# Cellar theory

A. Eremenko

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The temperature on the Earth surface at a given place experiences roughly speaking periodic fluctuation, daily and yearly. Let us try from a given periodic temperature on the surface to predict what happens on a small depth.

We consider the heat equation

$$u_t = ku_{xx},$$

where $x \geq 0$ is the depth, and suppose that the dependence of $u$ on time $t$ is expressed by a simple periodic function, say

$$u(x,t) = f(x)T(t),$$

where $f$ is bounded and $T$ periodic.

Separating the variables we obtain

$$\frac{T'}{T} = k\frac{f''}{f} = \lambda.$$

Function $T$ cannot be periodic unless $\lambda$ is pure imaginary, say $\lambda = i\omega$, where we may assume that $\omega > 0$. Then for $f$ we obtain $kf'' = i\omega f$, so

$$f(x) = c_1e^{\mu x} + c_2e^{-\mu x}, \quad \text{where} \quad k\mu^2 = i\omega.$$

We need square root of $i$. There are two of them,

$$\pm(1+i)/\sqrt{2}.$$

We can use only that with negative real part, because we want $f$ to be bounded. So

$$f(x) = c\exp\left(-\sqrt{\frac{\omega}{2k}}(1+i)x\right), \quad \text{and} \quad T(t) = e^{i\omega t}.$$

Taking the real part of the product we obtain

$$u(x,t) = \exp\left(-x\sqrt{\frac{\omega}{2k}}\right)\cos\left(\omega t - x\sqrt{\frac{\omega}{2k}}\right).$$

Notice the following features:

a) The amplitude of oscillation decreases with depth exponentially.

b) There is a phase shift at a depth.

For a typical thermal diffusivity of soil $k = 10^{-7} m^2/sec$, and period of the oscillation 1 year, find at what depth $x$, the shift of the seasons will be 1/2 of a year.

We have $x = \pi\sqrt{2k/\omega}$ meters. (One year is $60 \cdot 60 \cdot 24 \cdot 365 \approx 3 \cdot 10^7$sec.) So $\omega \approx (2\pi/3)10^{-7}$. So with this value of thermal diffusivity we obtain that in this kind of soil at the depth of about 3 meters the seasons switch: the maximum temperature happens in winter and the minimum in summer.

By what factor the amplitude of the oscillations is smaller at this depth than the amplitude on the surface?

This problem explains, by the way, why one can keep ice in a cellar in summer in the places where the average yearly temperature is somewhat above zero, but the fluctuation between winter and summer is large enough. Like in Wisconsin.