Cisotti formula

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During my office hours I was asked whether there exists a continuous analog of the Schwarz–Christoffel formula, for a conformal map of a disc onto any nice region.

Such a formula indeed exists; it was found by Umberto Cisotti (1921). The reference is apparently [1] but I am not sure, I have never seen this book. I follow Lavrentiev-Shabat [2, 3, 4], just translating a page from the book.

Let $w = f(z)$ be a conformal map of the unit disc onto a smooth Jordan region bounded by a curve C, and *suppose that we know the argument $\theta(t)$ of the tangent vector to the curve C at the point $f(e^{it})$.*

Think why this is the generalization of the data entering into the Schwarz–Christoffel formula.

On the unit circle we have $dz = ie^{it}dt$, and on the curve C we have $dw = |dw|e^{i\theta}$. Then

$$i\frac{dw}{dz} = e^{i(\theta-t)}\frac{|dw|}{dt}. \tag{1}$$

As f is conformal, $dw/dz \neq 0$ in the unit disc, so the function

$$-i\log\left(i\frac{dw}{dz}\right)$$

is analytic in the unit disc, and by (1), its real part on $|z| = 1$ equals $\theta - t$. On the other hand, if $z = e^{it}$, then

$$\Re\{-i\log[-(1-z)^2]\} = \pi + 2\arg(1-z) = t.$$

(Just make a picture to see this).

So the real part of the analytic function

$$g(z) = -i\log\left(-i(1-z)^2\frac{dw}{dz}\right) \tag{2}$$

on the unit circle coincides with θ. Thus g can be recovered from the Schwarz formula:

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \theta(t) \frac{e^{it} + z}{e^{it} - z} dt + ic,$$

where c is a real constant. Once we found g, we can find f from (2):

$$f(z) = i \int_{z_0}^{z} \frac{e^{g(\zeta)} d\zeta}{(1 - \zeta)^2} + w_0.$$

This is Cisotti's formula.

In general, it is as useless as the Schwarz–Christoffel formula, unless we know something about $\theta(t)$.

Exercise: derive the Schwarz–Christoffel formula from Cisotti's formula.

References

[1] U. Cisotti, Idromeccanica piana. I, II. Milano: Tamburini, 1921-22.

[2] М. А. Лаврентьев, Б. В. Шабат, Методы теории функций комплексного переменного, Москва, 1973 (4-th edition).

[3] M. A. Lawrentjew und B.V. Schabat, Methoden der komplexen Funktionentheorie, VEB, Berlin 1967.

[4] M. A. Lavrentiev, B. V. Shabat, Métodos de la teoria de las funciones de una variable compleja, Moscow, 1991.