Find maximum of one function of one variable

A. Eremenko

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1. Consider the rectangular lattice $\Lambda = \{an + ibm : n, m \in \mathbf{Z}\}$ where a and b are real numbers,

$$a^2 + b^2 = 1, \quad a \in (0, 1). \tag{1}$$

Let $\mathbf{U}$ be the unit disc, and $f_a : U \to \mathbf{C} \backslash \Lambda$ the universal covering map with $f_a(0) = (a + ib)/2, \ f_a'(0) > 0$.

Maximize $f_a'(0)$ for $a \in (0, 1)$.

2. Of course, this comes from the attempt to find the exact constant in the Landau theorem. Landau's theorem says that if f is analytic in the unit disc, and $f'(0) = 1$ then the image of f contains some disc of radius $L - \epsilon$, for every $\epsilon > 0$. The problem is to find the optimal constant L.

Let D be the image domain, and let R be the inner radius of of D (which is the sup of the radii of discs contained in D). Let λ be the (linear) density of the hyperbolic metric in D. Then for every analytic function $f : \mathbf{U} \to D$ we have $|f'(0)| \leq 1/\lambda(f(0))$, and this is optimal. So one needs the infimum of $\lambda(w)$ over all $w \in D$ and all D with fixed inner radius.

This seems hopeless. One can restrict the problem by considering only those regions D whose complements are lattices. A. Baernstein proved that among such regions the *local* minimum of λ occurs in the center of the complement of the hexagonal lattice.

Our problem is to find the similar infimum for *rectangular lattices*.

3. The function f_a in section 1 has almost explicit representation. Consider the quadrilateral Q in the unit disc bounded by arcs of circles orthogonal to the unit circle, symmetric with respect to the coordinate axes, and having one vertex $exp(i\theta)$, where $0 < \theta < \pi/2$. Then there is a unique $a \in (0, 1)$ such that Q is conformally equivalent to the rectangle $[0, a, a+ib, ib], \ b = \sqrt{1 - a^2}$,

by a conformal map which sends vertices to vertices and $\exp(i\theta)$ to $a + ib$. It is easy to see that f_a is the inverse to this map. One can show that our inverse map can be expressed in terms of solutions of the a Lamé differential equation

$$w'' = (\wp - c)w,$$

but determining the accessory parameter c is not a trivial matter.