tag{3}$$

The functions η and u satisfy the ordinary differential equations

$$\begin{aligned} &au'' + bk\eta'' + u - k\eta + u\eta = 0, \\ &dku'' + c\eta'' - ku + \eta + \frac{1}{2}u^2 = 0, \end{aligned} \tag{4}$$

where the derivatives are with respect to ξ. It is clear that a homoclinic solution about the origin of (4) will lead to a traveling-wave solution of (1). Therefore, the problem of finding traveling-wave solutions becomes that of finding homoclinic orbits of (4).

The paper is organized as follows. In Section 2, we prove first the existence of solitary-wave solutions with any phase speed $k > 1$ for the regularized Boussinesq system. By solitary-wave solutions, we mean, as in [37], a traveling-wave solution consisting of a *single hump of positive elevation*, and satisfying (3). This result offers a rigorous justification of the numerical experiments presented in [7], where a detailed study of the numerically generated solitary-wave solutions was presented. We then provide a general sufficient condition for the systems in (1) to have solitary-wave solutions. Together with the results in [19] where systems with exact analytical solitary-wave solutions were found, the existence of solitary-wave solutions is now proved for a large number of systems of type (1).

In Section 3, we discuss several special but well-known systems of type (1). These systems are special in the sense that the corresponding ordinary differential equation (4) degenerates to a two-dimensional dynamic system. Therefore, the investigation of homoclinic orbits can be conducted through a straightforward phase diagram analysis.

In Section 4, we first describe the numerical method used to search for even, multi-pulsed solutions and then apply the method to the regularized Boussinesq system. In comparison with solitary-wave solutions, multi-pulsed solutions are solutions with several crests and/or troughs. The initial evidence for their existence was provided in [19], where an exact solution with one trough was found. The numerical results in this section indicate that not only solutions with one-trough exist, but also solutions with any number of troughs.

In order to learn more about the regularized Boussinesq system and its solutions, a toolbox AUTO described in [22, 16] is used in Section 5 to study the bifurcation diagrams of the solitary-wave solution and multi-pulsed solution. Starting from a N-trough solution found in Section 4, the solution branch with respect to variations in the phase speed k was obtained. The results show that the only branch which continues through the entire range $(1, +\infty)$ is the solitary-wave branch. Other branches possess a turning point at a critical phase speed $k_c > 1$, below which the branches cease to exist. The critical speed k_c is monotonically increasing with respect to the number of troughs.

Since the KdV-type equations do not possess multi-pulsed solutions, our results show that there is a significant difference between the one-way model equations and the two-way model systems, although they are formally approximations of the same order to the Euler equation (cf. [7]). However, a detailed study in Section 5 reveals that the magnitude of all multi-pulsed solutions is quite large, so the difference happens in regimes where the solution is no longer small. Thus, our results do not call into question the validity of one or the other of KdV-type equations and Boussinesq systems, but does emphasize that when these models are used to simulate waves in a practical situation, they cannot both be accurate for large amplitude waves. Indeed, this interesting anomaly can be viewed as another cautionary lesson; with the conclusion that it is best to use these models well within the range they were derived to describe.

In Section 6, a preliminary study on the "structural stability" or the "well-posedness" of a system with respect to the perturbation of the system parameters a, b, c and d is conducted. With the change of system parameters, the properties of the system may change. For instance, a system may change from being integrable to non-integrable, from being Hamiltonian to non-Hamiltonian and from being well-posed to ill-posed. The study is aimed at investigating the relationship between various model systems when they are used in modeling the same situation.

2. Existence of solitary-wave solutions for $k > 1$

Since the full Euler equation have solitary-wave solutions [5, 3] and the systems in (1)-(2) are its approximations, and as solitary waves play a central role in evolution of certain types of disturbances, it is important to know if the equations (1)-(2) have solitary-wave solutions. In this section, we first prove the existence and then obtain *a priori* bounds on solitary-wave solutions. We start by recalling the results in [35, 4] which will be used in the proof.

Denote $\mathbf{u} = (u, \eta)^T$ and consider a system in the form of

$$S_1\mathbf{u}'' + S_2\mathbf{u} + \nabla g(u, \eta) = 0, \tag{5}$$

where S_1 and S_2 are 2×2 matrices and the derivatives are with respect to ξ, the main result in [35, 4] may be stated as

Theorem 2.1. *Assume that S_1 and S_2 are symmetric, $g \in C^2(I\!R^2, I\!R)$ has the property that $g, \nabla g$ and the second partial derivatives of g are all zero at $(0, 0)$. Define $Q(\mathbf{u}) = \mathbf{u}^T S_1 \mathbf{u}$ and $f(u, \eta) = \mathbf{u}^T S_2 \mathbf{u} + 2g$. Assume also that*

(I) *$det(S_1) < 0$, so Q is indefinite and there exist two linearly independent vectors $\mathbf{u}_1 = (u_1, \eta_1)^T$ and $\mathbf{u}_2 = (u_2, \eta_2)^T$ where Q vanishes, and*

(II) *there exists a closed curve $\mathcal{F}$ which passes through the origin of $I\!R^2$ such that*

  (i) *$f = 0$ on $\mathcal{F}$, and $\mathcal{F}\backslash\{(0,0)\}$ lies in the set $\{(u,\eta) : Q(u,\eta) < 0\}$,*

  (ii) *$f(u,\eta) > 0$ in the (non-empty) interior of $\mathcal{F}$,*

  (iii) *$\mathcal{F}\backslash\{(0,0)\}$ is strictly convex, which is to say*

$$D = f_{uu}f_\eta^2 - 2f_{u\eta}f_uf_\eta + f_{\eta\eta}f_u^2 < 0 \text{ on } \mathcal{F}\backslash\{(0,0)\},$$

  (iv) *$\nabla f(u,\eta) = 0$ on $\mathcal{F}$ if and only if $(u,\eta) = (0,0)$.*

Then there exists an orbit γ of (5) which is homoclinic to the origin and which has the following properties:

(a) *$(u(0), \eta(0)) \in \hat{\mathcal{F}}$, where $\hat{\mathcal{F}}$ is the segment of $\mathcal{F}$ between P_1 and P_2, with P_i satisfying $\nabla f(P_i) \cdot \mathbf{u}_i = 0$ for $i = 1$ and 2,*

(b) *$(u(\xi), \eta(\xi))$ is an even solution, namely $(u(\xi), \eta(\xi)) = (u(-\xi), \eta(-\xi))$, $(u'(\xi), \eta'(\xi)) = -(u'(-\xi), \eta'(-\xi))$ for any $\xi \in I\!R$,*

(c) *$(u(\xi), \eta(\xi))$ is in the interior of $\mathcal{F}$ for all $\xi \in I\!R\backslash\{0\}$,*

(d) *γ is monotone in the sense that $(u(\xi), \eta(\xi)) \le (u(s), \eta(s))$ if $\xi \ge s \ge 0$.*

Notice that for the regularized Boussinesq system, which is the member of the class (1) with $a = c = 0$ and $b = d = \frac{1}{6}$, the corresponding ordinary differential equation (4) can be written in the form of (5) with

$$S_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -\frac{6}{k} & 6 \\ 6 & -\frac{6}{k} \end{pmatrix}$$

and $g(u,\eta) = -\frac{3}{k}u^2\eta$. Therefore, one can prove the existence of homoclinic orbits about the origin by verifying the hypotheses (I) and (II).

Theorem 2.2. *The regularized Boussinesq system possesses a homoclinic orbit about the origin with any phase speed $k > 1$ and it has the additional properties (a)–(d) stated in Theorem 2.1.*

Proof. Since $Q(u,\eta) = -u\eta$, (I) is true with $\mathbf{u}_1 = (1,0)^T$ and $\mathbf{u}_2 = (0,1)^T$. The closed curve $\mathcal{F}$ (if it exists) has to be in quadrants I or III due to the sign of Q. Because

$$f(u,\eta) \equiv \frac{6}{k}\left(-u^2(1+\eta) - \eta^2 + 2ku\eta\right) = 0 \tag{6}$$

on $\mathcal{F}$, it must be the case that

$$u_\pm = \frac{k\eta \pm \sqrt{\eta^2(k^2 - 1 - \eta)}}{(1+\eta)} \quad \text{if} \quad \eta \ne -1.$$

Therefore when $k > 1$, $\{f \equiv 0\}$ forms a closed curve $\mathcal{F}^*$ passing through origin which lies in the first quadrant when $0 \le \eta \le k^2 - 1$. Factoring f as $f(u,\eta) = -\frac{6}{k}(1+\eta)(u - u_+)(u - u_-)$, it is clear $f > 0$ in the interior of $\mathcal{F}$ because u is between u_- and u_+. It is easy to check that u and η are not zero on $\mathcal{F}^*\backslash\{(0,0)\}$, so (II)(i) and (II)(ii) are verified. A straightforward calculation shows that

$$D = -2\left(\frac{6}{k}\right)^3 \left((u^2 - \eta^2)^2 + u^4\eta\right)\eta < 0$$

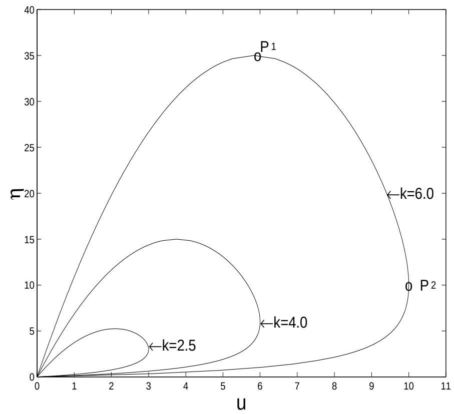

FIGURE 1. Curve $\mathcal{F}$ with $k = 2.5, 4.0$ and 6.0.

on $\mathcal{F}^* \backslash \{(0,0)\}$, so $\mathcal{F}^*$ is strictly convex. Condition II(iv) can be checked by solving the system $\nabla f = 0$ together with $f(u,\eta) = 0$, namely

$$-u(1+\eta) + k\eta = 0, \tag{7}$$

$$-u^2 - 2\eta + 2ku = 0, \tag{8}$$

$$-u^2(1+\eta) - \eta^2 + 2ku\eta = 0, \tag{9}$$

on $\mathcal{F}^* \backslash \{(0,0)\}$. Solving η and u with the restrictions $u > 0$ and $\eta > 0$, one finds $\eta = -\dfrac{1}{2}u^2 + ku$, $u = \dfrac{2}{k}(k^2-1)$ and $k^2 = 1$. Therefore II(iv) holds for $k > 1$, $\mathcal{F}^*$ is the desired curve $\mathcal{F}$ and the conclusions in Theorem 2.1 hold for the regularized Boussinesq system.

The curves of $\mathcal{F}$ for $k = 2.5, 4.0$ and 6.0 are shown in Figure 1. For each k, a solitary-wave solution $(u(x-kt), \eta(x-kt))$ exists with $(u(0), \eta(0))$ located on the segment of $\mathcal{F}$ between P_1 and P_2. Furthermore, the solution is monotonically decreasing to zero as $\xi \to \pm\infty$. Using Theorem 2.2, one can easily obtain the following *a priori* bounds for the solitary-wave solutions.

Corollary 2.1. *For any phase speed k larger than 1, the regularized Boussinesq system possess an even solitary-wave solution $u(x,t) = u(x-kt)$ and $\eta(x,t) = \eta(x-kt)$. Letting $\xi = x - kt$, the solution has the following properties:*

(i) $(u(\xi), \eta(\xi))$ *is positive and monotonically decreasing for* $\xi > 0$,

(ii) $\dfrac{k^2-1}{k} < |u(\xi)|_{L^\infty} = u(0) < 2(k-1)$; $2(k-1) < |\eta(\xi)|_{L^\infty} = \eta(0) < k^2 - 1$,

(iii) $u(0) = 2(k-1) + O\left((k-1)^2\right)$; $\eta(0) = 2(k-1) + O\left((k-1)^2\right)$ *as* $k \to 1$,

(iv) $u(0) = \dfrac{\eta(0)\left(k + \sqrt{k^2 - 1 - \eta(0)}\right)}{1 + \eta(0)}$,

(v) $\left(k - \sqrt{k^2-1}\right) u(\xi) < \eta(\xi) < \left(k + \sqrt{k^2-1}\right) u(\xi)$.

Proof. Notice that $\mathcal{F}$ is convex and in the first quadrant of the u–η plane; (i) is a direct consequence of (c) and (d) in Theorem 2.1. Since P_1, P_2 and $(u(0), \eta(0))$ are in quadrant I, P_1 satisfies (7) and (9) implies $P_1 = \left(\frac{k^2-1}{k}, k^2 - 1\right)$; P_2 satisfying (8) and (9) yields $P_2 = (2(k-1), 2(k-1))$. The results (ii) and

(iii) are therefore consequences of (a) and (b) in Theorem 2.1. Property (iv) is true because $f(u(0),\eta(0))=0$. Since $(u(\xi),\eta(\xi))$ is in the interior of $\mathcal{F}$ which is strictly convex, $\eta(\xi)$ is in a cone

$$s_1 u(\xi) < \eta(\xi) < s_2 u(\xi),$$

where s_1 and s_2 are the slopes of $\mathcal{F}$ at $(u,\eta)=(0,0)$. Considering η as a function of u and taking the derivative of $f(u,\eta)=0$ with respect to u yields

$$\eta'_u = \frac{2u-2k\eta+2u\eta}{-2\eta+2ku-u^2}. \tag{10}$$

For $k>1$ fixed, $f(u,\eta)=0$ implies $\eta=\frac{1}{2}\left(-u^2+2ku\pm\sqrt{(u^2-2ku)^2-4u^2}\right) = u\left(k\pm\sqrt{k^2-1}\right)+O(u^2)$ as $u\to 0$. Substituting into (10) gives

$$\eta'_u = \frac{u-k\eta}{ku-\eta}+O(u^2) = k\pm\sqrt{k^2-1}+O(u^2)$$

as $u\to 0$. Therefore, $s_1=k-\sqrt{k^2-1}$ and $s_2=k+\sqrt{k^2-1}$, so (v) is true and the corollary is proved.

Remark 1. Because no uniqueness result was obtained here, Corollary 2.1 is only valid for the solitary-wave solution whose existence was settled in Theorem 2.2.

Remark 2. In contrast to the KdV-type equations where a traveling-wave solution satisfying (3) is unique for a fixed k, such solutions are not unique for the regularized Boussinesq system. In particular, an analytical solution with the form

$$\begin{aligned} u(x,t) &= \pm\frac{15}{2}\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}\left(x\mp\frac{5}{2}t\right)\right) \\ \eta(x,t) &= \frac{15}{4}\left(2\operatorname{sech}^2\left(\frac{3}{\sqrt{10}}\left(x\mp\frac{5}{2}t\right)\right)-3\operatorname{sech}^4\left(\frac{3}{\sqrt{10}}\left(x\mp\frac{5}{2}t\right)\right)\right), \end{aligned} \tag{11}$$

and a solitary-wave solution exist for phase speed $k=2.5$. For phase speed $k>2.5$, the non-uniqueness is clearly seen in our numerical results presented in Section 5.

Using a similar approach, one can study other systems in (1). Toland in [33, 34] has studied the Bona-Smith system [9] where $b=d=-c=\frac{1}{3}, a=0$, and proved the existence of solitary-wave solutions for any $k>1$ and the uniqueness of even traveling-wave solutions satisfying (3) for k sufficiently large and for $k\in(1,\frac{3}{2}]$.

Writing the systems in (4) in the form of (5), one has

$$\begin{pmatrix} -\frac{6a}{k} & -6b \\ -6d & -\frac{6c}{k}\end{pmatrix}\begin{pmatrix} u \\ \eta\end{pmatrix}'' + \begin{pmatrix} -\frac{6}{k} & 6 \\ 6 & -\frac{6}{k}\end{pmatrix}\begin{pmatrix} u \\ \eta\end{pmatrix} + \begin{pmatrix} g_u \\ g_\eta\end{pmatrix} = 0, \tag{12}$$

where $g(u,\eta)=-\frac{3}{k}u^2\eta$. In order to apply Theorem 2.1, we restrict attention to the systems with $b=d$ (these systems have Hamiltonian structure), which can be expresses as two-parameter family of systems where

$$b=d=\frac{1}{2}\left(\theta^2-\frac{1}{3}\right)-a, \quad c=\frac{2}{3}-\theta^2+a, \quad 0\le\theta^2\le 1, \quad a\in I\!R \tag{13}$$

and consider the traveling-wave solutions with phase speed k satisfying $det(S_1)<0$ so $b^2k^2-ac>0$.

Since $f(u,\eta)$ is the same as in (6), $\mathcal{F}^*$ is in quadrant I for $k > 1$ and inside the cone $s_1 u \le \eta \le s_2 u$ (cf. Corollary 2.1 (v)), the next theorem is a direct consequence of Theorem 2.1.

Theorem 2.3. *For a given system in (1) with a, b, c, d satisfying (13) and for a phase speed $k > 1$ which satisfies $b^2k^2 - ac > 0$, let Q be the function*

$$Q(u,\eta) = -\frac{6}{k}\left(au^2 + c\eta^2 + 2bku\eta\right)$$

and $\mathcal{F}^*$ be the closed curve satisfying (6) with $0 \le \eta \le k^2 - 1$. If $\mathcal{F}^*$ is in the region of $Q < 0$, then there is a solitary-wave solution with phase speed k which satisfies (i)–(v) in Corollary 2.1.

Straightforward, but tedious calculation of Q and $\mathcal{F}^*$ shows that some of the systems in (1)-(13) have solitary-wave solutions. We content ourselves with working out a few of the more interesting examples.

Corollary 2.2. A system in (1) with

$$a = \theta^2 - \frac{2}{3}, \quad c = 0, \quad b = d = \frac{1}{2}(1 - \theta^2), \text{ where } 0 \le \theta^2 \le 1$$

has a solitary-wave solution with the properties expressed in Corollary 2.1 for the following phase speeds k:

(a) $k > 1$ when $\frac{2}{3} \le \theta^2 \le 1$;
(b) $1 < k < k_0$ when $0 \le \theta^2 < \frac{2}{3}$, where

$$k_0 = \frac{(\theta^2 - \frac{5}{6})^2 + \frac{1}{36}}{(\frac{2}{3} - \theta^2)(1 - \theta^2)}.$$

Proof. Notice that in both cases,

$$Q(u,\eta) = -\frac{6a}{k}u\left(u + \frac{2b}{a}\eta\right).$$

When $\frac{2}{3} \le \theta^2 \le 1$, one has $a > 0$ and $b > 0$ so $Q < 0$ in I quadrant which yields the conclusion (a).

If $0 \le \theta^2 < \frac{2}{3}$, one has $a < 0$ and $b > 0$ so $Q < 0$ as $\eta > -\frac{a}{2b}u$ and $u > 0$. Recall that $\mathcal{F}^*$ is in the cone $s_1 u \le \eta \le s_2 u$ with $s_1(k) = k - \sqrt{k^2 - 1}$. Hence, $s_1(k) > -\frac{a}{2b}$ would imply $\mathcal{F}^*$ is in the region of $Q < 0$. Since $0 < -\frac{a}{2b} < 1$, $s_1(1) = 1$, $\lim_{k\to+\infty} s_1(k) = 0$ and s_1 is monotonically decreasing, there is a k_0 such that $s_1(k_0) = -\frac{a}{2b}$. Conclusion (b) is proved by expressing k_0 explicitly.

Corollary 2.3. For the restricted two-parameter family of systems (1) with a, b, c and d satisfying (13), the system has solitary-wave solutions with the properties in Corollary 2.1 if θ, a and k satisfy

(a) $\dfrac{1}{3} \le \theta^2 \le \dfrac{2}{3}$, $\theta^2 - \dfrac{2}{3} < a < 0$ and $k > 1$;
(b) $0 \le \theta^2 < \dfrac{1}{3}$, $\theta^2 - \dfrac{2}{3} < a < \dfrac{1}{2}\left(\theta^2 - \dfrac{1}{3}\right)$ and $1 < k < k_0$ where k_0 is the zero of f_1 with

$$f_1(k) = \frac{1}{c}\frac{-bk + \sqrt{b^2k^2 - ac}}{k - \sqrt{k^2 - 1}} - 1.$$

Proof. Since a and θ satisfy (a) or (b), and a, b, c, d satisfy (13), so $a < 0,\ b > 0,$ $c > 0$ and $Q(u, \eta)$ can be factored into

$$Q(u,\eta) = -\frac{6}{k}c(\eta - k_1 u)(\eta - k_2 u)$$

with $k_1 = \frac{1}{c}\left(-bk + \sqrt{b^2k^2 - ac}\right)$ and $k_2 = \frac{1}{c}\left(-bk - \sqrt{b^2k^2 - ac}\right)$. Since $k_1 > 0 > k_2$, $k_1 \le s_1$ would lead to the existance of solitary-wave solutions. Define f_1 by

$$f_1(k) \equiv \frac{k_1}{s_1} - 1 = \frac{1}{c}\frac{-bk + \sqrt{b^2k^2 - ac}}{k - \sqrt{k^2 - 1}} - 1.$$

The restrictions in (13) yield $a + c + 2b > 0$ and $b + c \ge 0$. Hence, $\frac{1}{c}\left((b+c)^2 - (b^2 - ac)\right) > 0$ and

$$f_1(1) = \frac{1}{c}\left(-(b+c) + \sqrt{b^2 - ac}\right) < 0.$$

Because $b > 0$ and

$$f_1'(k) = \frac{1}{c}\frac{\left(-bk + \sqrt{b^2k^2 - ac}\right)\left(-b\sqrt{k^2 - 1} + \sqrt{b^2k^2 - ac}\right)}{\left(k - \sqrt{k^2 - 1}\right)\sqrt{b^2k^2 - ac}\sqrt{k^2 - 1}} > 0,$$

so $f_1(k)$ is monotone and

$$\lim_{k \to +\infty} f_1(k) = -\frac{1}{b}\left(a + b\right).$$

Hence, in case (a), $a + b > 0$, so $f_1(k) < 0$ and the solitary-wave solution exists for all $k > 1$; in case (b), $a + b < 0$, so $f_1(k) < 0$ for $1 < k < k_0$ where k_0 satisfies $f_1(k_0) = 0$ and the solitary-wave solution exists for $1 < k \le k_0$.

3. Solutions for several particular systems

When the ordinary differential equation (4) degenerates to a two-dimensional dynamical system, the traveling-wave solution of the corresponding partial differential equation can be analyzed using its phase diagram. Several well-known systems of (1) are in this class and we shall discuss them briefly in this section.

(i) Kaup system or integrable Boussinesq system

The so-called Kaup system studied in [26, 27, 30, 20] is the member of (1) with $a = \frac{1}{3}, b = c = d = 0$. The system is mathematically interesting because it is integrable, but not suitable for modeling because it is not linearly well-posed (cf. [8]).

Finding the solitary-wave or the multi-pulsed solutions is relatively simple in this case because the ordinary differential equations (4) corresponding to traveling waves can be written as

$$(14)\qquad \begin{aligned} (u')^2 = R_1(u) \equiv& \frac{3}{4}u^2\left(u - (2k-2)\right)\left(u - (2k+2)\right), \\ \eta =& u\left(k - \frac{1}{2}u\right), \end{aligned}$$

where the unknown function u is separated from η. By studying the function $R_1(u)$, one can show that there exists a unique solution $u(\xi)$ which is even and

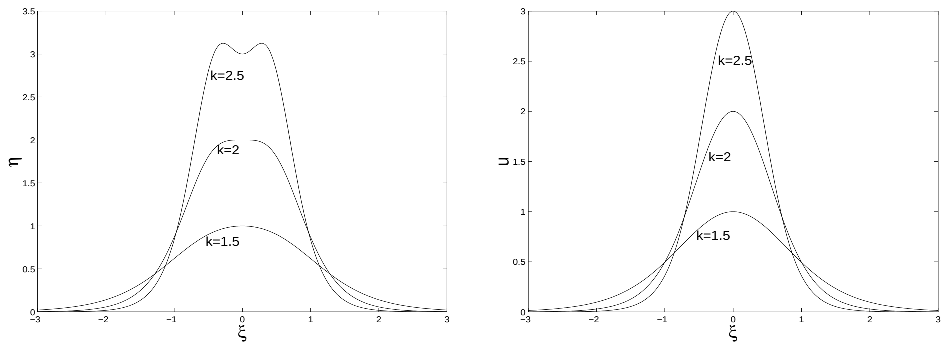

FIGURE 2. Solitary-wave like solutions of the Kaup system with phase speeds $k = 1.5, 2, 2.5$.

monotone for $\xi > 0$ for any $k > 1$. When $1 < k \leq 2$, the corresponding $\eta(\xi)$ is also monotonically decreasing; while for $k > 2$, that is no longer true. With the definition of solitary waves used in this paper, the system does not possess solitary-wave solutions for $k > 2$. The solutions with $k = 1.5, 2$ and 2.5 obtained by integrating (14) are shown in Figure 2. In reference to Corollary 2.1, we notice that for $1 < k \leq 2$, $|u|_{L^\infty} = |\eta|_{L^\infty} = 2(k-1)$, so they are the limiting case of result (ii).

(ii) Classical Boussinesq system

The classical Boussinesq system, which has $a = b = c = 0,\ d = \frac{1}{3}$ in (1), is the subject of study in [10, 29, 32, 1, 31, 36]. Traveling-wave solutions $(u(\xi), \eta(\xi))$ with the boundary conditions (3) satisfies the ordinary differential equation

$$(u')^2 = R_2(u) \equiv \frac{1}{k}\left(-u^3 + 3ku^2 + 6u + 6k\log\left|\frac{k-u}{k}\right|\right),$$
$$\eta = \frac{u}{k-u}.$$

By observing that

$$R_2'(u) = \frac{3u}{k(k-u)}(u-u_+)(u-u_-)$$

where $u_\pm = \frac{1}{2}\left(3k \pm \sqrt{k^2+8}\right)$ and $0 < u_- < k < u_+$, one can show that $R_2(u)$ is monotonically increasing in $(0, u_-)$ and monotonically decreasing in (u_-, k), $R_2(0) = 0$ and $R_2(k) = -\infty$. Therefore, for any $k > 1$, there exists a unique $u(\xi)$ which is even and monotonically decreasing for $\xi > 0$. The associated $\eta(\xi)$ is also monotonically decreasing because $\eta'(\xi)$ has the same sign as $u'(\xi)$. It is easy to check that $|u(\xi)|_{L^\infty} = u(0)$ which satisfies $R_2(u(0)) = 0$ and is in the interval $\left(\frac{1}{2}\left(3k - \sqrt{k^2+8}\right),\ k\right)$ and $|\eta(\xi)|_{L^\infty} = \eta(0) = \frac{u(0)}{k-u(0)}$.

(iii) KdV-regularized long-wave system

The system in (1) with $b = c = 0,\ a = d = \frac{1}{6}$ is called the KdV-regularized long-wave system, because the highest order term in the η_t equation consists of a third derivative with respect to x which is characteristic of the KdV-equation, and the highest order term in the u_t equation is the mixed derivative η_{xxt} which is characteristic of the regularized long wave equation [6]. The situation here is very similar to the case of the classical Boussinesq system. The traveling-wave solution with (3) satisfies

$$
\begin{aligned}
(u')^2 = R_3(u) \equiv k^{-3}u(-2k^2u^2 + 6k^3u - 3ku + 18k^2 - 6) \\
+ 6k^{-4}\left(3k^4 + 2k^2 - 1\right)\log\left|\frac{k^2+1-ku}{k^2+1}\right|, \\
\eta = \frac{u(4k-u)}{2(k^2+1-ku)},
\end{aligned}
$$

where

$$
R_3'(u) = \frac{6u}{k^2+1-ku}(u-u_+)(u-u_-)
$$

with $u_\pm = \frac{1}{2}\left(3k \pm \sqrt{k^2+8}\right)$ and $0 < u_- < \frac{1+k^2}{k} < u_+$. One can check that $R_3(u)$ is monotonically increasing in $(0, u_-)$ and monotonically decreasing in $\left(u_-, \frac{k^2+1}{k}\right)$, $R_3(0) = 0$ and $R_3(\frac{k^2+1}{k}) = -\infty$. Therefore for any $k > 1$, there exists a unique $u(\xi)$ which is even and monotonically decreasing for $\xi > 0$. The associated $\eta(\xi)$ is also monotonically decreasing because $\eta'(\xi)$ has the same sign as $u'(\xi)$.

4. Even multi-pulsed solutions of regularized Boussinesq system

In this section, we search for even multi-pulsed solutions of the regularized Boussinesq system. Multi-pulsed solutions have been studied for the equation

$$
u'''' + Pu'' + u - u^2 = 0 \tag{15}
$$

which is obtained from a fifth-order long-wave equation for gravity-capillary waves when traveling waves were considered [2, 15]. The same equation may also be obtained in modeling the buckling of a strut on a nonlinear elastic foundation [24, 25]. Solutions with multi-pulses were found when the origin of the phase space is a hyperbolic bi-focus point, that is, the eigenvalues of the linearized problem are of the form $\pm a \pm bi$, see [13, 18, 14, 17]. A more generalized equation in the form

$$
u'''' + \mu u'' - cu = h(u, u', u'')
$$

was studied in [23] for a certain class of nonlinear functions h. The existence of a countably infinite family of geometrically distinct solutions was proved for certain μ and c such that the origin of the phase space is again a hyperbolic bi-focus point.

To compare with the existing work, we first note that the systems in (1) are two-way models describing gravity waves, and they are first-order approximations to the Euler equations with respect to α, while (15) is obtained from a one-way model describing gravity-capillary waves which retains terms up to $O(\alpha^{7/2})$, where α is the ratio of wave height to water depth (cf. [21]). The characteristics of the origin in (1) and (15) could also be different. For example, the ordinary differential equation corresponding to the regularized Boussinesq system written as a single equation is

$$
u'''' + \frac{12}{k}uu'' - 12u'' + \frac{6}{k}(u')^2 + \frac{18}{k^2}u^3 - \frac{54}{k}u^2 + 36u - \frac{36}{k^2}u = 0, \tag{16}
$$

so the characteristic equation at the origin is $\lambda^4 - 12\lambda^2 + 36(1 - \dfrac{1}{k^2}) = 0$. The eigenvalues are $-\lambda_1,\ -\lambda_2,\ \lambda_2,\ \lambda_1$ with $\lambda_1 = \sqrt{6(1+\frac{1}{k})}$ and $\lambda_2 = \sqrt{6|1-\frac{1}{k}|}$. Therefore, we are searching for the multi-pulsed solutions when the origin is a hyperbolic saddle point instead of a bi-focus point. As a consequence, the associated multi-pulsed solutions will not have the oscillatory tails that are characteristic of equation (15) when the origin is a bi-focus point. Furthermore, previous work was carried out on equations involving only a quadratic polynomial in u while equation (16) has a cubic term.

The method we use here is similar to the one in [17]. Since $a = c = 0$ and $b = d = 1/6$, system (4) can be written as a first-order system by setting $x_1 = \eta,\ x_2 = \eta',\ x_3 = u,\ x_4 = u'$, thereby obtaining

$$\dot{\mathbf{x}}(\xi) \equiv \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ 6x_1 - \frac{6}{k}x_3 - \frac{6}{k}x_1x_3 \\ x_4 \\ 6x_3 - \frac{6}{k}x_1 - \frac{3}{k}x_3^2 \end{pmatrix} \equiv \mathbf{f}(\mathbf{x}). \tag{17}$$

The shooting method is used which integrates (17) starting from an initial condition and shoots for an even solution. Since a homoclinic solution $\mathbf{x}(\xi)$ about the origin lies on the unstable manifold $\mathcal{M}^+$ for ξ small enough, so

$$\mathbf{x}(\xi) = \epsilon\mathbf{v} + O(\epsilon^2),$$

where $\mathbf{v}$ is a unit vector spanned by $\mathbf{v}_3$ and $\mathbf{v}_4$, the eigenvectors corresponding to positive eigenvalues λ_2 and λ_1. It is therefore reasonable to start the integration from the unstable manifold by using

$$\mathbf{x}(0) = \epsilon\left(\cos(\delta)\frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} + \sin(\delta)\frac{\mathbf{v}_4}{\|\mathbf{v}_4\|}\right). \tag{18}$$

Fixing $\epsilon = 0.0001$, the problem becomes finding δ and T such that the solution $\mathbf{x}(\xi)$ of (17)-(18) satisfies $x_2(T) = x_4(T) = 0$, where $[0, T]$ is the interval of integration.

The procedure in finding δ and T consists of two steps: (i) choosing δ^0 and T^0 and (ii) iterating δ and T with Newton's method. The starting value δ^0 is taken to be 0 so the solution is close to the eigenspace associated to the smallest positive eigenvalue. The initial value T^0 is determined by first integrating (17)-(18) from 0 to a T_{end} (T_{end} is relatively big but before the solution blows up), and then finding the point T^0 which is well away from 0 and with small residual $E_0(T^0)$, where

$$E_0(\xi) \equiv \sqrt{x_2(\xi)^2 + x_4(\xi)^2}.$$

We now describe in detail the Newton's iteration procedure. Denoting $\mathbf{s} = \{\delta, T\}$, we introduce the new independent variable $\tau = \xi/T$ so that the integration of (17)-(18) is on the interval $[0, 1]$. Let $\mathbf{y} = (y_1(\tau), y_2(\tau),\ y_3(\tau), y_4(\tau)) = (x_1, x_2, x_3, x_4)$, equation (17) becomes

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial \tau} = T\mathbf{f}(\mathbf{x}) = T\mathbf{f}(\mathbf{y}) \equiv \mathbf{g}(\mathbf{y}), \\ &\mathbf{y}(0) = \mathbf{x}(0) = \epsilon\left(\cos(\delta)\frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} + \sin(\delta)\frac{\mathbf{v}_4}{\|\mathbf{v}_4\|}\right). \end{aligned} \tag{19}$$

k	2.5	4.0	5.5	7	8.5	10
T_{end}	8	10	15	15	15	15
T^0	7.23	8.38	9.68	11	12.4	13.8
$E_0(T^0)$	0.019	0.17	0.12	0.25	0.22	0.22

TABLE 1. For each k, one integrates (17)-(18) on the interval $[0, T_{end}]$ with $\epsilon = 0.0001, \delta = 0$ and find T^0 which is away from zero and where $E_0^2(T^0) = (x_2(T^0))^2 + (x_4(T^0))^2$ is relatively small.

Denoting the solution of (19) as $\mathbf{y}(\tau; \mathbf{s})$ and defining the target function as

$$\mathcal{B}(\mathbf{s}) = \begin{pmatrix} b_1(\mathbf{y}(\tau;\mathbf{s})) \\ b_2(\mathbf{y}(\tau;\mathbf{s})) \end{pmatrix} \Bigg|_{\tau=1} \equiv \begin{pmatrix} y_2(1;\mathbf{s}) \\ y_4(1;\mathbf{s}) \end{pmatrix}, \tag{20}$$

the goal of Newton's algorithm is finding $\mathbf{s}^*$ such that $B(\mathbf{s}^*) = 0$. Given $\mathbf{s}^{(n)}$, $\mathbf{s}^{(n+1)}$ is determined via Newton's iteration, namely

$$\mathbf{s}^{(n+1)} = \mathbf{s}^{(n)} - [D\mathcal{B}(\mathbf{s}^{(n)})]^{-1}\mathcal{B}(\mathbf{s}^{(n)}),$$

where

$$[D\mathcal{B}(\mathbf{s}^{(n)})]_{ij} = \left(\frac{\partial b_i(\mathbf{y}(\tau;\mathbf{s}))}{\partial s_j} \right) \Bigg|_{\tau=1,\mathbf{s}=\mathbf{s}^{(n)}} = \sum_{k=1}^{4} \left(\frac{\partial b_i(\mathbf{y}(\tau;\mathbf{s}))}{\partial y_k} \frac{\partial y_k(\tau;\mathbf{s})}{\partial s_j} \right) \Bigg|_{\tau=1,\mathbf{s}=\mathbf{s}^{(n)}}.$$

Denoting

$$z_{k,j} = \frac{\partial y_k(\tau;\mathbf{s})}{\partial s_j}$$

and using (20), one obtains

$$D\mathcal{B}(\mathbf{s}) = \begin{pmatrix} z_{2,1} & z_{2,2} \\ z_{4,1} & z_{4,2} \end{pmatrix}.$$

Taking the derivatives of $\mathbf{z}$ with respect to τ, one finds

$$\dot{z}_{k,j} = \frac{\partial \dot{y}_k}{\partial s_j} = \frac{\partial T f_k}{\partial s_j} = \begin{cases} T \sum_{l=1}^4 \frac{\partial f_k}{\partial y_l} z_{l,j}, & j = 1, \\ T \sum_{l=1}^4 \frac{\partial f_k}{\partial y_l} z_{l,2} + f_k, & j = 2, \end{cases}$$

which is a system of ordinary differential equations for $z_{k,j}$ where $k = 1, \cdots, 4$ and $j = 1, 2$. The initial condition can be obtained by taking the derivatives on the initial condition for $\mathbf{y}$

$$\begin{aligned} z_{l,1}(0;\mathbf{s}) &= \left(\frac{\partial \mathbf{y}(0;\mathbf{s})}{\partial \delta} \right)_l = \epsilon \left(-\sin(\delta) \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} + \cos(\delta) \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} \right)_l, \\ z_{l,2}(0;\mathbf{s}) &= \left(\frac{\partial \mathbf{y}(0;\mathbf{s})}{\partial T} \right)_l = 0, \end{aligned}$$

for $l = 1, \cdots, 4$. In summary, the values of $z_{k,j}$ can be evaluated by solving a system of ordinary differential equations. Denoting $y_{4+i} = z_{i,2}$, $y_{8+i} = z_{i,1}$ for

k	2.5	4.0	5.5	7.0	8.5	10
T^*	7.23	8.39	9.69	11.0	12.4	13.8
δ^*	$-5.4(-6)$	$-3.3(-4)$	$-3.0(-4)$	$2.4(-5)$	$-5.1(-4)$	$-1.1(-3)$
$E_0(T^*)$	$1.1(-7)$	$1.6(-6)$	$5.7(-8)$	$1.7(-8)$	$3.7(-8)$	$5.7(-8)$
η-type	1-trough	2-trough	3-trough	4-trough	5-trough	6-trough

TABLE 2. For each k, the solution $\mathbf{y}(\tau)$ of (19) with $\epsilon = 0.0001, \delta = \delta^*, T = T^*$, is a multi-pulsed even solution (cf. Figures 3-8).

$i = 1, \cdots, 4$, the system can be written as

$$
\begin{aligned}
&\dot{y}_5 = Ty_6 + f_1, && \dot{y}_6 = T\left(6y_5 - \frac{6}{k}(y_7 + y_3y_4 + y_1y_4)\right) + f_2,\\
&\dot{y}_7 = Ty_8 + f_3, && \dot{y}_8 = T\left(6y_7 - \frac{6}{k}(y_5 + y_3y_7)\right) + f_4,\\
&\dot{y}_9 = Ty_{10}, && \dot{y}_{10} = T\left(6y_9 - \frac{6}{k}(y_{11} + y_9y_3 + y_1y_{11})\right),\\
&\dot{y}_{11} = Ty_{12}, && \dot{y}_{12} = T\left(6y_{11} - \frac{6}{k}(y_9 + y_3y_{11})\right),
\end{aligned}
\tag{21}
$$

with initial conditions

$$
\begin{aligned}
&(y_5, y_6, y_7, y_8)\,\big|_{\tau=0} = (0,0,0,0),\\
&(y_9, y_{10}, y_{11}, y_{12})\,\big|_{\tau=0} = \epsilon\left(-\sin(\delta)\frac{\mathbf{v}_3^T}{\|\mathbf{v}_3\|} + \cos(\delta)\frac{\mathbf{v}_4^T}{\|\mathbf{v}_4\|}\right).
\end{aligned}
\tag{22}
$$

Therefore, the Newton's iteration, starting from a given $\mathbf{s}^{(n)}$, consists of evaluating
$(y_1, \cdots, y_{12})(1; \mathbf{s}^{(n)})$ by integrating ordinary differential equations (19)–(21)–(22) from 0 to 1 and then letting

$$\mathbf{s}^{(n+1)} = \mathbf{s}^{(n)} - \begin{pmatrix} y_{10} & y_6 \\ y_{12} & y_8 \end{pmatrix}^{-1} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix}\Bigg|_{\tau=1}.$$

In order for the Newton's iterations to converge, and in particular converge to the solution with a specified number of troughs, a bit of experimentation with different k, T_{end}, ϵ and δ^0 was necessary. By starting with different values, one can determine even, multi-pulsed solutions with various numbers of pulses.

Applying the procedure just described with $k = 2.5,\ 4,\ 5.5,\ 7,\ 8.5$ and 10, we integrate (17)-(18) on $[0, T_{end}]$ with T_{end} shown in Table 1, $\epsilon = 0.0001$ and $\delta = 0$, obtain T^0's and thereby E_0's which are listed in Table 1. These data are then used as the initial guesses in the Newton's iteration to find (δ^*, T^*). The results are shown in Table 2 along with the corresponding values $E_0(T^*)$. Since $E_0(T^*)$ is very small, the corresponding solution is indeed even and they are plotted in Figures 3–8. Solutions with wave profiles η consist of 1 to 6 troughs are obtained. The middle troughs and humps are slightly smaller than the outside ones. The velocity is always positive. It appears to us that solutions with any number of

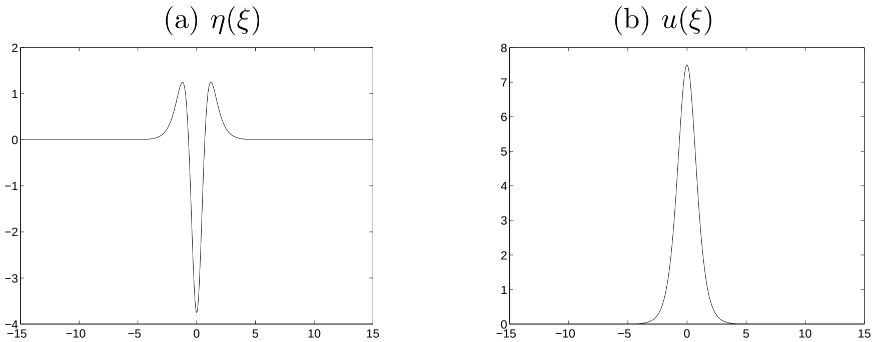

FIGURE 3. an one-trough solution ($k = 2.5$)

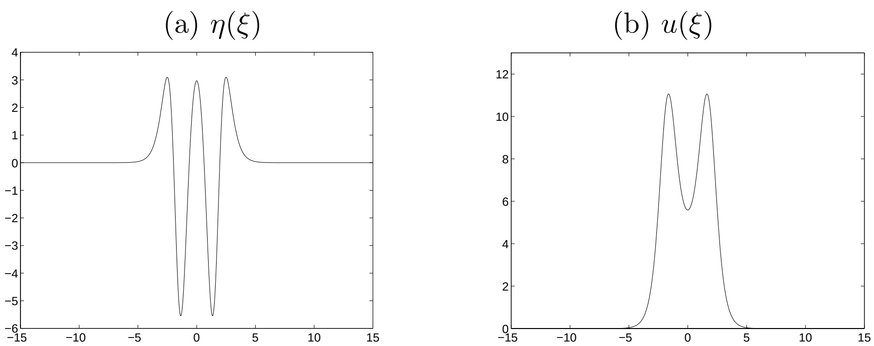

FIGURE 4. a two-trough solution ($k = 4.0$)

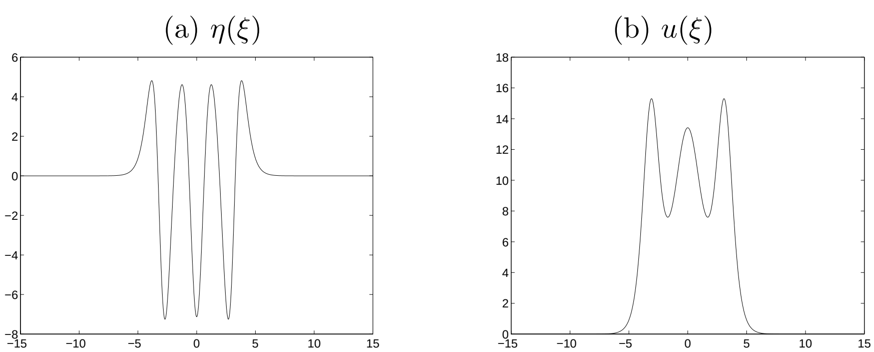

FIGURE 5. a three-trough solution ($k = 5.5$)

troughs can be found with exactly the same technique, so we conjecture that Tables 1 and 2 can be extended indefinitely.

5. Solution branches of the regularized Boussinesq system

To obtain a general picture of the multi-pulsed solutions of (1), it is helpful to study whole solution branches obtained by continuation from a given approximate solution. The solution branches can be calculated numerically by using a toolbox HomCont in AUTO [22, 16] and the precise quantitative properties of these branches can be obtained.

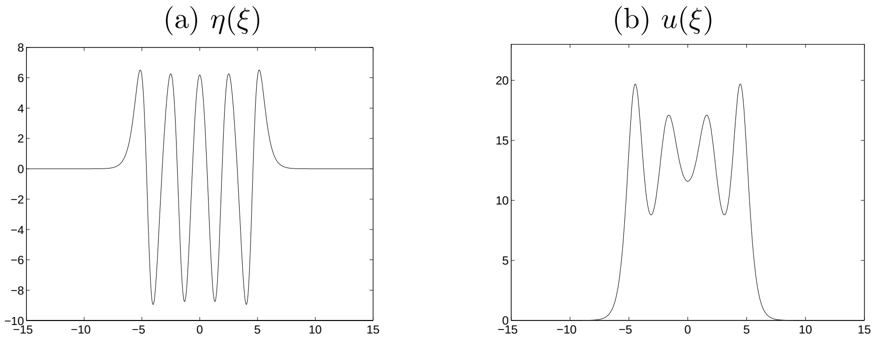

FIGURE 6. a four-trough solution ($k = 7.0$)

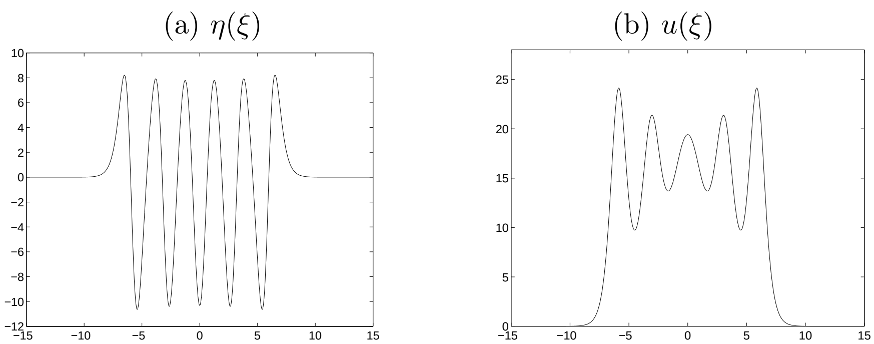

FIGURE 7. a five-trough solution ($k = 8.5$)

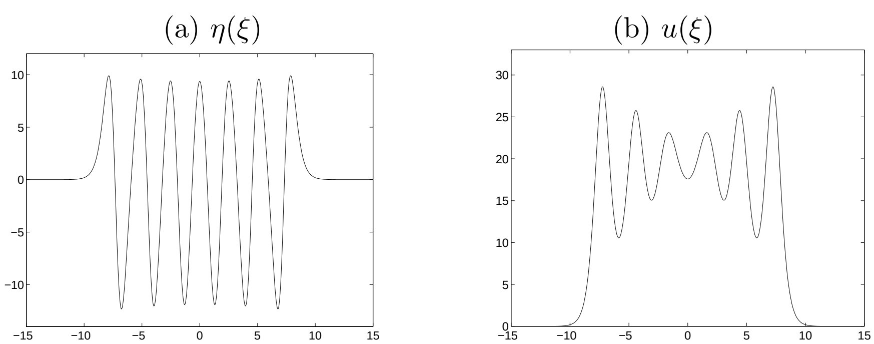

FIGURE 8. a six-trough solution ($k = 10$)

We start the investigation from analyzing the solitary-wave solution branch which has been proven to exist and its *a priori* estimate is given in Corollary 2.1. The initial solution used to continue the branch is

$$u_{int}(\xi) = \eta_{int}(\xi) = \eta_0 sech^2 \left(\frac{1}{2}\sqrt{\frac{3\eta_0}{k}}\ \xi \right), \text{ with } k = 1.05 \text{ and } \eta_0 = 0.1, \tag{23}$$

which is the first-order approximation of a solution (u, η) with respect to α (see [7]). There are other solutions, such as the accurate numerical solution generated in [7], which could also be used, but formula (23) is chosen because of its simplicity. For the single trough solution branch, the obvious choice as a starting point

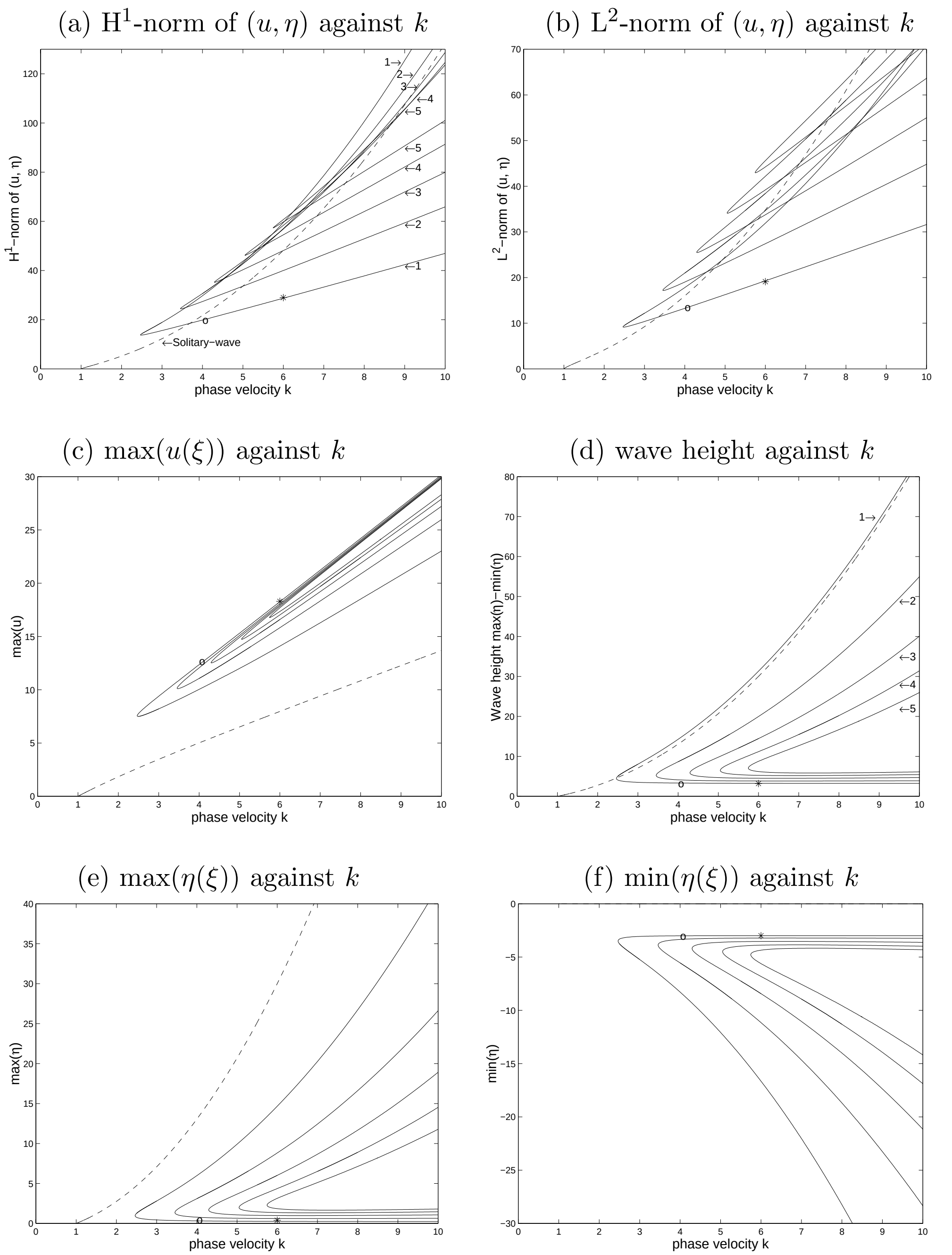

FIGURE 9. Bifurcation diagrams of regularized Boussinesq system.

for continuation is the solution (11). The continuation of multi-pulsed solutions starts from the solutions found in Section 4.

In Figure 9, the bifurcation diagrams of the solitary-wave solutions (dashed line) and the multi-pulsed solutions with 1 to 5 troughs (solid lines) are presented. The H^1-norm

$$\left(\int_{-\infty}^{\infty}\left[\eta^2+u^2+\eta_\xi^2+u_\xi^2\right]d\xi\right)^{\frac{1}{2}}$$

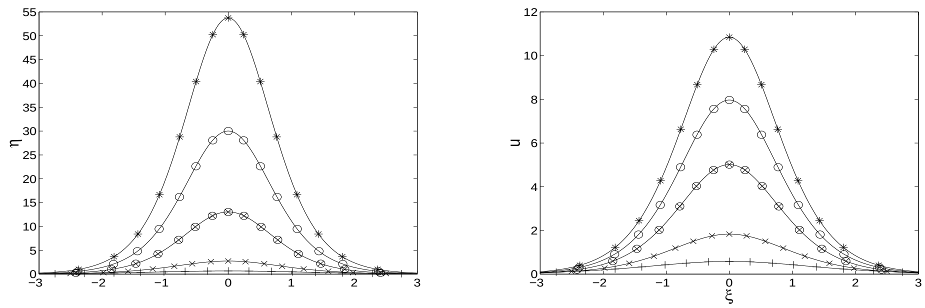

FIGURE 10. Solitary-wave solutions with phase speed $k = 1.3$, '+'; $k = 2$, 'x'; $k = 4$, '$\otimes$'; $k = 6$, 'o'; and $k = 8$, '$*$'.

and the L^2-norm of the solution (u, η), the amplitude of the velocity: $\max u(\xi)$, the wave height $\max \eta(\xi) - \min \eta(\xi)$, the height of the crests $\max \eta(\xi)$ and the depth of the troughs $\min \eta(\xi)$ with respect to k are plotted. The numbers marked in Figure 9(a) indicate the number of troughs. The two points marked by "o" and "$*$" correspond to two one-trough solutions which are plotted in Figure 11.

From Figure 9, one can see that the only branch which continues through the range $(1, +\infty)$ is the solitary-wave branch. Other branches possess a turning point at a point $k_c > 1$ and the branches cease to exist below that. There is an one-parameter family of solitary waves with wave height ranging from 1 to $+\infty$ which confirms the result in Theorem 2.2. Note from Figure 9(a) and 9(b) that the multi-pulsed solutions exist when the phase speed and amplitude are large, and the depth of the troughs is at least 2.5 measured from the still water level. Our numerical results show that the solutions with one trough exist for $k \geq 2.5$ and the waves with two troughs exist for $k \geq 3.5$. A solution from the "upper" branch in Figure 9(a) has larger H^1-norm, and also has larger L^2-norm and larger wave height but smaller magnitude of velocity u, than the corresponding solution from the lower branch with the same k. The velocity u is always positive. Solutions found in last section which are plotted in Figures 3-8 are from the lower branch. Their locations in Figure 9 can be determined by the values of the phase speed k. Figure 9 also indicates that at any given k, if there is a solution with N_0 troughs, then there is a solution with N troughs where N is any positive integer smaller than N_0.

In Figures 10–12, individual solutions along the branches are presented. Five solitary-wave, five one-trough and five three-trough solutions are plotted. Again, the locations of these solutions in Figure 9 can be determined by the values of the phase speed k. In Figure 10, we plot solitary-wave solutions with phase speed $k = 1.3, 2, 4, 6$ and 8 respectively. The amplitude of the wave and the magnitude of the velocity are monotonically increasing as k increases (cf. Figure 9). As a check on the theory, the *a priori* estimates obtained in Corollary 2.1 can be checked numerically. In Figure 11, five one-trough solutions with phase speed 2.466, 4 and 6 are plotted. There are two one-trough solutions at $k = 4$ and $k = 6$ respectively. Curves marked with "+" and "x" are from the upper branch and curves marked with "o" and "$*$" are from the lower branch. The solution marked by "$\otimes$" is near the turning point, which has the smallest H^1-norm, the smallest L^2-norm and the smallest L^∞-norm of velocity u among the solutions from the

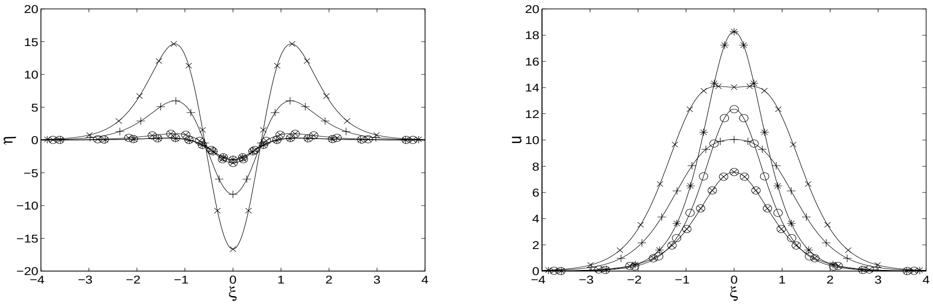

FIGURE 11. Solutions along the one-trough solution branch in Figure 9, "$\otimes$" is near the turning point with $k = 2.466$, "+" and "x" are from the upper branch with $k = 4$ and $k = 6$, "o" and "$*$" are from the lower branch with $k = 4$ and $k = 6$.

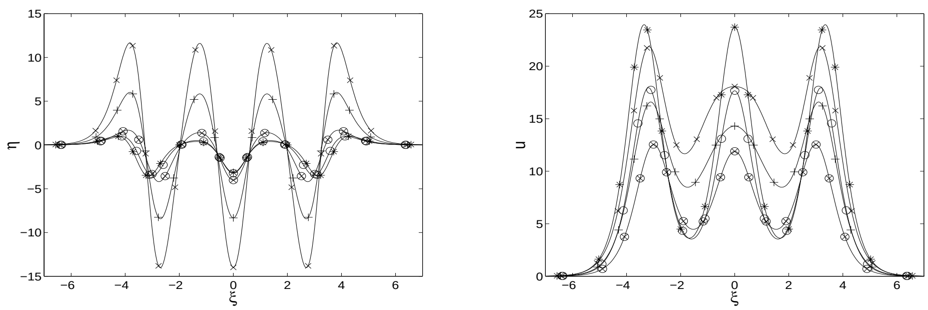

FIGURE 12. Solutions along the three-trough solution branch in Figure 9, "$\otimes$" is near the turning point with $k = 4.3$, "+" and "x" are from the upper branch with $k = 6$ and $k = 8$, "o" and "$*$" are from the lower branch with $k = 6$ and $k = 8$.

same branch. Figure 12 is similar to Figure 11, but five three-trough solutions are plotted. Solutions from the branches can also be viewed by plotting their trajectories. The solitary-wave, one-trough, and up-to-five trough solutions with $k = 6$ are plotted in Figure 13. The trajectories start from the curve $f(u, \eta) = 0$ (dashed line) where $f(u, \eta)$ is defined in (6) and end at the origin.

Remark 3. Since the model systems in (1) are appropriate approximations to the Euler's equations only for small amplitude and long waves, these multi-pulsed solutions are likely to be artifacts of the modeling rather than a reflection of physical reality since their amplitudes are well beyond those needed to derive the models.

Remark 4. It is interesting to note that even though the systems in (1) have the same formal accuracy as the uni-directional wave equations such as the KdV-equation and the regularized long wave equation, the solution set of traveling waves of these bi-directional wave systems is much more complex.

6. SOLUTION BRANCHES WITH RESPECT TO CHANGE OF SYSTEM PARAMETERS

In this section, we study the "structural stability" of the class of systems under change of system parameters a, b, c and d. Instead of studying the systems

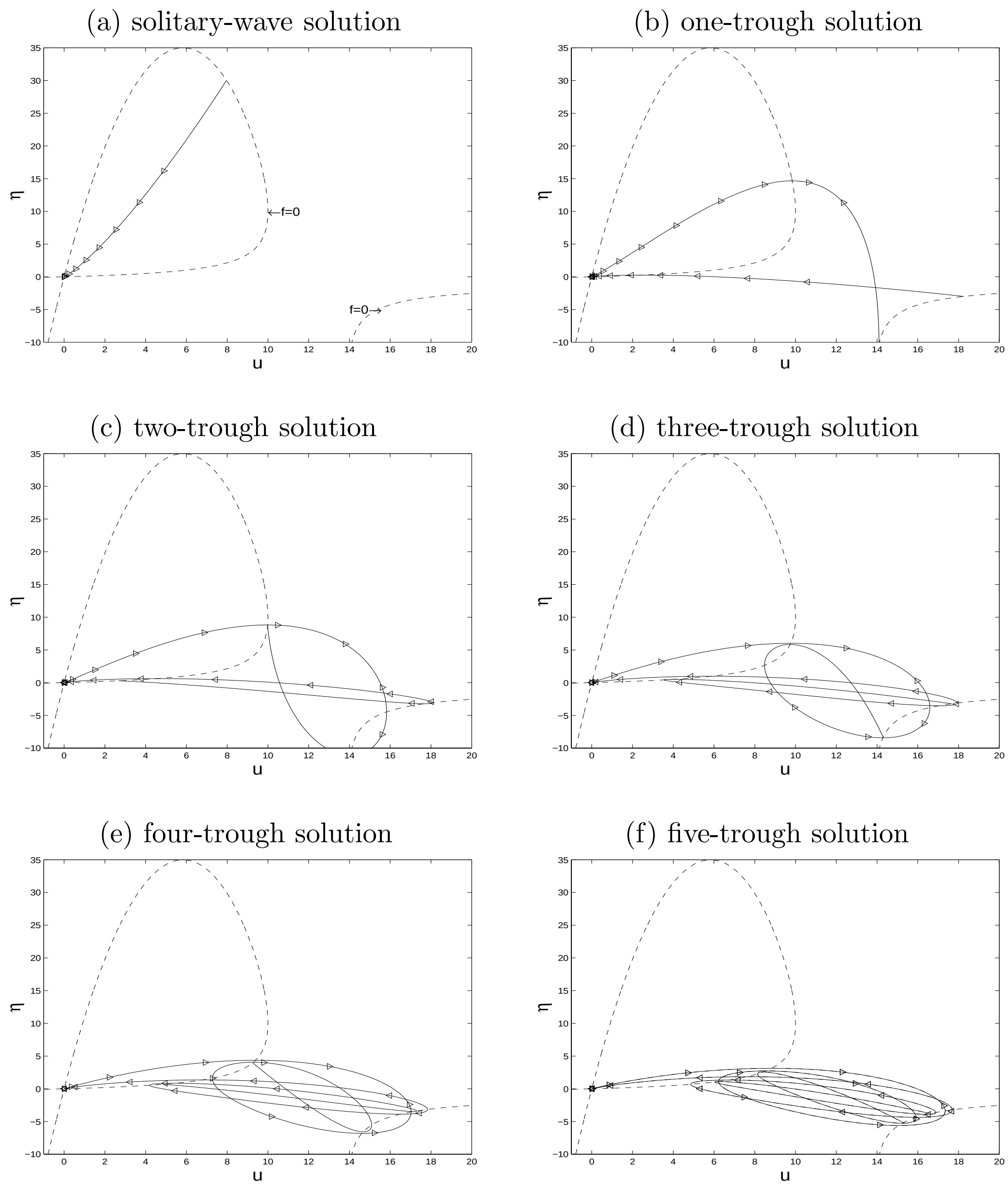

FIGURE 13. The trajectories of solutions (u, η) when $k = 6$.

individually, we will study an one-parameter family of them and investigate the relationship between them. Systems in (1)-(2) with different a, b, c and d have different mathematical properties in terms of the integrability, the Hamiltonian structure and the well-posedness, but they all model the same physical situation, that is the small amplitude long waves. A solution which is "stable" under changes of a, b, c and d will have a better chance to be physically meaningful. We therefore will study the effect of a, b, c and d on the solitary-wave and multi-pulsed solutions. For a given solution with a phase speed k, the solution branch will be continued by varying the system parameters.

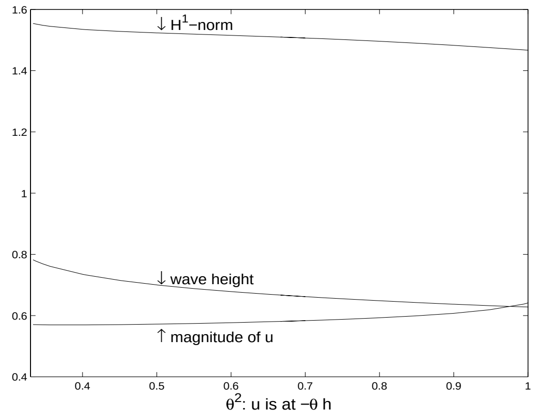

FIGURE 14. Bifurcation diagram of solitary-wave solutions when $k = 1.3$.

Letting $\Delta = ac - bdk^2$ and supposing $\Delta \neq 0$, the system of ordinary differential equations (4) is not degenerated and has dimension four. Denoting

$$g_1(u,\eta) = u - k\eta + u\eta, \quad g_2(u,\eta) = -ku + \eta + \frac{1}{2}u^2$$

and letting $x_1 = \eta$, $x_2 = \eta'$, $x_3 = u$, $x_4 = u'$, it can be written as

$$\dot{\mathbf{x}}(\xi) \equiv \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{1}{\Delta}(-ag_2 + dkg_1) \\ x_4 \\ \frac{1}{\Delta}(-cg_1 + bkg_2) \end{pmatrix} \equiv \mathbf{f}(\mathbf{x}), \tag{24}$$

which has parameters a, b, c, d and k. A complete theoretical or numerical investigation of (**??**), especially its bifurcation diagrams with respect to all the parameters, is beyond the scope of this paper. In this section, we will only investigate an one-parameter family of systems for a few given phase speeds. More precisely, we will consider systems with $\lambda = \mu = 0$ and various θ such that $\theta^2 \in (\frac{1}{3}, 1)$, so

$$a = c = 0, \quad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad d = \frac{1}{2}\left(1 - \theta^2\right) \tag{25}$$

according to (2). The point $\theta^2 = \frac{2}{3}$ corresponds to the regularized Boussinesq system and $\theta^2 = \frac{1}{3}$ corresponds to the classical Boussinesq system. The corresponding ordinary differential equation is

$$-bdk^2u^{(4)} + \frac{1}{3}k^2u'' + (1-k^2)u - bk(u')^2 - \frac{1}{3}kuu'' + \frac{3}{2}ku^2 - \frac{1}{2}u^3 = 0$$

and the eigenvalues about the origin are real, and of the form $-\lambda_1, -\lambda_2, \lambda_2, \lambda_1$ where

$$\lambda_1 = \sqrt{\frac{1}{2bd}\left(\frac{1}{3} + \sqrt{(b-d)^2 + \frac{4bd}{k^2}}\right)}, \quad \lambda_2 = \sqrt{\frac{1}{2bd}\left(\frac{1}{3} - \sqrt{(b-d)^2 + \frac{4bd}{k^2}}\right)}.$$

The software AUTO is again used to study the solution branches. The solutions of regularized Boussinesq system will be continued up to the surface $ac - bdk^2 = 0$ where the corresponding dynamical system becomes two-dimensional.

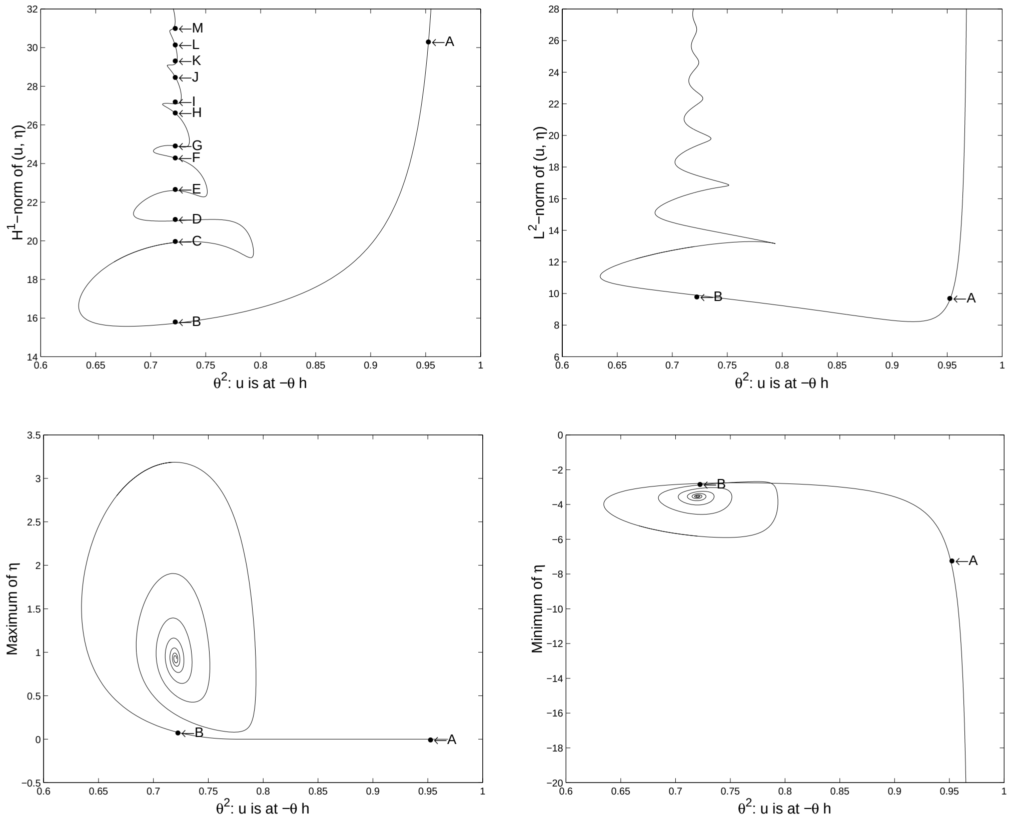

FIGURE 15. A bifurcation diagram of solutions with $k = 3$.

We start the investigation on the solitary-wave solution branch, since such solutions played an important role in the development of water wave theory. The results are shown in Figure 14 which demonstrates that every system in (1)-(25) has a solitary-wave solution with $k = 1.3$. Hence, the existence of solitary-wave solutions is shown for infinite many systems, which are obtained by taken velocities at different depths $y_\theta = -\theta h_0$. The H^1-norm of the solution (u, η), the wave height: $\max(\eta(\xi))$ and the amplitude of the velocity: $\max(u(\xi))$ with respect to θ^2 are plotted. It is observed that the solutions corresponding to different systems agree with each other in the same order of accuracy as the model systems. The wave heights are between 0.6 and 0.8 with the solitary wave associated to the classical Boussinesq system being higher than the one to regularized Boussinesq system.

In Figure 15, the solution branch with $k = 3$ obtained by continuation starting from the exact one-trough solution (11) is plotted. The H^1-norm and the L^2-norm of the solution (u, η), the height of the crests: $\max(\eta(\xi))$ and depth of the troughs: $\min(\eta(\xi))$ are plotted. The result is very different from that in Figure 14. The curve does not extend to all θ^2 in $(\frac{1}{3}, 1)$ and the solutions along the branch consist of one trough, two troughs, and many troughs. Solutions at points A to M are plotted in Figures 16-28. Towards one end of the curve (toward point A), the solution becomes a single trough and the depth is getting deeper and deeper which leads to a blow-up solution. Towards other end of the curve, the number

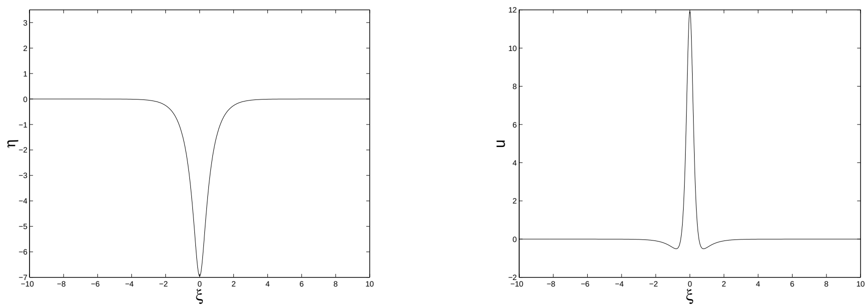

FIGURE 16. Solution at point A in Figure 15 with $\theta^2 = 0.9$ and $k = 3$.

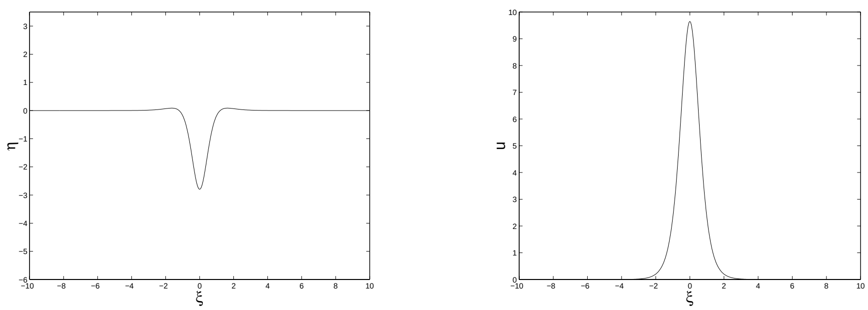

FIGURE 17. Solution at point B in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

of pulses is increasing, so is the spread of the wave which leads to a blow-up solution in H^1- and L^2-norm.

Figure 14 and 15 are only two snapshots of the complicated structure of the four-dimensional dynamical system (4). More figures can be generated on solution branches with other phase speed k and on solution branches started from solutions of a different system by restarting the process in Section 4 and 5. Further study on the qualitative and quantitative behavior of the dynamic system (4), such as the structures of these bifurcation diagrams, the structures of the blow-up solutions, the stability of these special solutions and also the solution branches with respect to the change of other parameters, will be carried out elsewhere.

It is worth to note that the continuation technique presented in this paper is effective such that after finding one solution of a system for a phase speed, it is possible to find a similar solution of another system with another phase speed by varying the system parameters and the phase speed.

7. Conclusions

In this paper, we studied the solitary-wave and multi-pulsed solutions on the systems which are formally equivalent to the famous KdV equation, but can model waves traveling in both directions.

In Theorem 2.3, we proved a sufficient condition for a system in (1) to have solitary-wave solutions with phase speed k. It was verified that the regularized Boussinesq system satisfies such condition and hence has solitary-wave solutions

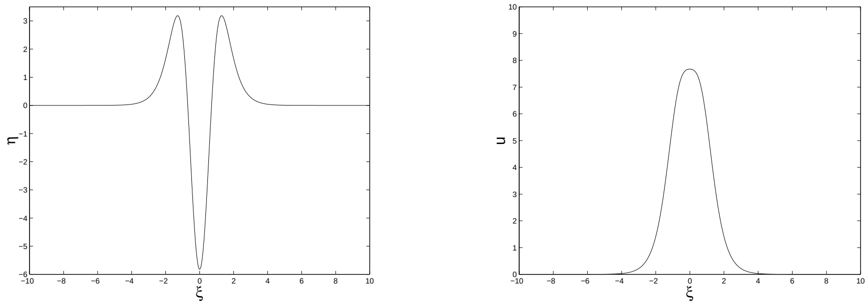

FIGURE 18. Solution at point C in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

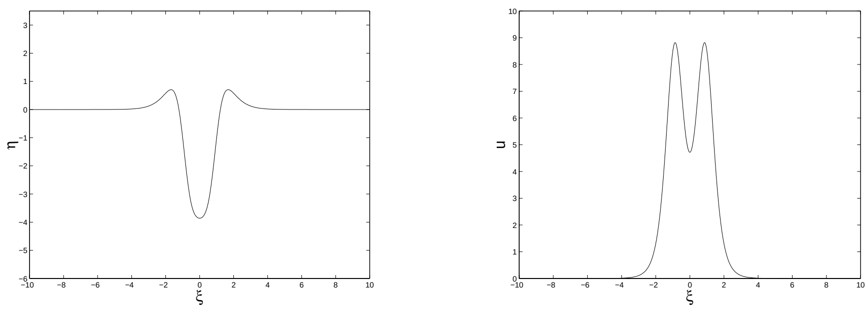

FIGURE 19. Solution at point D in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

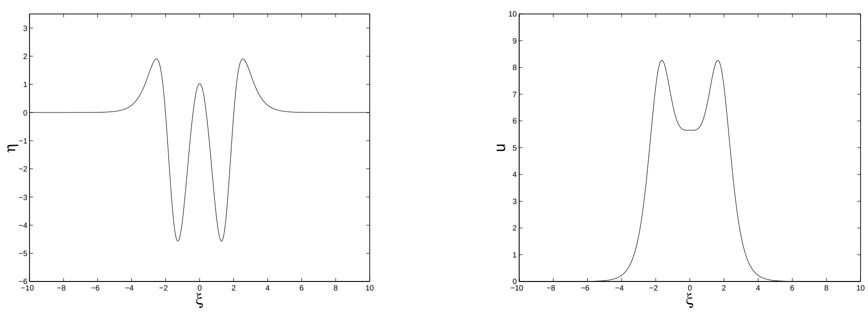

FIGURE 20. Solution at point E in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

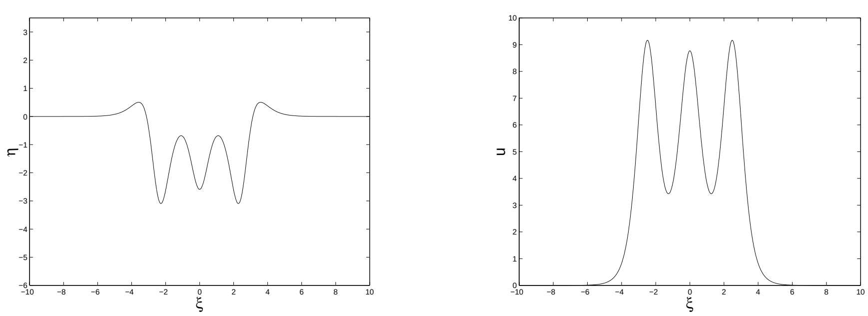

FIGURE 21. Solution at point F in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

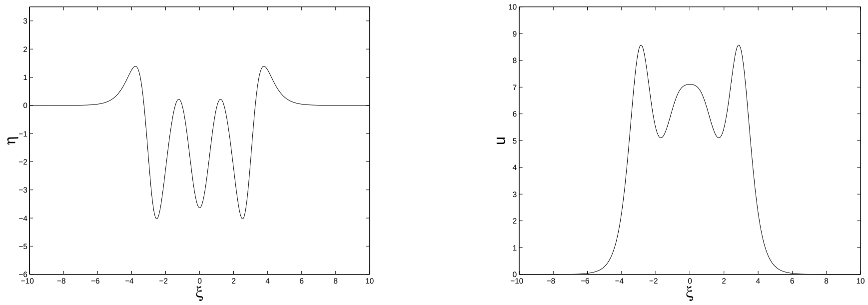

FIGURE 22. Solution at point G in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

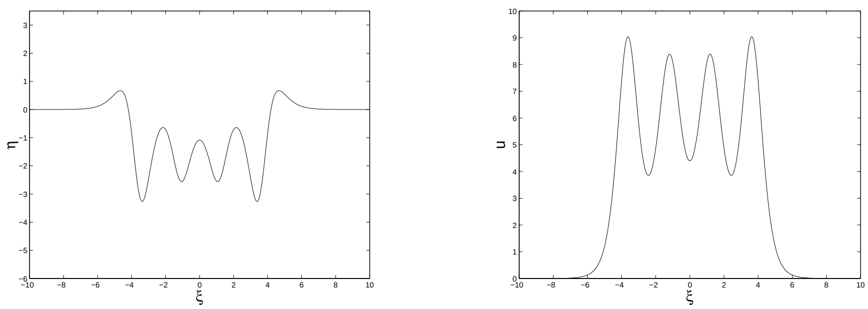

FIGURE 23. Solution at point H in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

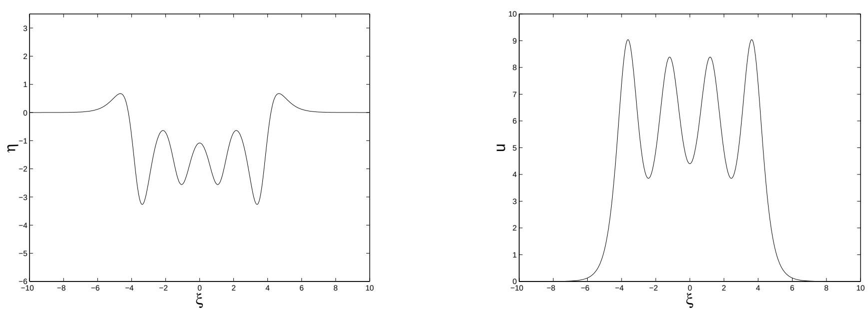

FIGURE 24. Solution at point I in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

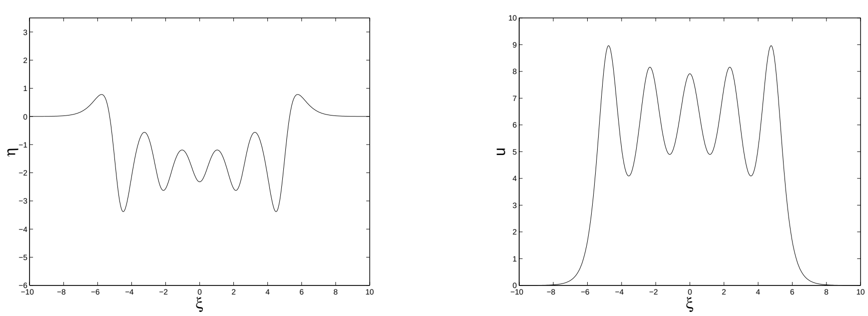

FIGURE 25. Solution at point J in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

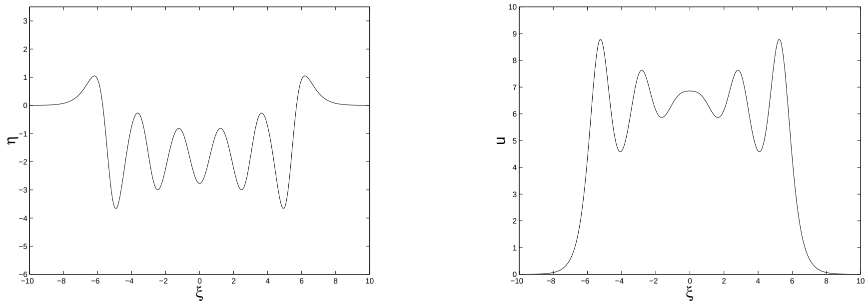

FIGURE 26. Solution at point K in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

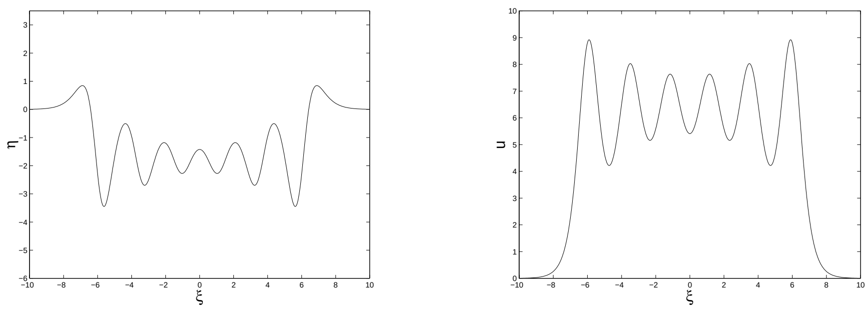

FIGURE 27. Solution at point L in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

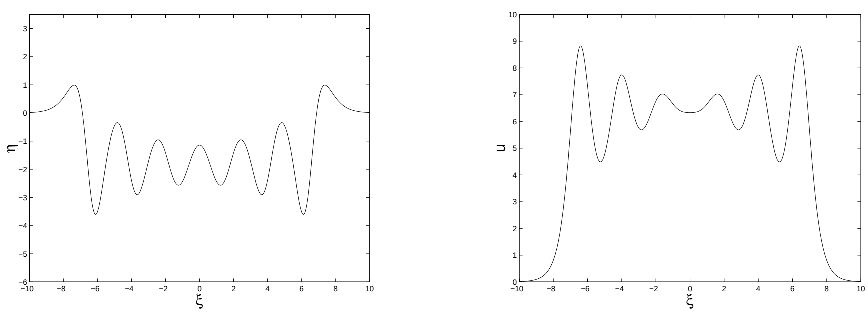

FIGURE 28. Solution at point M in Figure 15 with $\theta^2 = 0.72$ and $k = 3$.

with any $k > 1$. A priori estimates on such solutions were presented in Corollary 2.1.

In Section 3, we studied the Kaup system, the classical Boussinesq system and the KdV-regularized long-wave system. By investigating the phase diagrams of these systems, we proved that a unique traveling-wave solution which vanishes at infinity exists for each system with any given $k > 1$.

In Section 4, we found solutions with multiple pulses for the regularized Boussinesq system. The bifurcation diagrams of these solutions were studied in Section 5. The numerical results indicate that multi-pulsed solutions exist for any phase speed larger than a critical phase speed k_c.

In Section 6, we studied an one-parameter family of systems, obtained by taking velocities at different depths in the water tank. It was demonstrated that every one of these systems has a solitary-wave solution. In addition, the solutions, which corresponding to different systems, are consistent with each other in the same order of the accuracy as the model systems. This result is very encouraging since it indicates that each one of these systems (infinite many) can model solitary waves as they are suppose to, and the result is accurate for small-amplitude long waves. For large amplitude waves, the situation is very different and different systems behave very differently. It is demonstrated that some systems have multi-pulsed solutions and some don't. For the systems which do have multi-pulsed solutions, the solutions are very different.

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