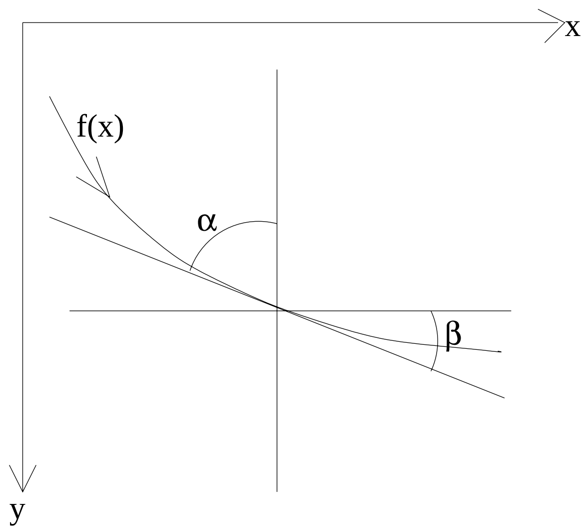

We consider the vertical plane P, containing the given points, A and B, and assume that A is higher than B. Then we introduce the rectangular coordinates in this plane, with the origin at A, horizontal x axis, and the y axis pointing *down*. Let $y = f(x)$ be the function whose graph is the curve of minimal time from A to B.

If the motion starts from rest at A, and there is no friction, the speed v of the particle at a point (x, y) can be found from the energy conservation law

$$mv^2/2 = mgy \quad \text{that is} \quad v = \sqrt{2gy}.$$

On the other hand, Snell's law and Fermat's principle tell us that $\sin\alpha$ (see the picture) should be proportional to the speed v. (Think about this, this is a crucial point of Bernoulli's argument!) We compute $\sin\alpha$ in terms of the derivative,

$$\sin\alpha = \cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}}.$$

Combining these two equations we obtain

$$y\left(1+(y')^2\right) = k,$$

where k is a constant. This equation is separable, and we have

$$dx = \sqrt{\frac{y}{k-y}}dy.$$

Such integrals are evaluated by trigonometric substitutions. In our case the proper change of variables is

$$\sqrt{\frac{y}{k-y}} = \tan t,$$

that is

$$y = \frac{k}{2}(1 - \cos 2t). \tag{1}$$

Using this substitution and integrating, we obtain

$$x = \frac{k}{2}(2t - \sin 2t) + C. \tag{2}$$

From the condition that our curve poasses through the origin, we conclude that $C = 0$. The equalities (1) and (2) five a parametric description of our curve. Putting $\theta = 2t$ and $a = k/2$ we get a simpler parametrization:

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(1 - \cos\theta).$$

This curve is a cycloid. It is easy to visualize as the trajectory of a point on the circumference of a wheel, rolling along the x axis.

Source: G. Simmons, *Differential equations with applications and historical notes*, McGraw Hill, NY, 1072.