Modified Cartan's Conjecture

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Denote $D(R) = \{z : |z| < R\}$. For a fixed integer $p \geq 3$, consider the set $V(D)$ of all vectors $f = (f_1, \ldots, f_p)$, where f_j are functions, holomorphic and zero-free in a region D in the complex plane, and satisfying

$$f_1 + f_2 + \ldots + f_p = 0.$$

If S is an infinite sequence of such vectors, and $D_1 \subset D$, we call a subset $I \subset \{1, \ldots, p\}$ a C-class for S in D_1, if
(a) for some $k \in I$ all sequences $(f_j/f_k),\ f \in S,\ j \in I$ are uniformly bounded on compact subsets of D_1, and
(b) $\sum_{j \in I} f_j/f_k \to 0$ for $f \in S$, uniformly on compact subsets of D_1.

It follows from (b) that every C-class contains at least 2 elements.

Conjecture. *Given an infinite sequence S of vectors in $V(D(1))$, there exists and infinite subsequence S' of S, such that the set $\{1, \ldots, p\}$ is a union of disjoint C-classes for S' in $D(R_p)$, where $R_p > 0$ depends only on p.*

For $p = 3$ one can take $R_3 = 1$, and the Conjecture is equivalent to Montel's Theorem. For $p = 4$ one can also take $R_4 = 1$, and in this case the Conjecture is a consequence of the following result of H. Cartan, which is true for every p:

Cartan's Theorem [1,3] *Given an infinite sequence of vectors in $V(D(1))$, there exists a subsequence S' of S, such that either the set $\{1, \ldots, p\}$ constitutes a C-class, or it contains at least two disjoint C-classes.*

When $p = 5$ one cannot take $R = 1$ anymore, but the Conjecture is true with $R_5 = 1/64$ [2]. If the Conjecture is true, can one take $R_p > 0$

independent of p? What is the geometric interpretation of the Conjecture? Apparently it says something on the Kobayashi pseudometric in $p-2$ dimensional projective space minus p hyperplanes in general position.

[1] H. Cartan, Ann. Sci. Ecole Norm. Super., 45 (1928), 255-346.
[2] A. Eremenko, Amer. J. Math., 118 (1996), 1141-1151.
[3] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer, 1987.